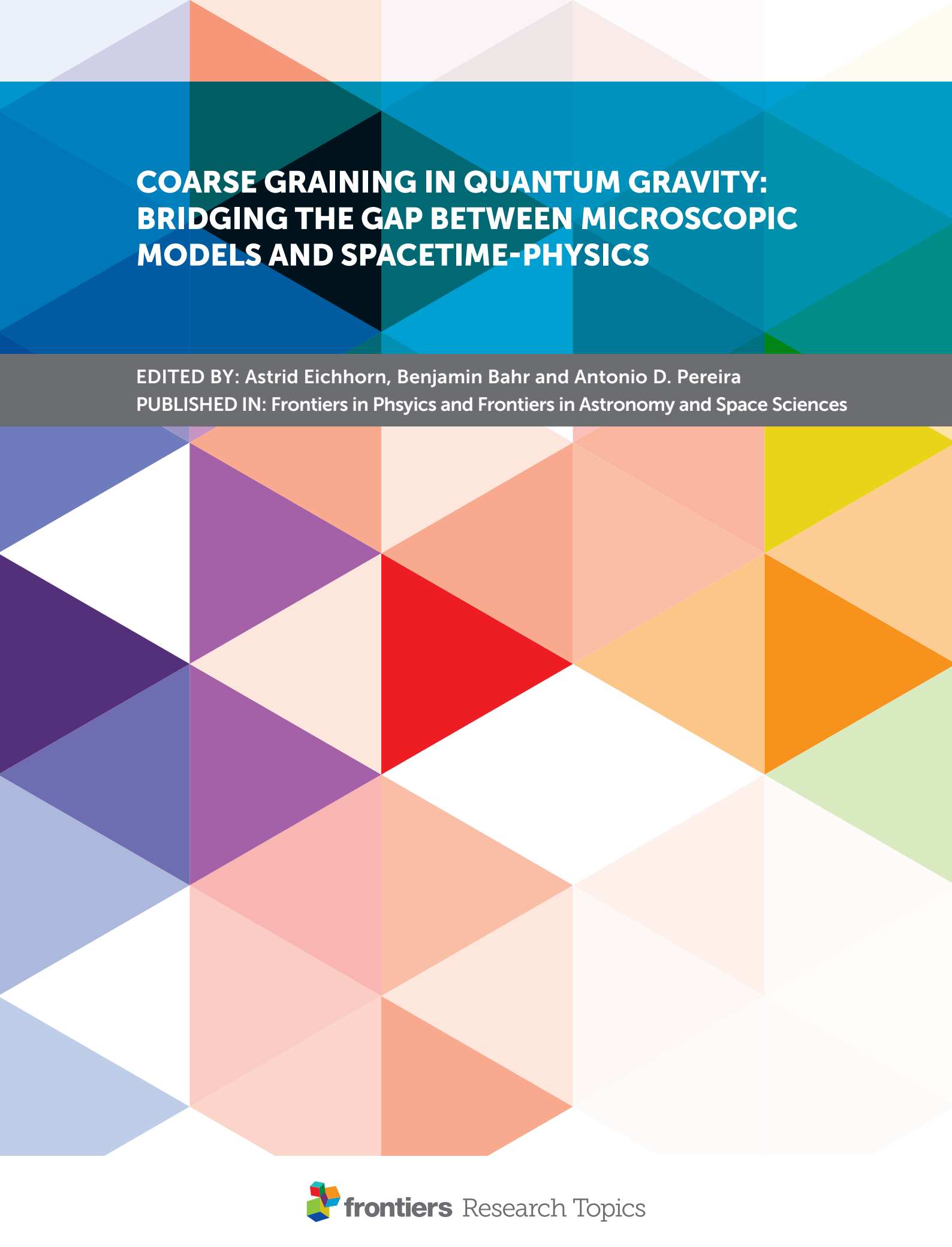




COARSE GRAINING IN QUANTUM GRAVITY: BRIDGING THE GAP BETWEEN MICROSCOPIC MODELS AND SPACETIME-PHYSICS

EDITED BY: Astrid Eichhorn, Benjamin Bahr and Antonio D. Pereira

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COARSE GRAINING IN QUANTUM GRAVITY: BRIDGING THE GAP BETWEEN MICROSCOPIC MODELS AND SPACETIME-PHYSICS

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Editorial: Coarse Graining in Quantum Gravity: Bridging the Gap Between Microscopic Models and Spacetime-Physics

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Coarse Graining in Quantum Gravity: Bridging the Gap Between Microscopic Models and Spacetime-Physics

Historically, a large number of approaches to quantum gravity take their starting point in the observation that perturbative renormalizability breaks down for the Einstein Hilbert action. This has triggered developments of various paradigms and frameworks that in a more or less radical sense depart from more standard quantum field theoretic ideas. Yet, early on, several ideas already existed on how to formulate quantum gravity as a local quantum field theory with a focus on its Renormalization Group (RG) behavior, including Weinberg's asymptotic-safety idea and Stelle's asymptotically free higher-derivative gravity. Today, the RG is experiencing a renaissance in quantum gravity. This goes significantly beyond the concept of renormalization as a systematic procedure for removing divergences. Rather, the RG plays a central role in the very definition of the quantum theory. In short, technical breakthroughs in various formulations of coarse graining enable a search for universality and scale symmetry in a broad range of setups. In the modern formulation of RG tools for quantum gravity, the notion of scale is implemented in several distinct, intricate ways that are even applicable to a pre-geometric setting.

In summary, a range of distinct approaches to quantum gravity are converging toward the point of view that coarse-graining and the associated notion of scale symmetry could enable us to probe properties of quantum space-time. This convergence in itself could prove to be a catalyst for breakthroughs: while every single approach to quantum gravity is facing open questions and challenges, both of conceptual and of technical nature, many insights obtained within the distinct approaches are in fact complementary. Questions that are seen as technically and/or conceptually challenging in one given approach, might be more easily tackled in another one. Therefore, the development of a common language like the RG and the associated unified conceptual framework holds the promise that important insights could be translated between approaches.

Moreover, there is the distinct possibility that what we now perceive of as different approaches to quantum gravity are in fact simply mathematically different formulations of the same physics. More specifically, different approaches could give rise to the same universality class, thus resulting in the same infrared physics. At the same time, a continuum limit encoded in that universality class could ensure the restoration of diffeomorphism symmetry in discrete approaches. We consider

the confluence of quantum-gravity approaches at this RG-vantage point a promising research area that is still in its relatively early stages. The special issue “Coarse graining in quantum gravity: Bridging the gap between microscopic models and spacetime physics” provides an incomplete snap-shot of this evolving field, highlighting novel ideas, pointing out open challenges and reviewing recent developments. The diverse perspectives brought together in this issue highlight the broad set of research lines converging toward each other as well as the broad range of research opportunities that are opening up.

A series of papers focuses on various aspects of the asymptotic-safety approach, both with continuum and lattice techniques.

In Donoghue provides a constructive criticism of the asymptotic-safety program and discusses a key open question of the current state of the art of the asymptotic safety program, namely the Lorentzian signature of space time. In Bonanno et al. the authors critically reflect on the state of the art in asymptotically safe quantum gravity, providing a comprehensive list of open questions and critically reviewing potential pathways to finding answers both within the functional RG approach and lattice techniques. These two papers exemplify the usefulness of constructive criticism across research lines in quantum gravity.

In Ambjorn et al. use the Causal Dynamical Triangulation approach as a concrete framework to search for asymptotic safety in quantum gravity. They use measurements of the correlation function of the spatial volume profile of the emergent effective spacetime to define lines of constant physics and search for a UV fixed point.

Two papers deal with two aspects of background independence in functional RG techniques for quantum gravity. In Pawłowski and Reichert provide a comprehensive review of fluctuation field calculations in the functional RG framework for asymptotically safe quantum gravity. The fluctuation field arises within the background field methods that enables the definition of a local coarse graining procedure in a background independent manner. In Pagani and Reuter explore the peculiarities and physical implications of background-independent RG flows using field theoretic degrees of freedom like the metric. The requirement of background independence is encoded in self-consistent backgrounds. Both technically and conceptually this constitutes a departure from standard coarse-graining procedures on a fixed background. Specifically, the authors discuss how the so-called naturalness problem regarding the cosmological constant is profoundly different from this perspective.

In Platania reviews developments that tackle a central question in any approach to quantum gravity, namely its phenomenological viability. More specifically, she explores potential consequences of gravitational anti screening, associated with the UV limit in asymptotically safe quantum gravity, in early-universe cosmology. The current state of the art in the field does not yet allow a robust derivation of effects in cosmology from a fundamental gravity theory, instead one can develop and analyze quantum-gravity inspired models. The tentative nature of the link between a fundamental theory and cosmological observations also makes a quantitative

comparison between distinct approaches to quantum gravity challenging. In the absence of experimentally measurable observables, the comparison of characteristic properties of the quantum geometry between approaches to quantum gravity is a potential pathway to find commonalities or differences. In this spirit, Kurov and Saueressig link functional RG techniques to the analysis of geometric operators and observables in quantum gravity.

In Held discusses the notion of effective asymptotic safety. This can be viewed as an additional step in Weinberg’s translation of interacting RG fixed points from statistical physics, as the asymptotically safe fixed point is approached at scales that are infrared scales when viewed from a more fundamental model (while they remain UV scales compared to, e.g., the electroweak scale). In that scenario, an interacting fixed point is therefore approximately realized over an intermediate regime of scales, focusing trajectories starting from different microphysics onto common predictions for the macrophysics. This qualitative idea—that has the potential to provide a unification between asymptotically safe gravity and other approaches—is made quantitatively precise with a calculable notion of predictive power worked out in detail in gauge-Yukawa models.

Other approaches to quantum gravity in which RG ideas and techniques play a role are then discussed in the following list of papers.

In Steinhaus provides an updated review on the implementation of coarse-graining techniques in spin foam models. He focuses on modern techniques such as tensor network renormalization. Such techniques could allow to search for and probe the continuum limit of such models. In the closely related group field theories, in Finocchiaro and Oriti report on the status of simplicial group field theories and their renormalization. In particular, they present a perturbative computation of corrections to correlation functions and provide a road map for the field.

In Thiemann takes the canonical quantum gravity approach as a mathematically well-defined setup to tackle the problem of quantizing gravity. He applies the constructive quantum field theory program and, as such, advocates that renormalization is an essential ingredient to fix some ambiguities in the framework. This constitutes a review of a series of papers written by the author and collaborators, where Hamiltonian Renormalization is established in the canonical language by means of constructive quantum field theory. One of the papers in that series is also published in the present special issue, see by Liegener and Thiemann.

A different solution to the coarse-graining problem in gravity is implemented in the discrete, pre-geometric setting of matrix models by Castro and Kosłowski. In Castro and Kosłowski, they apply discrete functional RG techniques to matrix models which encode a preferred foliation. They discover fixed points which are compatible with known results from causal dynamical triangulations in two dimensions and thereby establish the applicability of these techniques in a causal setting. This work constitutes a compelling example of how numerical Monte-Carlo simulations can meet the computationally less expensive techniques arising from the functional RG.

In Steinwachs reviews developments that aim at formulating quantum gravity as a perturbatively renormalizable and ghost-free quantum field theory, enabled by a breaking of full diffeomorphism invariance to foliation-preserving diffeomorphisms.

In summary, these papers provide a partial snapshot of the state-of-the-art of the RG framework to quantum gravity that brings together previously disconnected approaches and is suitable to tackle both formal as well as phenomenological questions.

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A Critique of the Asymptotic Safety Program

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The present practice of Asymptotic Safety in gravity is in conflict with explicit calculations in low energy quantum gravity. This raises the question of whether the present practice meets the Weinberg condition for Asymptotic Safety. I argue, with examples, that the running of Λ and G found in Asymptotic Safety are not realized in the real world, with reasons which are relatively simple to understand. A comparison/contrast with quadratic gravity is also given, which suggests a few obstacles that must be overcome before the Lorentzian version of the theory is well behaved. I make a suggestion on how a Lorentzian version of Asymptotic Safety could potentially solve these problems.

Keywords: asymptotic safety, quantum gravity, effective field theory, quadratic gravity, Lorentzian

1. PREFACE

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Asymptotic freedom describes the situation where the coupling constants of a quantum field theory run to zero at asymptotically high energy. For renormalizable theories, this running is logarithmic in the momentum.

Asymptotic Safety (AS) describes the situation where the coupling constants run to an ultraviolet fixed point where the couplings are finite but where the beta functions vanish. While this can happen in a renormalizable field theory [1] where the running is logarithmic, its most common application is in the study of gravity [2–5]. In this case, the running is generically power-law, because of the dimensional coupling constants. In this paper I am discussing only the gravitational case with power-law running.

There is a conflict between the much of the present practice in AS and known explicit calculations of quantum processes in quantum gravity. This was originally pointed out in work with Anber [6]. At low energy calculations of quantum gravity processes can be carried out in the rigorous Effective Field Theory (EFT) treatment [7, 8] and we can compare these observables with the practice of Asymptotic Safety. The EFT is valid at low energies, which in this case means below the Planck scale. The major action in Asymptotic Safety happens around the Planck scale. Nevertheless, the AS techniques also apply below this scale, and predictions only emerge by running the cutoff to zero energy. Therefore in the overlap region we can make this comparison. More recently, explorations of quadratic gravity [9–21], which involves curvature-squared terms in the action, also shed light on the connection to AS. Quadratic gravity is a renormalizable theory for quantum gravity in the ultraviolet. It is somewhat more tentative and needs further exploration itself. However, it provides a calculational framework which is reasonably close to AS, such that it provides an interesting lessons for AS.

The present paper is an attempt to explain many of the issues involved. It has been invited to be part of a volume describing an overview of running couplings in gravity. It is meant both as a summary of concerns aimed at the AS community, and as an explication of the core issues for an outsider audience. As such it will contain comments which are unnecessary for an AS practitioner, as well as occasional technical details aimed only at the experts. I hope that this document can serve this dual purpose.

The reader will also notice that I often use the phrase “present AS practice.” This is because I want to differentiate between what is often done in the present AS literature from what could be the ultimate understanding of Asymptotic Safety. The AS paradigm is potentially an attractive resolution to the puzzle of quantum gravity. However, the present status is not yet a successful resolution. This article is then an attempt to point out shortcomings in the present practice as well as to point to future directions which may be fruitful.

1.1. Key Contrasts: Euclidean vs. Lorentzian, Powers vs. Logarithms, Cutoffs vs. Dimensional Regularization

As a preview to the more technical discussion which follows, let me mention some of the important issues which are central to that discussion.

The foundational technique of AS practice is the Euclidean functional integral. One studies this with an infrared cutoff and integrates out quantum effect in an energy scale around the cutoff. This is a variation of our usual way of using cutoffs in that the cutoff is introduced to keep the quantum effects *above* the cutoff and removes those with scales below the cutoff. The variation of the coupling parameters with that scale gives the renormalization group flow of the couplings. It is understood that running the cutoff down from the UV fixed point down to a zero value for the cutoff will then include all of the quantum corrections.

However, it is also common practice in the community to assign a meaning to the parameters at given values of the cutoff. For example, the running Newton constant in AS is often parameterized as

$$G(k) = \frac{G}{1 + Gk^2/g_*} \quad (1)$$

where k is the cutoff, and g_* is related to the fixed point in a way that will be described below. The use of the symbol k makes it tempting to think of k as a momentum (in practice it is closer to a mass cutoff) and to think of the resulting $G(k)$ as one that depends on the momentum scales in a reaction. This is incorrect, as we will see from direct examples in section 3.1. Moreover, even if it were a Euclidean momentum, its Lorentzian counterpart would be ill-defined. A large Euclidean momentum can translate to a massless on-shell Lorentzian particle if $k_0^2 - \mathbf{k}^2 = 0$ or to positive or negative values of the various kinematic invariants in reactions (i.e., $s > 0$ or $t < 0$). The basic question then is whether $G(k)$ at finite values of the cutoff has any physical meaning. Explicit calculations suggest that it does not.

A second point to watch is that the important features of AS do not occur when dimensional regularization is used. For example, if one truncates to the Einstein action, then the Newton constant does not run in dimensional regularization, contradicting Equation (1). At one level, this can be blamed on a known weakness of dimensional regularization. Near $d = 4$ it cannot identify quadratic divergences as it includes integrations over all scales. So it is perfectly allowable to use cutoffs to identify effects at a particular scale around the cutoff. But in the

end, *real physics should not depend on the regularization scheme*. I take it as given that dimensional regularization provides an acceptable regularization scheme to describe physical processes in field theory. I know of no counter-example. Moreover, I use dimensional regularization in the perturbative regime where it use in scattering amplitudes is unquestioned. So in the end, any scheme which uses cutoffs to define the theory should give the same physical predictions in such reactions. We need to understand how AS can do that. This is not a trivial constraint. In fact, we can understand how this occurs, but the resolution tells us that the running $G(k)$ is not valid for physical processes.

The other feature to be aware of, before we start describing the details, is the difference between logarithmic running constants and power-law running. Our experience in renormalizable field theories is with logarithmic running. The need to use running couplings comes from the existence of large logarithms. If we measure the coupling at a renormalization scale μ_r and apply it at an energy scale s , there will be large corrections of order $\alpha(\mu_r) \log(s/\mu_r^2)$. Use of the renormalization group lets us take that original measurement up to the scale $\mu_r^2 \sim s, t$, in which case there are no longer any large logarithms. Note that the signature of the kinematic invariants does not matter as $\log s/\mu_r^2 \sim \log t/\mu_r^2$ up to small factors as long as s and t are both of order μ_r^2 , even though s and t have opposite signs. Moreover, μ_r is an unphysical parameter. In the end, μ_r disappears from physical processes.

However, AS applied to gravity requires something different, which is power-law running. Because most of the couplings in the most general Lagrangian are dimensionful, one multiplies them by powers of the scale in order to define dimensionless variables. For example the Newton constant is modified by

$$g(k) = Gk^2 \quad (2)$$

The running of this dimensionless coupling is that which defines the fixed point. In this case

$$g \xrightarrow{k \rightarrow \infty} = g_* \quad (3)$$

hence the notation of Equation (1). However, now we must make contact with physical processes. If we imagine measuring G at some scale μ_r^2 , one is faced with the question of making the measurement of at some values of s or t of order μ_r^2 . But s and t generally carry opposite signs, and $g(s)$ and $g(t)$ are wildly different quantities in a way that does not occur in logarithmic running. Moreover, as we will see, there is no reason to expect that something like $G(s)$ captures the actual effect of quantum corrections to G . Higher order momentum dependence generally refers to new operators, where the factors of s or t come from extra derivatives on the fields. These new operators need not enter reactions in the same way as the lowest order operator.

2. FOUNDATIONAL ISSUES

In this section, we discuss several issues associated with running coupling constants. Therefore, let me be clear what I mean by a running coupling constant. It is a coupling defined to depend on

a scale which captures essential quantum corrections in physical processes for physics around that scale. The fact that it is useful in physical processes is important. We will see that this aspect is also part of the original formulation of Asymptotic Safety by Weinberg [2]. A useful running coupling should also apply to more than one process—it should be universally valid. If there is a scale dependence in some function which however does not have a direct physical meaning, we do not refer to this as a running coupling.

2.1. There Is No Gravitational Running of Regular Coupling Constants

There are obviously gravitational corrections to ordinary reactions which occur in the Standard Model. Robinson and Wilczek suggested that it could be useful to define the gravitational correction to the running coupling constants of the theory [22]. For example, for the gauge couplings, this could take the form

$$\beta(g, E) \equiv \frac{dg}{d \ln E} = -\frac{b_0}{(4\pi)^2} g^3 + a_0 \frac{E^2}{M_{\text{P}}^2} g \quad (4)$$

After a large number of papers in the literature [23–33], on various sides of this issue, it has become clear that this does not occur. The reasons are instructive for our discussion of Asymptotic Safety.

The first significant reason is kinematic. In Lorentzian reactions, the variable E^2 , can have either a positive or negative sign. For example, if the reaction $e^+e^- \rightarrow \mu^+\mu^-$ has the gravitational correction

$$\mathcal{M} \sim \frac{e^2(1 - aGs)}{s} \quad (5)$$

where $s = (p_1 + p_2)^2 > 0$ and a is some constant. For the reaction $e^+\mu^- \rightarrow e^+\mu^-$, related to it by crossing symmetry, will have the form

$$\mathcal{M} \sim \frac{e^2(1 - aGt)}{t} \quad (6)$$

with $t = (p_1 - p_3)^2 < 0$ having the opposite sign from s . The gravitational corrections will go in different directions in the two reactions. If the first reaction has a decreasing coupling, the second one will have an increasing coupling. In more complicated QED reactions, there will be many kinematic invariants which span the range of sizes and signs. These effects cannot be captured by a running coupling constant. If one attempts to measure the effective electric charge at a renormalization scale $s = \mu_r^2$ using $e^+e^- \rightarrow \mu^+\mu^-$, such as $e^2(\mu_r) = e^2(1 - aG\mu_r^2)$ that coupling will not be useful in describing the crossed reaction or in other more complicated reactions.

The other significant reason is universality. The gravitational corrections carrying powers of the energy are not actually a renormalization of the electric charge, but are described by new

operators with extra derivatives. For example, if we take the bare QED Lagrangian to be

$$\mathcal{L} = \frac{1}{4e_0^2} F_{\mu\nu} F^{\mu\nu} \quad (7)$$

then after loop corrections the energy dependent terms would be reflected in operators such as

$$\begin{aligned} \mathcal{L} = & \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + a G F_{\mu\nu} \square F^{\mu\nu} + b G \bar{\psi} \sigma_{\mu\nu} i \not{D} \psi \partial_\mu A_\nu \\ & + c G \bar{\psi} i \not{D} D^2 \psi + \dots \end{aligned} \quad (8)$$

These operators can enter different reactions in different ways, depending on the particle content and kinematics of those processes. Their contribution is not generally in the same manner as the original renormalized charge, and then is not generally able to be described by a running charge.

It should be noted that because the graviton is massless, not all the gravitational corrections are described by local operators. There can be non-local effects reflecting the long distance propagation of the graviton. However, this feature does not change the discussion above.

This brief discussion follows most closely Anber et al. [29] where further examples are given, but is also reflected in different ways in Pietrykowski [23], Toms [24], Ebert et al. [25], Tang and Wu [26], Rodigast and Schuster [27], Daum et al. [28].

2.1.1. Using a Cutoff Does Not Imply the Running of a Coupling Constant

In response to criticisms such as the above, some authors suggested that using a cutoff regularization scheme would produce a running coupling [30–32]. This is not correct, and again it is useful for our purposes to understand why.

We first note that using dimensional regularization there is no gravitational renormalization of the electric charge when neglecting the masses of the fermions. This follows from power-counting with a dimensional coupling G . Temporarily neglecting the fermion masses, the only dimensional factor in dimensional regularization comes from the factor μ^{4-d} inserted in Feynman integrals in order to keep the dimensions correct. This yields factors of $\log \mu^2$ in intermediate steps in calculations but could never produce a factor $G\mu^2$ in gravitational calculations. With fermion masses, the gravitational corrections are of the form

$$\mathcal{L} = \frac{1 + aGm^2}{4e_0^2} F_{\mu\nu} F^{\mu\nu} + \dots \quad (9)$$

where a is some constant and the ellipses refer to the momentum dependent corrections discussed above. When measuring the electric charge one finds

$$\frac{1 + aGm^2}{4e_0^2} = \frac{1}{4e_r^2} \quad (10)$$

and one expresses predictions in terms of the renormalized charge e_r . One is left only with the momentum dependent operators described above.

Real physics does not depend on the nature of the regularization scheme. However, the authors [30–32] suggested that the use of a cutoff regularization could be used to define a running coupling which would capture the quantum gravitational effects at a given scale. That is, by using a cutoff Λ one would define the beta function

$$\beta(g, \Lambda) \equiv \frac{dg}{d \ln \Lambda} = -\frac{b_0}{(4\pi)^2} g^3 + a_0 \frac{\Lambda^2}{M_{\text{P}}^2} g \quad (11)$$

This would get around the kinematic and universality problems of the Robinson-Wilczek suggestion. The reasoning is vaguely Wilsonian—by using a cutoff one includes effects which occur below that scale. One rebuttal is that one must also include effects which occur above that scale, and the overall physics is independent of the separation scale. However, even if one neglects this, the cutoff effect disappears in renormalization procedure. The introduction of a cutoff does lead to a renormalization of the bare electric charge, of the form

$$\mathcal{L} = \frac{1 + a_0 G \Lambda^2}{4e_0^2} F_{\mu\nu} F^{\mu\nu} + \dots \quad (12)$$

with the suggestion that

$$\frac{1 + a_0 G \Lambda^2}{4e_0^2} = \frac{1}{4e(\Lambda)^2} \quad (13)$$

However when one calculates a physical process, this effect enters the amplitude just like the renormalized charge, and the correct identification is

$$\frac{1 + a_0 G \Lambda^2}{4e_0^2} = \frac{1}{4e_r^2} \quad (14)$$

and this manifestation of Λ disappears from the physical amplitude [33]. In the end, cutoff regularization and dimensional regularization do agree in physical amplitudes.

Here we have seen the definition of a coupling constant which depends on a scale—the cutoff Λ . In that sense it is a truism that it “runs.” However, it does not qualify as a “running coupling constant” because that running is not relevant for physical processes at energies around that scale. Indeed the cutoff dependence is completely unphysical—it disappears from all amplitudes. If we wish to describe its scale dependence we should come up with a different name for it. Perhaps “incomplete coupling constant” is appropriate, as it is defined to include only quantum corrections below the cutoff scale. When used as a UV regulator, we do not care about the incompleteness, as the true physics beyond is unknown and in any case irrelevant for low energy processes. But if we are trying to use the cutoff as a running parameter at the scale of the energies being studied, we do care about the incompleteness. The full coupling constant does not have such a division.

2.1.2. Log Running vs. Power-Law Running

The above sections illustrate a truism—*There are no power-law running coupling constants in 4D Minkowski quantum field theory.*

Logarithmic running works because the logarithm is directly tied to renormalization. In the QED case, photon exchange with the vacuum polarization leads to a factor of

$$\mathcal{M} \sim \frac{e_0^2}{q^2 [1 + \Sigma(q)] + i\epsilon} \quad (15)$$

where $\Sigma(q)$ is scalar part of the vacuum polarization. No matter how one chooses to regularize it, the vacuum polarization contains a divergent term and a logarithm of q^2 . The divergence and the logarithm share the same coefficient. If we measure the charge using $e^+ e^- \rightarrow \mu^+ \mu^-$ at a renormalization scale $s = \mu_R^2$ with $s = (p_1 + p_2)^2 \gg m_e^2$, this result becomes

$$\frac{e^2(\mu_R)}{s [1 - \frac{\alpha}{3\pi} \log \frac{-s}{\mu_R^2}] + i\epsilon} \quad (16)$$

Because the logarithm comes along with charge renormalization, it occurs in every reaction in the same fashion. And because of the properties of the logarithm, the same running coupling would apply to the crossed reaction $e^+ \mu^- \rightarrow e^+ \mu^-$ with the change $s \rightarrow t$.

Power-law effects do not share these features. There is no universal connection of power-law corrections to the renormalization of the charge. And because of Minkowski kinematics, the effects in different channels can go in opposite directions.

That being said, it is possible in any one calculation to define a running coupling *for that particular process*. This may be a useful procedure. However, in field theory, a coupling constant has multiple duties. It not only describes that one particular process, but also must describe a multitude of others. These can differ in the arguments, i.e., $\lambda(\phi)$ vs. $\lambda(q^2)$, and also on the nature of the process. The same coupling needs to describe not only space-like vs. time-like reactions such as we have used as examples above, but also multi-particle reactions which involve many more particles than the simplest reaction. It is this multiplicity of uses where attempts to define power-law running couplings fail. The same definition which works in one setting will in general fail in the these other settings. The logic and mathematics which tell us that logarithmic running coupling constants are useful does not apply to power-law running.

The reader may object that Wilson has taught us the value of coarse-graining as a way to define couplings at different scales, and that this procedure has been verified in condensed matter systems even including power-law re-scalings. However, the couplings in these condensed matter examples do not have as many applications as the couplings in scattering processes. And the 3D setting for condensed matter systems does not display the kinematic variety of Minkowski reactions. It is easy to understand how the Wilsonian rescaling in condensed matter may be useful, while corresponding Minkowski QFT applications are more complicated.

2.2. Weinberg Formulation of Asymptotic Safety

The vision for Asymptotic Safety for gravity was formulated by Weinberg [2]. He invokes a situation where all the coupling

constants run to fixed values at high energy. This includes the dimensionful couplings, when rescaled by a universal dimension. He defines dimensionless variables g_i by multiplying by a scale μ . For example, one would have $g_G = G\mu^2$ and $g_\Lambda = \Lambda_{vac}/\mu^4$, where Λ_{vac} is the vacuum energy density¹.

Specifically, in his 1979 paper [2] Weinberg formulates the hypothesis using scattering processes and other reactions. Using these dimensionless coupling he suggests that these rates could have the form

$$R = \mu_R^D f \left[\frac{E}{\mu_R}, X, g_i(\mu_R) \right] \quad (17)$$

where X stands for all the other dimensionless physical variables. Here μ_R is meant to be a renormalization point, as used above. Because physics cannot depend on the arbitrary choice of the renormalization point, one can choose $\mu_R = E$ and have the result that the rate behaves as

$$R = E^D f[1, X, g_i(E)] \quad (18)$$

Aside from the pre-factor (which would involve $D = -2$ for a total cross-section) the rates would then depend on the couplings $g_i(E)$ as $E \rightarrow \infty$. Asymptotic safety is defined by the condition that the running couplings go to constant values $g_i(E) \rightarrow g_{i*}$ at high energy, or equivalently that their beta functions vanish

$$\beta(g_i) = E \frac{\partial}{\partial E} g_i = 0 \quad (19)$$

This is the UV fixed point. The implication here is that instead of blowing up with the energy, as GE^2 would, these factors go to constant values. I will refer to Equations (17)–(19) as the Weinberg conditions for Asymptotic Safety.

We can see from the discussion of coupling constants in the previous subsection that this needs to be generalized somewhat, as there is no unique energy E in Minkowski reactions. We do not want to include the kinematic variables in the running parameters, such as $g_i(s)$, $g_i(t)$, ... because of the kinematic ambiguity and differing signs. The best that we can hope for is to choose all of the kinematic variables of order the renormalization point, $|s| \sim |t| \sim \dots \sim \mu_R^2$ and write the rate as

$$R = \mu_R^D f \left[\frac{s}{\mu_R^2}, \frac{t}{\mu_R^2}, \dots, X, g_i(\mu_R) \right]. \quad (20)$$

In this formulation it is not clear how the renormalization scale μ_R drops out of physical observables. However, that can work out *in a given process* by explicitly performing the renormalization and demanding that the result is independent of μ_R . That demand then identifies the renormalization group flow of the couplings. The larger question is whether, having done this renormalization in one process, the result generalizes to other processes and is useful in describing the quantum effects of the full theory. This raises the possibility that the Weinberg

conditions themselves are unworkable when applied to a full set of reactions with many kinematic variables of differing signs. Our comparison with explicit reactions below will be discouraging in this respect when applied to G and Λ . However, if Asymptotic Safety is to be successful there must be a modified version of these conditions which applies for the high energy limit of physical processes. I will continue to use the Weinberg formulation as the vision for the AS program.

In our discussion of the present practice of Asymptotic Safety, it is important to point out that the Weinberg proposal is for true running couplings in the sense that we are using that phrase in this paper. That is, these are couplings that apply in physical reactions (in particular as functions of energy) and which in a useful way capture relevant quantum corrections appropriate for those energies.

2.3. The Practice of Asymptotic Safety

This section is clearly meant primarily for readers outside the AS community. It tries to very briefly explain the formalism and physics of the calculations. However, there are important comments toward the end of section 2.3.1 that are intended for all readers.

The present practice of Asymptotic Safety does not study reaction rates, but rather evaluates the flow of the Euclidean functional integral in a background field formulation—the Euclidean functional renormalization group (FRG). That is, the functional integral is a function of the metric, curvatures and covariant derivatives. The logic here is that once all quantum corrections are included in the Euclidean functional integral, the result can be continued to Lorentzian spaces, and the metric and curvatures expanded in the external fields in order to obtain the amplitudes that the Weinberg criterion envisions. I will call this the “ideal FRG program.”

However, for the most part in present applications this logic is not followed in practice². Rather than evaluating the full functional integral, one evaluate the evolution from the UV fixed point down to some cutoff k including quantum corrections above k . Without evaluating the quantum corrections below the cutoff, it is then assumed that the resulting $g_i(k)$ are the appropriate couplings to use in something like the Weinberg criterion in real world applications at the scale k . That is, $g_i(k) \sim g_i(E \sim \mu_R \sim k)$. There is also necessarily a truncation of the basis (to be discussed soon) in such applications. There is an extra logical step required if these assumptions are to be true. This can be called the practical AS program.

One complication of the AS program is that the basis set of operators is infinite, with a corresponding infinite number of coupling constants. The renormalization flow for a theory such as gravity mixes operators of all dimensions, with the only restriction being that of general covariance. In the action, there will be local terms of the form

$$\mathcal{L} = \sqrt{-g} \left[-\Lambda_{vac} - \frac{1}{16\pi G} R + c_1 R^2 + c_2 C_{\mu\nu\alpha\eta} C^{\mu\nu\alpha\eta} \right]$$

¹I will try to keep separate the vacuum energy density Λ_{vac} (which much of the particle physics community refers to as the cosmological constant) from other definitions of the cosmological constant. Much of the Asymptotic Safety community uses the symbol Λ for a different version $\Lambda = -\Lambda_{vac}/8\pi G = -\Lambda_{red}$. For this combination, I will use Λ_{red} (with *red* standing for “reduced”).

²Codello et al. [34] have pursued the ideal FRG program to reproduce some of the results of chiral perturbation theory. The chiral logs emerge in the IR limit as $k \rightarrow 0$.

$$+d_1 R^3 + d_2 R \square R + \dots] \quad (21)$$

This series can be ordered by powers of derivatives, such that only the operators with few derivatives are relevant for the low energy limit. This is what is done in the effective field theory treatment. However, Asymptotic Safety concerns the high energy limit and all operators become active as the energy goes to infinity. The ideal FRG program then would involve all possible operators with their coefficients³. However in the ideal FRG program these coefficients are not all independent. The infinite set of couplings would be described by a few *relevant* couplings and only special values of the parameters would be consistent with the Asymptotic Safety hypothesis.

Practicality requires that this be truncated at some order. The AS community has explored a remarkable range of such truncations, and the overall picture that emerges has so far been independent of the truncation. For the purposes of this paper, I will assume that the truncation problem is not a fundamental obstacle. Nevertheless, we can examine truncations to see what might be issues for the full program, as in sections 4.1 and 4.2.

The fundamental equation of AS practice, the Wetterich equation [35], describes the change of the Euclidean functional integral Γ_k , again defined to include quantum fluctuations above the scale k , under a change in scale⁴.

$$k \frac{\partial}{\partial k} \Gamma_k = \frac{1}{2} \text{Tr} \left[\left(\frac{1}{\frac{\delta^2 \Gamma_k}{\delta g \delta g} + R_k} \right) k \frac{\partial}{\partial k} R_k \right] \quad (22)$$

Here R_k is the cutoff function which suppresses momentum modes below k . Conceptually, it is like a mass below the scale k and zero above k , chosen in some smooth way so that there is not a discontinuity. An example is

$$R_k = (k^2 - D^2)\theta(k^2 - D^2) \quad (23)$$

In understanding the variation $\delta^2 \Gamma_k / \delta g \delta g$, one notes that g schematically represents the metric and any other fields in the theory. If the functional contained $D_\mu g D^\mu g$ then the variation would be $-D^2$. So conceptually, this equation is similar to $k \partial_k \text{Tr} \log(D^2 + m_k^2)$. Of course the real case is very much more complicated by the interactions and all the indices. A positive feature of the flow equation is that the flow only depends on the physics near the cutoff scale k . Higher scales have already been included and no longer enter because of the vanishing of $\partial_k R_k$ at high k , while lower scales are suppressed by the cutoff. Qualitative results have so far been independent of the choice of the function, although numerical results do depend modestly on the choice.

Weinberg in his Erice lectures on critical phenomena [36] also expressed a similar structure for the running coupling.

Much work has gone into exploring the existence and properties of the UV fixed points. To do this one first identifies

³There are also non-local contributions to the functional integral. It is assumed that these are fully parameterized by the coefficients of the local operators.

⁴The Wetterich equation is more general than its application to AS, and Asymptotic Safety could in principle be addressed without the Wetterich equation (i.e., see section 4.3 for a possibility). However, present practice in AS involves this equation.

a truncation in the basis. One starts at finite k and uses the Wetterich equation to flow to higher scales. In the infinite dimensional space of coupling constants, the fixed points live on finite dimensional “critical surface.” Common expectation is that this is two or three dimensional. This leaves a two or three dimensional family of solutions. When one flows from the fixed point to the IR at $k = 0$, one will have two or three undetermined constants. In particular Λ_{vac} and G at $k = 0$ are not predicted. But in principle there are predictions for an infinite number of other constants in the local effective Lagrangian.

2.3.1. AS at One-Loop

In order to see the FRG machinery at work, we can look at the illuminating calculation of Codello and Percacci [37], which is described as a one-loop evaluation including terms up to the order curvature-squared. This example also allows a comparison with a conventional treatment of quadratic gravity, which will be given in section 4.

The Euclidean action is parameterized by five couplings, in the form

$$S = \int d^4x \sqrt{g} \left[\frac{1}{8\pi G} \Lambda_{red} - \frac{1}{16\pi G} R + \frac{1}{2\lambda} C^2 - \frac{\omega}{3\lambda} R^2 + \frac{\theta}{\lambda} E \right]. \quad (24)$$

Here C^2 is the Weyl tensor squared, and E is the Gauss-Bonnet term. The vacuum energy is defined by $\Lambda_{vac} = -\frac{1}{8\pi G} \Lambda_{red}$. In four dimensions, E is a total derivative and does not influence any local physics. This will be evidenced in the flow as the parameter θ does not influence any of the other physical parameters. The dimensionful parameters are Λ_{red} and G , while λ , ω , θ are dimensionless. To create dimensionless parameters one defines $\tilde{G} = Gk^2$ and $\tilde{\Lambda} = \Lambda_{red} k^{-2}$.

The evolution of the curvature-squared coefficients is exactly the same as was previously calculated in dimensional regularization [10, 11].

$$\begin{aligned} \beta_\lambda &= -\frac{1}{(4\pi)^2} \frac{133}{10} \lambda^2 \\ \beta_\omega &= -\frac{1}{(4\pi)^2} \frac{25 + 1098\omega + 200\omega^2}{60} \lambda \\ \beta_\theta &= \frac{1}{(4\pi)^2} \frac{7(56 - 171\theta)}{90} \lambda \end{aligned} \quad (25)$$

These run only logarithmically in the usual way. In particular, the coefficient of the Weyl-squared term is asymptotically free and runs logarithmically to zero. The coefficient ω runs to a fixed point $\omega_* = -0.023$. Note however that in this evaluation the coefficient of the R^2 term $\omega/3\lambda$ is also indicative of asymptotic freedom because λ is asymptotically free.

The remaining two couplings have an evolution

$$\begin{aligned} \beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda} + \frac{1}{(4\pi)^2} \left[\frac{1 + 20\omega^2}{256\pi^2 \tilde{G}\omega^2} \lambda^2 + \frac{1 + 86\omega + 40\omega^2}{12\omega} \lambda \tilde{\Lambda} \right] \\ &\quad - \frac{1 + 10\omega^2}{64\pi^2 \omega} \lambda + \frac{2\tilde{G}}{\pi} - q(\omega) \tilde{G} \tilde{\Lambda} \\ \beta_{\tilde{G}} &= 2\tilde{G} - \frac{1}{(4\pi)^2} \frac{3 + 26\omega - 40\omega^2}{12\omega} \lambda \tilde{G} - q(\omega) \tilde{G}^2 \end{aligned} \quad (26)$$

with $q(\omega) = (83 + 70\omega + 8\omega^2)/18\pi$. The initial factor in each beta function (± 2) is due to the explicit factor of k used to make the couplings dimensionless. The remaining are due to perturbative interactions and these need to be large in order to cancel the ± 2 if the beta function is to vanish. These perturbative terms are not found in dimensional regularization because they require powers of the cutoff.

If we follow Codello and Percacci [37] and set ω and λ to their fixed point values, the flow can be solved exactly. Expressing the result in terms of the Newton constant G and vacuum energy density Λ_{vac0} defined at $k = 0$, one finds,

$$G(k) = \frac{G}{1 + \frac{Gk^2}{g_*}} \quad (27)$$

with $g_* \approx 1.4$ and

$$\Lambda_{vac}(k) = \Lambda_{vac0} - \frac{1}{16\pi^2} k^4 \quad (28)$$

The quartic k dependence of Λ_{vac} is particularly striking. Evaluated at LHC energies, it would imply

$$\Lambda_{vac}(10\text{TeV}) \sim -10^{14} \rho_N \sim -10^{61} \Lambda_{vac0} \quad (29)$$

where ρ_N is the density of the nucleus and $\Lambda_{vac0} \sim (10^{-3}\text{eV})^4$ is the present experimental vacuum energy. It is also notable that the vacuum energy itself does not run to a UV fixed point. It increases without bound, and only the rescaled value $\tilde{\Lambda} \sim \Lambda_{vac}(k)/k^4$ stays finite.

However, this dependence is k^4 is actually illusory when it comes to applications of this parameter. Recall that $\Lambda_{vac}(k = 0) = \Lambda_{vac0}$ is meant to describe the vacuum energy density with all quantum corrections included, and $\Lambda_{vac}(k)$ is meant to describe that parameter with only quantum effects above the scale k included. This implies that when we use $\Lambda_{vac}(k)$ we need also to add in the quantum corrections below k . For the vacuum energy this is seen to be related to

$$\int^k \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_p = \frac{4\pi}{(2\pi)^3} \int_0^k p^2 dp \frac{1}{2} p = \frac{1}{16\pi^2} k^4 \quad (30)$$

If we add this back into Equation (28) we get the full vacuum energy⁵. The running value is seen to be the full value with the effects of the momentum scales up to k removed.

Similar considerations apply for the running $G(k)$. When using $G(k)$ one is instructed to also add in the quantum corrections from scales 0 up to k . When this is done, one obtains the full G , which is the measured value.

The functions $G(k)$ and $\Lambda(k)$ by default “run” because they depend on the scale k . However, we will see in the next section that they do not behave as gravitational running couplings in

the sense of Weinberg, because they do not apply to physical processes. We will also explain the reason for this. Instead, $\Lambda(k)$ and $G(k)$ are *incomplete* coupling constants. From their definition they include physics above the cutoff scale but not below. Indeed, insights from effective field theory indicate that the lower energy physics is the region that is dynamically important. Because of the uncertainty principle, physics from high energy scales beyond the active scale k appears as local effects, parameterized by coefficients in a local action. Low energy physics can influence those local coefficients also (such that the cutoff scale disappears from physical observables) but also include dynamical effects from low energy propagation. The momentum dependence that we will see in the reactions to be described in section 3.1 all comes from low energy, as the high energy effects are only seen in the occasional unknown coefficient, such as d_1 in Equation (37). Because they are incomplete, parameters such as $\Lambda(k)$ and $G(k)$ do not know about this low energy physics, and it is therefore not surprising that they do not capture the quantum physics seen in physical observables.

The AS running is an iterated one-loop calculation. The renormalization group is used to iterate the the matching at the scale k , which is itself performed at one loop order. For example, the full program has been performed in the quadratic truncation approximation of this section in Benedetti et al. [40]. This is an appropriate way to improve on the one-loop result of Codello and Percacci, but it does not change the fundamental interpretation of the cutoff dependence.

3. THE CASE AGAINST A RUNNING G_N AND Λ

Quantum corrections and matter effects will clearly modify the physical value of G and of the other parameters. However it is not a requirement that these organize themselves in a functional form that is usefully described by a running coupling. We can look at observables to see if this is the case.

The function $G(k)$ is defined to include all of the quantum effects above the cutoff scale k . In principle, it is designed to be supplemented by including all of the quantum effects *below* the scale k also when using it to calculate some observable. The matching scale k is unphysical and should drop out from physical observables once all quantum effects are included. Nevertheless, it is common AS practice to use G_k as if it were the effective Newton constant at an energy of order k . However, one can see by direct calculations that this is not the case [6]. The attempt to compare the form of $G(k)$ to low energy results is a valid test because the FRG predicts not only a UV fixed point but also the approach to the fixed point at lower energies with effective field theory calculations are performed. The same techniques which predict the fixed point also predict running at lower energies which overlaps with the validity of the EFT calculations.

3.1. Explicit Calculations

Let us start by listing a series of physical amplitudes which have been calculated to one loop order. All of these have been calculated with the assumption that the value of the cosmological

⁵The apparently missing factor of 2 in Equation (30)—for the 2 graviton helicity states—appears to come from the fact Equation (30) involves a non-covariant cutoff, while the Wetterich equation is a (Euclidean) covariant treatment. See also Ossola and Sirlin [38] and Akhmedov [39]. Nevertheless, the principle remains the same. I thank Roberto Percacci for this observation.

constant at low energy can be neglected. The results are then functions of G and in some, but not all, cases contain coupling constants which are equivalent to a four-derivative truncation of the effective action. These reactions are observables. The question is whether we can define a useful running G from these observables.

The most elemental quantum gravity process is the scattering of two gravitons. The lowest order scattering amplitude involves a large number of individual tree diagrams but is given by the simple form

$$A^{tree}(++;++) = i \frac{\kappa^2}{4} \frac{s^3}{tu}, \quad (31)$$

where the signs $+, -$ refer to helicity indices and s, t, u are the usual Mandelstam variables. In power counting, this is a dimensionless amplitude of order GE^2 . This was calculated at one loop order with the result. The one loop amplitudes have been calculated by Dunbar and Norridge [41]. These are of order G^2E^4 and take the form

$$\begin{aligned} \mathcal{A}^{1-loop}(++;--) &= -i \frac{\kappa^4}{30720\pi^2} (s^2 + t^2 + u^2), \\ \mathcal{A}^{1-loop}(++;+-) &= -\frac{1}{3} \mathcal{A}^{1-loop}(++;--) \\ \mathcal{A}^{1-loop}(++;++) &= \frac{\kappa^2}{4(4\pi)^{2-\epsilon}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \mathcal{A}^{tree}(++;++) \times (stu) \\ &\quad \times \left[\frac{2}{\epsilon} \left(\frac{\ln(-u)}{st} + \frac{\ln(-t)}{su} + \frac{\ln(-s)}{tu} \right) + \frac{1}{s^2} f\left(\frac{-t}{s}, \frac{-u}{s}\right) \right. \\ &\quad \left. + 2 \left(\frac{\ln(-u)\ln(-s)}{su} + \frac{\ln(-t)\ln(-s)}{tu} + \frac{\ln(-t)\ln(-s)}{ts} \right) \right], \end{aligned} \quad (32)$$

where

$$\begin{aligned} f\left(\frac{-t}{s}, \frac{-u}{s}\right) &= \frac{(t+2u)(2t+u)(2t^4+2t^3u-t^2u^2+2tu^3+2u^4)}{s^6} \left(\ln^2 \frac{t}{u} + \pi^2 \right) \\ &\quad + \frac{(t-u)(341t^4+1609t^3u+2566t^2u^2+1609tu^3+341u^4)}{30s^5} \ln \frac{t}{u} \\ &\quad + \frac{1922t^4+9143t^3u+14622t^2u^2+9143tu^3+1922u^4}{180s^4}. \end{aligned} \quad (33)$$

Other amplitudes can be obtained from these by crossing. I have discarded some purely infrared effects, including the expected IR radiative divergence. As noted by 't Hooft and Veltman, this reaction and all pure graviton processes will be independent of any coupling constants other than G at this order, because the possible terms in the action vanish by the equations of motion $R_{\mu\nu} = 0$.

Another core process is the gravitational potential for heavy masses. Including the leading quantum correction the potential has the form [42, 43]

$$V(r) = -G \frac{Mm}{r} \left[1 + \frac{41}{10\pi} \frac{G}{r^2} \right], \quad (34)$$

This particular definition is derived from the low energy limit of the scattering amplitude. I have dropped the leading classical correction. The quantum correction is universal, independent of the spin of the heavy particles.

The bending of light around a massive object can also be reliably calculated [44–46].

$$\begin{aligned} \theta &\simeq \frac{4G_N M}{b} + \frac{15}{4} \frac{G_N^2 M^2 \pi}{b^2} \\ &\quad + \left(8bu^S - 47 - 64 \log \frac{b}{2b_0} \right) \frac{\hbar G_N^2 M}{\pi b^3} + \dots \end{aligned} \quad (35)$$

Here $1/b_0$ in the logarithm is the infrared cutoff which removes the IR singularities of the amplitude. Here there is not a universal behavior. The coefficient bu^S is a parameter which depends on the intrinsic spin of the particle. It has values $371/120$, $113/120$, $-29/8$ for scalars, the photon and the graviton, respectively.

Dunbar and Norridge have also calculated the gravitational scattering of a massless scalar particle, $\phi + \phi \rightarrow \phi + \phi$ [47]. At tree level, this has the form.

$$\mathcal{M}_{tree} = i \frac{\kappa^2}{4} \left[\frac{st}{u} + \frac{su}{t} + \frac{tu}{s} \right]. \quad (36)$$

with as usual $\kappa^2 = 32\pi G$. In this process there is a higher order operator which is needed to absorb the divergences which arise at one loop. This is

$$\mathcal{L}_2 = d_1 (D_\mu \phi D^\mu \phi)^2 \quad (37)$$

Including the renormalization of this higher order operator, the one loop hard amplitude is decrease in strength at high energy. Of course, since we are here using perturbation theory, we only should be obtaining the first

$$\begin{aligned} \mathcal{M}_h = i \frac{\kappa^4}{(4\pi)^2} \left\{ \frac{(s^4 + t^4)}{8st} \ln(-s) \ln(-t) + \frac{(s^4 + u^4)}{8su} \ln(-s) \ln(-u) + \frac{(u^4 + t^4)}{8tu} \ln(-t) \ln(-u) \right. \\ + \frac{(s^2 + 2t^2 + 2u^2)}{16} \ln^2(-s) + \frac{(t^2 + 2s^2 + 2u^2)}{16} \ln^2(-t) + \frac{(u^2 + 2t^2 + 2s^2)}{16} \ln^2(-u) \\ + \frac{1}{16} \left(\frac{st}{u} + \frac{tu}{s} + \frac{us}{t} \right) (s \ln^2(-s) + t \ln^2(-t) + u \ln^2(-u)) \\ + \left[-\frac{(163u^2 + 163t^2 + 43tu)}{960} \ln\left(\frac{-s}{\mu}\right) - \frac{(163u^2 + 163s^2 + 43us)}{960} \ln\left(\frac{-t}{\mu}\right) \right. \\ \left. \left. - \frac{(163s^2 + 163t^2 + 43ts)}{960} \ln\left(\frac{-u}{\mu}\right) + d_1^{ren}(\mu)(s^2 + t^2 + u^2) \right] \right\}, \quad (38) \end{aligned}$$

where μ is an infrared scale. Again a purely infrared effect has been removed.

Anber and I have used the Dunbar-Norridge method to find the amplitudes for two different species of particles [6]. In the reaction $A + B \rightarrow A + B$ we find that the hard amplitude is

$$\begin{aligned} \mathcal{M}_h = i \frac{\kappa^4}{(4\pi)^2} \left[\frac{1}{8} (s^4 \ln(-s) \ln(-t) + u^4 \ln(-u) \ln(-t)) \right. \\ - \frac{1}{16t} (s^3 + u^3 + tsu) \ln(-t) \\ + \frac{1}{16} (s^2 \ln^2(-s) + u^2 \ln^2(-u)) \\ + \frac{us}{16t} (s \ln^2(-s) + t \ln^2(-t) + u \ln^2(-u)) \\ + \frac{1}{240} (71us - 11t^2) \ln(-t) \\ \left. - \frac{1}{16} (s^2 \ln(-s) + u^2 \ln(-u)) \right], \quad (39) \end{aligned}$$

For the crossed process, $A + \bar{A} \rightarrow B + \bar{B}$, one exchanges $s \leftrightarrow t$, which yields a significantly different functional form.

It is easy to see by inspection that there are no common factors for the power-law corrections to these processes. This is an immediate indication that there will not be a useful definition of a running G which is useful in all processes. This is not a surprise as these kinematic effects do not amount to a direct renormalization of G . However, we can still proceed with an attempt to define a renormalization of G at a higher renormalization scale μ_R and look at the outcome.

First consider graviton-graviton scattering. If we wish to renormalize this at high energy, we would like a kinematic configuration where all the kinematic variables are of the same large energy. In this case, we chose the central physical point $s = 2E^2$, $t = u = -E^2$. If we use the amplitude $\mathcal{A}(++; ++)$ and use this point to determine $G(E)$, we find

$$G^2(E) = G^2 \left[1 + \frac{\kappa^2 E^2 \left(\ln^2 2 + \frac{1}{8} \left(\frac{2297}{180} + \frac{63\pi^2}{64} \right) \right)}{8\pi^2} \right]. \quad (40)$$

We see that this definition leads to a growing running coupling $G(E)$, as opposed to the expectation from asymptotic safety of a

order term in the expansion. Nevertheless the disagreement on the sign is clear.

We could alternatively consider the crossed reaction $\mathcal{A}(+, -; +, -)$ which is obtained from $\mathcal{A}(+, +; +, +)$ by the exchange $s \leftrightarrow t$. This makes the quantum corrections somewhat different, with the corresponding kinematic factor being

$$\begin{aligned} 1 + \frac{\kappa^2 t}{16\pi^2} \left[\ln \frac{-s}{t} \ln \frac{-u}{t} + \frac{su}{2t^2} f\left(\frac{-s}{t}, \frac{-u}{t}\right) \right] \\ = 1 + \frac{\kappa^2 E^2 \left(\frac{29}{10} \ln 2 - \frac{67}{45} \right)}{16\pi^2} \quad (41) \end{aligned}$$

instead of the factor in Equation (40).

If we used identical scalar particle scattering at the same kinematic point to identify a running coupling the result would be

$$\begin{aligned} G(E) = G \left[1 - \frac{\kappa^2 E^2}{360(4\pi)^2} \right. \\ \left. \left(609 \ln \frac{E^2}{\mu^2} + (340\pi^2 + (123 - 340 \ln 2) \ln 2) \right) \right]. \quad (42) \end{aligned}$$

The single log term which appears in Equation (43) could reasonably be associated with the higher order operator d_1 , and perhaps should be removed from this expression. Using the scattering of non-identical particles, one would find for $A + B \rightarrow A + B$,

$$\begin{aligned} \mathcal{M}_{total} = \frac{i\kappa^2 E^2}{2} \left[1 - \frac{\kappa^2 E^2}{10(4\pi)^2} \left((19 + 10 \ln 2) \ln \left(\frac{E^2}{\mu^2} \right) \right. \right. \\ \left. \left. + 5(\pi^2 - (\ln 2 - 1) \ln 2) \right) \right]. \quad (43) \end{aligned}$$

which would lead to yet a different running $G(E)$. On the other hand, using $A + \bar{A} \rightarrow B + \bar{B}$ we would have

$$\begin{aligned} \mathcal{M}_{total} = \frac{i\kappa^2 E^2}{8} \left[1 + \frac{\kappa^2 E^2}{10(4\pi)^2} \left(9 \ln \left(\frac{E^2}{\mu^2} \right) \right. \right. \\ \left. \left. - 5\pi^2 + (19 + 5 \ln 2) \ln 2 \right) \right]. \quad (44) \end{aligned}$$

The crossing problem is obvious here.

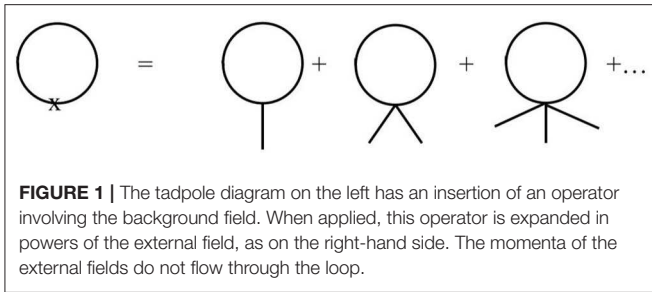


FIGURE 1 | The tadpole diagram on the left has an insertion of an operator involving the background field. When applied, this operator is expanded in powers of the external field, as on the right-hand side. The momenta of the external fields do not flow through the loop.

There is not much point to continue. It is clear that any application to other processes will yield yet other discordant results. Even if we have an operational definition of a running G at a higher renormalization point in one process, this definition does not apply to other reactions. This is not surprising, as the quantum corrections here are not related to a renormalization of G .

We note also that having set the cosmological constant to zero at low energy, it stays zero in the scattering amplitudes. All the corrections come in at higher powers in the energy, in accord with the power counting theorems of the effective field theory. The cosmological constant also does not run in these scattering amplitudes.

The examples here are evidence that the Weinberg criterion for AS is false, *as applied to the parameters Λ and G* . Even if we do not attempt to use the FRG form of the running G , there is no other form that does the job either. Nature does not organize itself like that at low energy. Perhaps a revision of the Weinberg criterion is possible in which other parameters more important to the high energy limit have the flow envisioned by Weinberg.

It is possible that in one given process—say, FLRW cosmology for example—it could be useful to define power-law running parameters for use in that setting and those running parameters might asymptote to a non-trivial UV fixed point. However, even if this is the case it would not imply that this defines a consistent quantum field theory of gravity. Such a field theory would have to be broadly applicable to all observables, and we have seen above a broad class of observables which do not share a useful running G .

3.2. The Driving Force of the Tadpole Graph

We can look beyond the formalism and identify what is going wrong in the functional RG approach to the running G . The diagram driving the flow for this operator is the tadpole diagram of **Figure 1**. This diagram vanishes in dimensional regularization for massless particles. It is non-vanishing when evaluated with a cutoff. The issue is not really whether it vanishes or not, but that is a symptom. Since physical processes *can* be regularized dimensionally, we should not be surprised that there is not a signal of this diagram in the physical amplitudes. The more important feature is that this diagram does not feel the values of the external momenta, and here cutoff and dimensional regularization agree. Even with a cutoff, there is no external momentum flowing in loop. This tells us that the diagram does not know about the momentum scales of the physical reactions, and so cannot correspond to the use of running coupling

depending on those scales. Once we identify how to treat this diagram, we will be able to bring the cutoff regularized result into agreement with dimensional regularization. To demonstrate this we need to look at the physics of the background field method.

With background field methods, one can capture the quantum effects using the heat kernel [48–53], defined as

$$H(x, \tau) = \langle x | e^{-\tau \mathcal{D}} | x \rangle \quad (45)$$

for some differential operator $\mathcal{D}$. For example the functional determinant can be evaluated using

$$\Delta S = \int d^4x \text{Tr} \langle x | \log \mathcal{D} | x \rangle \quad (46)$$

with

$$\langle x | \log \mathcal{D} | x \rangle = - \int_0^\infty \frac{d\tau}{\tau} \langle x | e^{-\tau \mathcal{D}} | x \rangle + C \quad (47)$$

The local heat kernel is expanded in powers of τ with the Seeley-DeWitt coefficients a_i , with the result

$$H(x, \tau) = \frac{i}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} [a_0(x) + a_1(x)\tau + a_2(x)\tau^2 + \dots] \quad (48)$$

in an arbitrary dimension d . The contribution to the action is then

$$\begin{aligned} \langle x | \log \mathcal{D} | x \rangle = & \frac{-i}{(4\pi)^{d/2}} \left[m^d \Gamma(-d/2) a_0(x) \right. \\ & + m^{d-2} \Gamma(1-d/2) a_1(x) \\ & \left. + m^{d-4} \Gamma(2-d/2) a_2(x) + \dots \right] \quad (49) \end{aligned}$$

As an example which is simpler than the graviton itself consider a scalar coupled to gravity with the Lagrangian

$$\sqrt{-g} \mathcal{L} = \sqrt{-g} \frac{1}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2] \quad (50)$$

in which the coefficients have the form

$$\begin{aligned} a_0(x) &= 1 \\ a_1(x) &= \frac{1}{6} R \\ a_2(x) &= \frac{1}{180} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{72} R^2 \quad (51) \end{aligned}$$

From this we see that a_0 is associated with the cosmological constant, a_1 is associated with the renormalization of G and a_2 is associated with curvature-squared terms. In the AS beta functions this dependence is convoluted with the influence of the cutoff function, but this association remains true. I have included both a mass and a dimension d in order to make the following points. In dimensional regularization for the massless graviton, we would set $m = 0$ and the coefficients of a_0 and a_1 would vanish. The divergence in the coefficient a_2 is non-vanishing in the massless limit and is the usual divergence that one finds at

one loop order. But also, in this evaluation the mass m serves as a proxy for the IR cutoff of AS, with $m^2 \sim k^2$. So we see that the k^4 and k^2 dependence of the running couplings comes from the a_0 and a_1 coefficients, respectively.

In 4D flat space, the Passarino-Veltman theorem [54] says that all one loop diagrams can be reduced to scalar tadpole, bubble, triangle and box diagrams. The “scalar” part of this statement says that any momentum factors in the numerator can be removed and replaced by external momenta, leaving behind only the tadpole, etc diagrams with no momenta in the numerator. The heat kernel performs this operation describing the result using derivatives in the local operators, in our case R , R^2 , $R_{\mu\nu}R^{\mu\nu}$, etc. The scalar tadpole, bubble, etc diagrams then contribute to the coefficients of the local operators. Each is readily identifiable by its dimension and divergence structure. In particular, in 4D the scalar tadpole has dimension E^2 and the scalar bubble is dimensionless, which is why they carry the k^2 and $\log k$ cutoff dependence. In curved spacetime, the use of the equivalence principle means that the short distance behavior of loops is equivalent to that of flat space. The use of Riemann normal coordinates can be used to describe the heat kernel and the AS RG flow using the same classification of tadpole, bubble, etc. diagrams [53] including the non-local components of the heat kernel. The k^2 cutoff dependence of the a_1 coefficient is characteristic of the scalar tadpole diagram.

We can learn a bit more by looking at the ingredients to these heat kernel coefficients. Working in flat space for simplicity, we consider the differential operator as

$$\begin{aligned}\log \mathcal{D} &= \log[d_\mu d^\mu + m^2 + \sigma(x)] = \log[\square + m^2 + V(x)] \\ &= \log\left[(\square + m^2)\left(1 + \frac{1}{\square + m^2}V\right)\right] \\ &= \log[\square + m^2] + \frac{1}{\square + m^2}V \\ &\quad + \frac{1}{2}\frac{1}{\square + m^2}V\frac{1}{\square + m^2}V + \dots\end{aligned}\quad (52)$$

where $d_\mu = \partial_\mu + \Gamma_\mu(x)$ and $\sigma(x)$ describe some interactions. Inserting a set of momentum eigenstates, we see that the first two terms in the heat kernel expansion are tadpole loops

$$\sim \int \frac{d^4 p}{(2\pi)^4} \times \log[\square + m^2] \sim \int dm^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2} \quad (53)$$

and

$$\sim \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2} \times V(x) \quad (54)$$

These two are represented in **Figure 1**. The key point here is that the tadpole has no external momenta flowing in these loops. This implies that when matrix elements are taken of the resulting effective Lagrangian, there will be no external momentum dependence coming from the a_0 and a_1 coefficients. This is already evident in the discussion of the one-loop running contributions to Λ and G in section 2.3.1. In contrast, the a_2 term is given by a bubble diagram, with two vertices and

two propagators. It does involve the external momenta because it involves the interaction V at different spacetime points. In addition to the local divergence which is contained in a_2 there is a non-local $\log q^2$ dependence. This can also be identified by a non-local version of the heat kernel method [52, 53].

Combined with the discussion of section 2.3.1, we arrive at an understanding of how the cut-off regularization can agree with dimensional regularization. The dimensional regularization case integrates over all momenta with no separation of scales. The result is that the physical values of Λ_{vac} and G are not modified. In the cutoff regularization case, the so-called running couplings of $\Lambda(k)$ and $G(k)$ represent these parameters with quantum effects only above the scale k included. They are actually incomplete couplings, where the physics below the scale k is missing. Technically, they are described by the tadpole diagram in which no momentum flows. When supplemented by the rest of the loop below k we again get the physical values of the parameters as the dependence on the separation scale must vanish. There is no external momentum flowing through these loops so that there is no net effect on the kinematic features of scattering amplitudes. This confirms that the k dependence in $G(k)$ does not correspond to running in any kinematic sense. In contrast, the bubble diagram, associated with a_2 will contain logarithmic momentum dependence. Both dimensional regularization and cutoff regularization will agree on this and logarithmically running couplings associated with the a_2 coefficient will be physical.

4. COMPARISON WITH QUADRATIC GRAVITY

In this section, I discuss the AS result for the truncation including terms of order curvature squared, summarized above in section 2.3.1, with work on quadratic gravity, which uses the same operator basis but which does not use the AS machinery.

There are three points to be made in this comparison. (1) At least at one loop, this AS truncation is unsatisfactory in that when continued to Lorentzian spaces it contains a tachyon. It also contains a ghost state and violates causality on short time scales, although these may be less disastrous. (2) Further analysis of the ghost state indicates that there is an obstruction to the continuation from Euclidean space to Minkowski space, as there is a pole in the upper right quadrant of the complex q_0 plane. These are both problems that could be due to the specific truncation, but which could in principle surface at any order of truncation in AS. (3) The third point is more positive: A focus on higher order terms in the graviton propagator may be useful for a Lorentzian variant of Asymptotic Safety.

4.1. Tachyons and Ghosts

Because there are higher order terms in the most general action, the gravitational propagator will contain higher powers of q^2 . With a truncation at order of the curvature-squared, this implies terms up to q^4 in the propagator. Normally these are forbidden by the Källén-Lehmann representation of the propagator,

$$D(q) = \frac{1}{\pi} \int_{4m_f^2}^{\infty} ds \frac{\rho(s)}{q^2 - s + i\epsilon} \quad (55)$$

with the spectral function $\rho(s)$ being positive definite, which says that the propagator can fall by at most q^{-2} at high momentum⁶. It then becomes clear that some of the usual assumptions of QFT (which forms the basis of the KL representation) must be given up in Asymptotic Safety (also in quadratic gravity). Some of the dangers are evidenced in the partial fraction decomposition of the propagator

$$iD(q) = \frac{i}{q^2 - aq^4/M^2} = \frac{i}{q^2} - \frac{i}{q^2 - M^2/a} \quad (56)$$

Here, M is the intrinsic scale of the higher order terms, and I have included a parameter $a = \pm 1$ because the higher order behavior can come with either sign. For both signs of a , the second term in the partial fraction decomposition automatically comes with the “wrong” overall sign—it is a ghost. For $a = -1$ the ghost is also tachyonic in that it occurs for spacelike values of the four-momenta⁷. As far as I know, there is no way to rescue this situation. It leads to an unstable state with runaway production of tachyons. The $a = +1$ ghost is non-traditional in QFT, but seems to be more manageable. When treated properly, it can lead to a unitary theory [16], but one which violate microcausality [17, 55]. However, these options are ones which any truncation of AS will be forced to confront.

The parameters of the one-loop AS solution given in section 2.3.1 imply a tachyon in the spin-zero propagator and a $a = +1$ ghost in the spin-two propagator. Let us defer the discussion of the spin-two ghost to the next subsection. The spin-zero tachyon is a serious problem if it were to survive at higher order truncations. There is a bit of history/physics to understand concerning the tachyon. The first ingredient is that in this case, the high mass state is not ghost-like. It is the massless pole in the spin-zero channel which is ghost-like. That is, instead of Equation (56), one has an overall minus sign,

$$iD_0(q) = \frac{-i}{q^2 - aq^4/M^2} = \frac{-i}{q^2} + \frac{i}{q^2 - M^2/a} \quad (57)$$

That the massless pole is ghost-like is acceptable because the massless spin-zero component can be shown to be a gauge artifact [56]. The historical aspect is that several early works on the renormalization of quadratic gravity use what is now recognized to be the “wrong” sign without recognizing that this lead to tachyons. Adopting a modern parameterization for the quadratic terms, we have

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{6f_0^2} R^2 - \frac{1}{2f_2^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \right] \quad (58)$$

in Lorentzian space. These signs lead to a normal massive spin-zero state, and the $a = +1$ spin two ghosts. Early work used

⁶There is the caveat that the KL representation does not necessarily apply to gauge-variant fields because the spectral function then does not correspond to the insertion of physical states.

⁷Reminder: my metric convention is $(+, -, -, -)$.

the opposite sign on the $1/6f_0^2$ term, and concluded that both f_0 and f_2 are asymptotically free [10, 11]. With the non-tachyonic sign, f_0 is no longer asymptotically free [14]. The Euclidean action of section 2.3.1 shares yields asymptotic freedom for the overall R^2 coupling, and then would share the tachyonic property when continued to Lorentzian space.

It is possible that the tachyonic state could be removed using a higher order truncation or no truncation at all. There are a few special functions whose Taylor expansion would show these poles when truncated at a fixed order, but which is well-behaved without the truncation. However, this is already an indication that simply obtaining a UV fixed point in the Euclidean FRG is not sufficient to claim that one has a well-behaved Lorentzian theory. Each truncation must be checked separately. It is even more difficult to understand the ideal case, with no truncation.

4.2. Obstacles to Analytic Continuation

The spin-two ghost in the quadratic truncation presents a more generic problem. There can be unexpected obstacles to the analytic continuation from Euclidean to Lorentzian spaces. There has been some work on analytic continuation of the FRG in scalar theories [57], which however does not address the issue raised in this section.

The location of the poles in the propagator has been explored in the quadratic gravity literature. I am particularly biased toward my own recent work with Menezes [16, 17], which is representative of the present status. The heavy ghost state will necessarily be unstable due to the coupling with the light gravitons and other light degrees of freedom. Including that coupling leads to a self-energy term in the propagator

$$iD_2(q) = \frac{i}{q^2 + \Sigma(q) - q^4/M^2} \quad (59)$$

where $\Sigma(q)$ is the self energy. In gravity, there is a cut starting at $q^2 = 0$ where the self energy develops an imaginary part $\text{Im } \Sigma(q) = \gamma(q)$. Unitarity requires $\gamma(q) \geq 0$. The ghost resonance then has the form near $q^2 = M^2$

$$\begin{aligned} iD_2(q) &= \frac{i}{q^2 - \frac{q^4}{M^2} + i\gamma(q)} \\ &= \frac{i}{\frac{q^2}{M^2} [M^2 - q^2 + i\gamma(q)(M^2/q^2)]} \\ &\sim \frac{-i}{q^2 - M^2 - i\gamma_M} \end{aligned} \quad (60)$$

This puts the resonance pole above the real axis

$$q^2 = M^2 + i\gamma_M \quad (61)$$

rather than usual resonances which occur below the real axis. In Donoghue and Menezes [17] we have labeled ghost resonances with this pole location as Merlin modes as they propagate backwards in time. We note that this construction would also work for higher order ghosts in the spin two channel. The fact that unitarity requires that $\gamma(q) \geq 0$, implies that all further ghost states would also live above the real axis.

For the purposes of quadratic gravity, this is an arguably acceptable result. The resulting theory is unitary and stable near Minkowski space [16], but violates microcausality on timescales of order the width [17, 55], which is proportional to the inverse Planck scale. A look at the underlying calculations shows that this would appear to continue to happen if the propagator was defined with yet higher order dependence even if there were other unstable ghosts induced, as long as there were no tachyonic states allowed. An AS theory defined in Lorentzian space would presumably share these acceptable features.

The danger for the present program of Asymptotic Safety is somewhat different. The original AS theory is defined in Euclidean space. To reach the real world, this needs to be continued to Lorentzian space. In amplitudes, this is accomplished by a rotation of the momentum space contour from the real axis to the imaginary axis, and is legitimate because there are no poles crossed by the rotation. The usual QFT rotation from Minkowski to Euclidean space is a tool which proves to be useful because of the usual analyticity properties of amplitudes. In the presence of higher derivatives, these analyticity properties are upset. This implies that there is no longer any guarantee that the Euclidean theory and the Minkowski theory share the same properties. The spin-two ghost found above is such a problem as would be any further ghosts.

There has been recent work which attempts to keep the momentum dependence separate from the k dependence and which addresses specific gravity amplitudes such as the propagator [58–62]. It appears that the spin-two ghost state is not just an artifact of the quadratic truncation. In a recent study by Bosma et al. [61], the spin-two sector was parameterized much more generally,

$$C_{\mu\nu\alpha\beta} W(\square) C^{\mu\nu\alpha\beta} \quad (62)$$

where $W(\square)$ is an arbitrary function, referred to as a form-factor. This directly impacts the spin-two propagator which becomes

$$iD_2(q) = \frac{i}{q^2 - q^4 W(q^2)} \quad (63)$$

Within the approximations of the calculation [61], the result is approximated by

$$W(q^2) = w_\infty + \frac{\rho}{\alpha - q^2} \quad (64)$$

where $\rho \simeq 0.015$, $\alpha \simeq 1.8$ in Planck units and w_∞ is a constant which is not determined by the calculation. In writing this result, I have made the continuation to Minkowski space in the most naive fashion—just changing the sign on the momentum. The result in Bosma et al. [61] is an approximate fit to the Euclidean numerical results and its full analytic structure is not precisely defined. Moreover, the comments above about analytic continuation would also be applicable to this form-factor, and it is not clear how open channels would influence this continuation. In any case, this will have ghost poles when

$$q^2 W(q^2) = 1 \quad (65)$$

Assuming that there are no tachyonic states, this is still a ghost pole. The form-factor description [61, 62] is a welcome new direction, because the functions of $\square$ have direct physical relevance, in contrast with the unphysical parameter k .

4.3. The Graviton Propagator and Lorentzian Asymptotic Safety

The higher order momentum dependence in the graviton propagator actually presents an opportunity for version of AS which is defined from the start in Lorentzian space. Potentially this could circumvent some of the problems which we have been discussing. However, it would require a reinterpretation of the program.

We have learned that low energy quantum effects involving Λ and G do not organize themselves in the way implied by present AS practice, or indeed of that suggested by the general Weinberg criterion.

However, we can also see that this may be irrelevant to the high energy behavior of the theory. In quadratic gravity, the propagator is modified by q^4 terms, such that the effects of Λ and G (of order q^0 and q^2) are sub-dominant at high energy, and the result is a renormalizable theory. So the fact that there is not a good definition of a running Λ and G is not important for the overall structure of the theory. The parameters of the quadratic curvature terms are the essential ones for the renormalizability and running of the theory. In an AS framework, one could truncate at yet higher orders. This produces higher powers of momenta in the graviton propagator which determine its high energy behavior.

Let us look at the potential for divergences in diagrams with these higher powers of the momenta. Consider the graviton propagator with the high energy behavior $1/q^n$. For consistency, we need to keep vertices with powers of momentum running up to q^n , as the same operator which gives momentum dependence to the propagator will also give new vertices. The most divergent diagrams are the ones with the highest powers of momentum in the vertices, so we will consider that all vertices carry this maximal momentum factor. Let N_V be the number of vertices, N_I be the number of internal propagators, and N_L be the number of loops. Then the overall high-momentum dependence of the diagram will be

$$(d^4 q)^{N_L} (q^n)^{N_V} \frac{1}{(q^n)^{N_I}} \quad (66)$$

from loop momenta, vertices and propagators⁸. However, the number of internal propagators can be eliminated in favor of the number of vertices and loops. The relation is

$$N_I = N_L + N_V - 1 \quad (67)$$

⁸The factors of q will in general involve external momenta, $q - p_i$ and after integration the amplitude will be expressed in terms of these p_i . Using dimensional regularization is useful here as it does not introduce extra dimensionful parameters, and the dimension in any divergence will be realized in terms of the external momenta.

This converts the high energy behavior into

$$q^{D_n} = (q)^{4N_L} (q^n)^{N_V} \frac{1}{(q^n)^{N_L+N_V-1}} = q^{(n+N_L(4-n))} \quad (68)$$

which summarizes the divergence structure.

For two derivative actions, $n = 2$ and we recover the well known power counting behavior of general relativity and chiral perturbation theory [63]

$$q^{D_2} = q^{(2+2N_L)} \quad (69)$$

with tree level being q^2 , one loop having divergences at q^4 , two loop at q^6 , etc. For $n = 4$, such as for quadratic gravity, we recover power-counting renormalizability, with

$$q^{D_4} = q^4 \quad (70)$$

independent of the number of loops. For larger values of n we get super-renormalizable behavior, with the diagrams becoming less divergent with higher loops. For example, for $n = 6$, the power-counting gives

$$q^{D_6} = q^{6-2N_L} \quad (71)$$

As the loop order increases, the amplitudes are increasingly focused on the infrared and are no longer divergent. Phrased differently, tree-level amplitudes are always of order q^n by assumption. For any n there will be potential divergences at one loop order involving effects at order q^4 . But then for larger $n > 4$ the diagrams become more convergent at higher loop order.

This allows a possible reinterpretation of the AS program. Perhaps only some of the couplings need to be have the running behavior implied by the Weinberg criterion. Sub-dominant couplings such as Λ and G are not important for the program. The important operators are those which dominate in the high energy limit. While there are in general there are an infinite number of these, the power counting above indicates that the damping provided by the higher powers of the graviton propagator may make a truncation at higher order feasible. This inverts the present practice. Instead of a focus on low dimensional operators, one is more interested in higher dimensional operators that influence the graviton propagator. I note a similarity with the “form-factors program” [62] in which the operators in the form factor, such as Equation (64), are higher powers of momentum in the graviton propagator. It would be interesting to see if this program could be formulated in Lorentzian spacetime.

Of course, this suggestion is still somewhat vague and needs to be better developed. One still needs to avoid tachyons and deal with ghosts. But it does point to a form of Asymptotic Safety that can be described from the start in Lorentzian spaces, and which can be in agreement with explicit calculations at low energy. Moreover, it is clear that the high momentum behavior of the graviton propagator is of special significance as it determines the UV properties of loop diagrams.

5. OVERALL ASSESSMENT

We have examined in particular the running Newton constant $G(k)$ within AS and argued that it is not valid for use in the real world. The reasons for that include:

1. It does not capture the energy dependence in explicit observables. There are kinematic and universality obstacles to any such use. Note that these examples are also counter-examples to the Weinberg conditions for Asymptotic Safety if applied to G , Λ . If the Weinberg vision for Asymptotic Safety is to continue, the conditions need to be modified to exclude the low energy parameters G , Λ .
2. The definition of the $G(k)$ and $\Lambda(k)$ are such that they include quantum effects beyond the scale k . They should be supplemented with the quantum effects below k . When this is done, the intermediate scale k should disappear.
3. We can also see that the values of $G(k)$ and $\Lambda(k)$ arise from the tadpole diagram, which (a) vanishes in dimensional regularization and (b) does not contain any external momentum flow through the loop. This loop will not influence the kinematic behavior of reactions.

Points 2 and 3 indicate that these couplings are what I have referred to as incomplete couplings rather than running couplings in the sense of the Weinberg criterion. They become complete only in the $k \rightarrow 0$ limit. In this sense there is a disconnect between essentially all of present AS practice and the Weinberg conditions of Equations (17)–(19). It needs to be recognized that the cutoff dependence of $G(k)$, $\Lambda(k)$ and likely many of the higher power couplings is not the same as the running couplings in physical reactions. These features are most problematic in attempts to apply Asymptotic Safety in phenomenological settings. Some of the previous phenomenological applications have been discussed in the surveys of the subject [3, 5]. The use of these couplings is not appropriate for phenomenological applications and does not satisfy the goals of Asymptotic Safety.

In the process of making these comparisons, it can be recognized that at least a portion of Weinberg's conditions for Asymptotic Safety fails at the energies which we have considered—that which applies to the proposed running of G and Λ . Not only does the FRG version of running fail to match explicit calculations, but even operationally there is no form that will work at scales below M_P . Nature does not organize itself this way. This need not be a fatal flaw, as these couplings describe operators which are sub-dominant in the high energy limit. Higher powers of curvatures and derivatives will dominate at high energy, and so it is possible that even if G and Λ do not run, the important couplings at high energy do. This is what happens in quadratic gravity, where the curvature squared terms make the theory renormalizable and their coefficients do have logarithmic running. However, there still needs to be a reformulation of the Weinberg criterion which takes into account the multiple kinematic variables of different magnitudes and signs which complicate to running of non-logarithmic power-law couplings.

This leaves the “ideal FRG program” as a possibility. Here one integrates in Euclidean space down from the UV fixed point

all the way to $k = 0$. The couplings have “run” in the theory space of coupling constants not in the real space of energies and momenta, and have completed their evolution by taking the $k \rightarrow 0$ limit. At intermediate values of k these couplings are not considered to be physical, but their $k = 0$ limit defines an action with an infinite number of terms, which is then to be applied in Lorentzian space. The action is described by an infinite number of parameters such as G and Λ , which are themselves just constants defined by their $k \rightarrow 0$ limit. These couplings are correlated—fixed by a smaller number defined at the fixed point. This appears to be the situation advocated in section 6.18 of the Wetterich review [64]. However, it is a very different situation than the Asymptotic Safety envisioned by the Weinberg conditions in Equations (17)–(19), where the running couplings were functions of energy applied in physical reactions. Here I have raised two cautions:

1. Any truncation of this ideal action will have ghosts, and possibly tachyons. These have to be understood and managed.
2. Any truncation without tachyons will likely have one or more obstacles to the analytic continuation from Euclidean to Lorentzian space. These are poles in the graviton propagator that occur in the quadrants needed for the Euclidean rotation.

There can be a significant difference between a Euclidean theory and a Lorentzian one in the presence of operators with higher derivatives/curvatures.

It is possible that both of these points can be overcome. However, even if this occurs, we do not have any indication on why the resulting theory would satisfy the Weinberg criterion or lead to finite results in physical observables. The Weinberg criterion gave an intuitive rationale for the finiteness of the theory. But if this ideal FRG program does not generate running parameters in physical reactions, we need a new rationale. If the

cutoff dependence in $G(k)$ etc is not the same as the running of couplings in physical reactions, what reason do we have to expect that we get finite high energy limits for such reactions? The existence of a Euclidean UV fixed point is not sufficient by itself for this result. Indeed, existing truncations do not satisfy this despite all having such fixed points. One needs to obtain finite results for an infinite number of processes at an infinite number of kinematic points. One does have an infinite number of couplings, but the mechanism for success is unknown.

On the more positive side, I have argued that maybe a Lorentzian version of AS could occur through a focus on the higher order terms contributing to the graviton propagator. The basic point here is that Λ and G become unimportant at high energy in the graviton propagator when higher powers of q^n appear in the propagator. This is seen in quadratic gravity where the inclusion of q^4 terms in the propagator lead to a renormalizable theory, and is encountered in Euclidean AS through the inclusion of form-factors [62]. I have used power counting to argue that one could perhaps get a Lorentzian theory with these higher order terms.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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From Renormalization Group Flows to Cosmology

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According to the asymptotic-safety conjecture, the gravitational renormalization group flow features an ultraviolet-attractive fixed point that makes the theory renormalizable and ultraviolet complete. The existence of this fixed point entails an antiscreening of the gravitational interaction at short distances. In this paper we review the state-of-the-art of phenomenology of Asymptotically Safe Gravity, focusing on the implications of the gravitational antiscreening in cosmology.

Keywords: cosmology–theory, quantum gravitation, renormalization group, singularity, inflation

1. GRAVITATIONAL ANTISCREENING: A HISTORICAL PERSPECTIVE

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Similarly to the case of Quantum Chromodynamics (QCD), the gravitational interaction might exhibit an antiscreening behavior at high energies [1]. A form of gravitational antiscreening was introduced in the 80s by Markov [2, 3] as a mechanism to cure the longstanding problem of gravitational singularities in General Relativity. In Markov and Mukhanov [4] and Markov [5], an *ad hoc* modification of the Einstein-Hilbert Lagrangian has been proposed, such that the corresponding field equations

$$G_{\mu\nu} = G(\rho) T_{\mu\nu} + \Lambda(\rho) g_{\mu\nu}, \quad (1)$$

admit a Newton coupling $G(\rho)$ and cosmological constant $\Lambda(\rho)$ whose strengths depend on the proper energy density ρ of matter fields, with $G(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. This latter assumption, which in Markov and Mukhanov [4] and Markov [5] is referred to as “asymptotic freedom of gravity,” has been introduced to render the gravitational interaction weaker at high energy densities. In a cosmological context, this could lead to a singularity-free cosmological evolution characterized by a deSitter initial state and a minimum radius of the order of the Planck length, $a_{\min} \sim L_{\text{Pl}}$. In this case the resolution of the cosmological singularity is due to the violation of the energy conditions, thus invalidating one of the key assumptions leading to the Hawking-Penrose singularity theorems [6]. Similar arguments could also apply to the case of black holes, where the gravitational antiscreening could lead to singularity-free black-hole spacetimes [7].

The “asymptotic freedom of gravity” discussed in Markov and Mukhanov [4] and Markov [5] was originally introduced as a modification of General Relativity at the classical level. It turns out that the gravitational antiscreening advocated in Markov and Mukhanov [4] and Markov [5] could be a natural consequence of the quantum properties of gravity. The “asymptotic safety” scenario for Quantum Gravity [8–10] aims at constructing a consistent quantum theory for the gravitational interaction within the well-established framework of Quantum Field Theory (QFT). As originally proposed by Weinberg [11, 12], in the light of the Wilsonian renormalization group [13, 14] and the related, generalized notion of renormalizability [15], a consistent QFT of gravity could be constructed if the gravitational renormalization group (RG) flow attains an

interacting—non-Gaussian—fixed point (NGFP) in the ultraviolet limit. In this case, in the ultraviolet regime, gravity approaches a scale invariant regime where the dimensionless counterparts of all gravitational couplings attain finite, generally non-zero, values. The theory is thus interacting in the ultraviolet regime and the presence of the NGFP ensures the “non-perturbative” renormability [15] of gravity. While asymptotic safety of gravity remains a formally unproved conjecture, there are strong indications that a suitable gravitational fixed point indeed exists [16–34]. Moreover, despite its interacting nature, the fixed point could lie in the vicinity of the free-theory fixed point (Gaussian fixed point, GFP), making the theory near-perturbative [35]. Similarly to the case of non-abelian gauge theories like QCD, the existence of an ultraviolet fixed point is guaranteed whenever the “paramagnetic interactions” of the action dominate over the diamagnetic ones [1]. This mechanism is at the basis of the (quantum) gravitational antiscreening. The realization of the latter can be understood more intuitively from the RG running of the Newton coupling. The structure of the beta function of the dimensionless Newton coupling $g(k) = G(k)k^2$ in $d = 4$ spacetime dimensions, k being the RG scale, is

$$\beta_g = (2 + \eta_N)g, \quad (2)$$

where $\eta_N = k\partial_k \log G(k)$ is the anomalous dimension of the Newton coupling. The function η_N depends on g as well as on all other (dimensionless) gravitational couplings. Further note that η_N depends on the RG scale k only implicitly, i.e., only through the RG running of the dimensionless gravitational couplings. A necessary condition for the existence of a non-trivial fixed point is that $\eta_N = -2$ for some values of the gravitational couplings. Assuming that a non-trivial fixed point indeed exists in the full (not truncated) theory space, the simple fact that $\eta_N < 0$ at the non-trivial fixed point implies that the dimensionfull Newton coupling decreases with the RG scale k and vanishes as $G(k) \sim g_*k^{-2}$ when $g(k) \rightarrow g_*$, where g_* denotes the fixed-point value of the dimensionless Newton coupling.

Although in the context of asymptotically safe gravity (ASG) the gravitational antiscreening has a quantum origin and despite the different semantics [Markov’s asymptotic freedom of $G(k)$ vs asymptotic safety of $g(k)$], this is the same principle advocated by Markov and Mukhanov [4] and Markov [5] as one possible way to soften or even remove the singularities affecting the solutions of the Einstein equations. In the case of stellar black holes the mechanism of singularity-avoidance is very intuitive: when a collapsing star disappears behind its event horizon and its density reaches planckian values, the gravitational interaction driving the collapse becomes weaker. Therefore, under certain conditions, the gravitational antiscreening can potentially halt the collapse and prevent the formation of a spacetime singularity. Hints that the mechanism of singularity-avoidance could be realized within the Asymptotic Safety scenario for Quantum Gravity have been found in Bonanno and Reuter [36–38], Torres [39, 40], Torres and Fayos [41], Bonanno and Koch [42, 43], Bonanno et al. [44], Adeifeoba et al. [45], Platania [46], and Bonanno et al. [47] via the so-called

RG-improvement procedure [see, e.g., [37, 48] and references therein], in Bosma et al. [49] by means of non-perturbative computations of quantum corrections to the Newtonian potential based on the functional renormalization group (FRG) method, and in Marunovic and Prokopec [50, 51] through a one-particle irreducible resummation of one-loop vacuum fluctuations of non-minimally coupled, massless, scalar matter. In analogy with the case of black-hole singularities, the gravitational antiscreening could provide a solution to the problem of the initial singularity [52–54] in cosmology. Moreover, the existence of a regime where gravity is approximately scale-invariant could be relevant in cosmology to provide a natural explanation for the nearly-scale-invariant distribution of temperature anisotropies in the Cosmic Microwave Background (CMB) radiation [55–61] [see also [62] for a recent review].

In this paper we review some of the main cosmological implications of ASG based on the running of the gravitational couplings. The rest of the present review is organized as follows. Section 2 summarizes the mechanism behind the renormalization group improvement and the scale-setting procedure. In sections 3 and 4 we review the main implications of the gravitational antiscreening in cosmology and inflation respectively. Finally, in section 5 we summarize the state-of-the-start of phenomenology of ASG, its main problems, and future perspectives.

2. RUNNING COUPLINGS AND RENORMALIZATION GROUP IMPROVEMENT

2.1. Decoupling Mechanism

One of the strengths of ASG is the possibility of constructing a quantum theory of gravity using the “language” of Quantum Field Theory—the standard framework to describe matter and all known fundamental interactions within the Standard Model of particle physics. On the one hand, this makes the connection between gravity and matter more straightforward than in other approaches to quantum gravity and allows to constrain the ultraviolet details of quantum gravity by verifying systematically its consistency with low-energy experiments and observations on the matter sector [see [63, 64] for recent reviews]. On the other hand, the computation of the quantum corrections to the classical solutions of General Relativity requires the knowledge of the gravitational quantum effective action $\Gamma_0^{grav}[g_{\mu\nu}]$. The classical Einstein equations are replaced by the fully quantum field equations

$$\frac{\delta \Gamma_0^{grav}[g_{\mu\nu}]}{\delta g_{\mu\nu}} = 0. \quad (3)$$

These are effectively classical field equations. Nonetheless, its solutions $\langle g_{\mu\nu} \rangle$ actually incorporate all quantum gravitational effects.

The computation of the effective action comes along with several technical and conceptual issues. First, computing the effective action exactly would require either to solve the gravitational path integral over globally hyperbolic spacetimes

or, equivalently, to solve the Functional Renormalization Group (FRG) equation for the effective average action Γ_k [65–67]

$$k\partial_k\Gamma_k = \frac{1}{2}\text{STr}\left\{\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right\} \quad (4)$$

and take the limit $k \rightarrow 0$. The mathematical tools currently available do not allow for an exact computation of the effective action, even if important progress in this direction has been made in Knorr and Saueressig [68] and Knorr et al. [69]. Secondly, the effective action is a gauge- and parametrization-dependent object: only physical observables will be independent of any gauge choice, parametrization and regularization schemes. A third key issue, related to the previous one, is that defining physical observables in quantum gravity is still an outstanding open problem [70, 71].

The *decoupling mechanism* [72] might provide a solution to some of the issues raised above. In what follows we will summarize how this mechanism works, mostly following the arguments and nomenclature in Reuter and Weyer [72]. As is clear from Equation (4), the RG running of the effective action is determined by the modified inverse propagator $\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)$. The regulator $\mathcal{R}_k \propto k^2$ is an effective mass-square term that implements the Wilsonian shell-by-shell integration of fast fluctuating modes: only fluctuations with momenta $p^2 \gtrsim k^2$ are integrated out, resulting in the partially-quantized effective action Γ_k . In the limit $k \rightarrow 0$ all quantum fluctuations are integrated out, so that Γ_0 coincides with the ordinary effective action. At the basis of the decoupling mechanism is the possibility that some infrared physical scales appearing in $\Gamma_k^{(2)}$, such as a physical mass term, could compete and eventually overcome the effect of the unphysical mass term in $\mathcal{R}_k$ in the infrared. In this case there will be a threshold value of k —a decoupling scale k_{dec} —below which the running of Γ_k is essentially frozen. The “threshold effective action” $\Gamma_{k_{dec}}$ and the ordinary effective action are thus expected to be approximately the same. The identification of the infrared scale k_{dec} , if any, could provide some of the (typically non-local) terms in Γ_0 . An emblematic example of this mechanism is massless scalar electrodynamics, where $\Gamma_k^{(2)} = p^2 + \lambda(k)\phi^2 + \dots$. The running of the quartic coupling $\lambda(k)$ is logarithmic, $\lambda(k) \sim \log k$ and therefore the decoupling should occur at $k_{dec}^2 \sim \log(k_{dec})\phi^2$. The effective action Γ_0 is thus expected to involve an effective non-local interaction of the form $\phi^4 \log(k_{dec}(\phi))$. To leading order, this leads to the $\phi^4 \log(\phi)$ -interaction appearing in the famous Coleman–Weinberg effective potential [73]. For other examples in QED and QCD see, e.g., [74–76] and references therein. When the effective action contains several competing infrared scales (particle momenta, field strengths, spacetime curvature, etc), identifying the threshold scale k_{dec} becomes a more involved task: k_{dec} might be a complicated non-linear function of all these scales or, in the best case, it might be given by the one physical infrared scale which dominates over the others. Conversely, a decoupling scale might not exist at all: this is the case if there are no dominant infrared scales acting as an actual physical cutoff. Therefore, the existence of

k_{dec} and its specific form strongly depend on the physical system under consideration.

The procedure of identifying and replacing the RG scale k with a physical infrared scale, supposedly acting as a decoupling scale k_{dec} , is known as *RG improvement*. It aims at using the RG running in order to incorporate leading-order quantum effects in the dynamics of a classical system. In the example mentioned above, the RG improvement is used *at the level of the action* to obtain the leading-order terms in the quantum effective action Γ_0 . Other forms of RG improvement are the RG improvement *at the level of the field equations* and *at the level of the classical solutions*; the latter allows, e.g., to derive the Uehling potential from the RG-running of the electric charge [76]. Even if the idea behind the decoupling mechanism seems to suggest that the RG improvement should be performed at the level of the action, this is typically considered as a source of ambiguity. In the next subsection we will discuss the RG improvement in the case of gravity and how to constrain the scale setting $k = k(x)$ based on the symmetries of the theory.

We remark that the RG running of the gravitational couplings extrapolated from FRG computations relies on the use of Euclidean metrics. The implications of ASG obviously involve Lorentzian spacetimes. It is thereby assumed that the scaling of the couplings and the existence of an ultraviolet-attractive fixed point are not affected, at least qualitatively, by the metric signature. Hints that this might indeed be the case have been found in Manrique et al. [77].

2.2. Renormalization Group Improvement and Scale-Setting Procedure

ASG relies on the existence of an ultraviolet-attractive fixed point of the gravitational RG flow. In the Einstein–Hilbert truncation, the scaling of the dimensionless Newton coupling $g(k) = G(k)k^2$ and cosmological constant $\lambda(k) = \Lambda(k)k^{-2}$ about the NGFP (g_*, λ_*) reads

$$\begin{cases} g_k = & g_* + c_1 e_1^1 \left(\frac{k}{M_{Pl}}\right)^{-\theta_1} + c_2 e_2^1 \left(\frac{k}{M_{Pl}}\right)^{-\theta_2} \\ \lambda_k = & \lambda_* + c_1 e_1^2 \left(\frac{k}{M_{Pl}}\right)^{-\theta_1} + c_2 e_2^2 \left(\frac{k}{M_{Pl}}\right)^{-\theta_2} \end{cases}, \quad (5)$$

where $M_{Pl} \sim (8\pi G_0)^{-1/2}$ is the reduced Planck mass, the c_i are integration constants labeling all possible RG trajectories, $\mathbf{e}_i$ are the eigenvectors of the stability matrix $\partial_{g_i}\beta_j|_{g_*}$ constructed using the beta functions β_j of all dimensionless couplings g_j , and $(-\theta_i)$ are its eigenvalues. The real part of the critical exponents θ_i determine the stability properties of the NGFP. In the case of pure gravity, the critical exponents θ_1 and θ_2 are typically a pair of complex conjugate numbers with positive real part: this implies that the NGFP is ultraviolet-attractive in the Einstein–Hilbert truncation. In extended truncations, involving higher-derivative operators, it has been shown that the NGFP comes with three relevant directions associated with the volume, R , and 4th-order derivative operators [20–22, 26, 30, 34, 78–81]. The number of relevant directions coincides with the number of free parameters (the integration constants c_i) to be fixed by comparison with observations, e.g., by requiring that in the

infrared $8\pi G_0 \sim M_{pl}^{-2}$ and $\Lambda_0 \sim 3 \cdot 10^{-122} M_{pl}^2$. Fixing the free parameters in this way allows to select the “RG trajectory realized by Nature” [82]. As it will be important in the applications of ASG in cosmology, it worth mentioning that the values of the critical exponents are influenced by the presence of matter: for instance, in the Einstein-Hilbert truncation, the presence of minimally-coupled free matter fields makes the critical exponents $\theta_{1,2}$ real [32, 81, 83–85].

Neglecting the running of the matter couplings, the scale-dependent Einstein-Hilbert action reads

$$S_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{-g} (R - 2\Lambda_k) + S_{matter}. \quad (6)$$

At some intermediate scale $k = k(x)$, the RG running modifies the classical field equations by an effective energy-momentum tensor $\Delta t_{\mu\nu} \equiv G_k(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) G_k^{-1}$ which could encode, at an effective level, the vacuum polarization effects of the quantum gravitational field [86],

$$G_{\mu\nu} = 8\pi G_k T_{\mu\nu} - \Lambda_k g_{\mu\nu} + \Delta t_{\mu\nu}. \quad (7)$$

If there is no energy-momentum flow between the gravitational and matter components of the theory, i.e., if the energy-momentum tensor $T_{\mu\nu}$ is separately conserved, the cutoff function $k = k(x)$ is constrained by the modified (contracted) Bianchi identities [82, 86–89]

$$\nabla^\mu G_{\mu\nu} = (8\pi G'_k T_{\mu\nu} - \Lambda'_k g_{\mu\nu}) \nabla^\mu k(x) + 8\pi G_k \nabla^\mu T_{\mu\nu} + \nabla^\mu \Delta t_{\mu\nu} = 0. \quad (8)$$

The above equation provides a “consistency condition” [82, 86–89] which can be used, under certain assumptions and/or approximations, to determine the scale-dependence $k = k(x)$. Specifically, we can identify the following cases

- CASE I: *RG improvement at the level of the field equations*

Performing the RG improvement at the level of the field equations or solutions is equivalent to neglecting $\Delta t_{\mu\nu}$. Assuming that the matter energy-momentum tensor is covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$, the shape of the function $k(x)$ is dictated by the condition

$$(8\pi G'_k T_{\mu\nu} - \Lambda'_k g_{\mu\nu}) \nabla^\mu k(x) = 0. \quad (9)$$

The solution to this equation depends on the form of the energy-momentum tensor. As it will be important in cosmology, let us focus on the case of a perfect fluid with energy density ρ and pressure p . In this case $T^\nu_\mu = \text{diag}(-\rho, p, p, p)$, with $p = w\rho$, and the above equation yields the condition

$$\frac{G'(k)}{G(k)} (\rho + \rho_\Lambda(k)) + \rho'_\Lambda(k) = 0, \quad (10)$$

with $\rho_\Lambda = \frac{\Lambda(k)}{8\pi G(k)}$. In the fixed point regime, the scaling of the dimensionfull Newton coupling and cosmological constant reads

$$G(k) = g_* k^{-2}, \quad \Lambda(k) = \lambda_* k^2, \quad (11)$$

so that k can be related to the matter energy density ρ [87, 90],

$$k^4 = \left(\frac{8\pi g_*}{\lambda_*} \right) \rho. \quad (12)$$

In this setup the running gravitational couplings depend on the energy density ρ of the matter degrees of freedom: this is exactly the *ad hoc* assumption employed in Markov and Mukhanov [4] and Markov [5] to obtain a classical modification of General Relativity free of the problem of spacetime singularities.

The relation between k and the energy density ρ can also be understood in terms of the decoupling mechanism. If the spacetime is filled with a perfect fluid with energy density ρ , the gravitational action is complemented by a matter action $S_{matter} = -\int d^4x \sqrt{-g} \rho$. Due to the (minimal) coupling between the gravitational and matter degrees of freedom, the energy density ρ enters the modified inverse propagator $(\Gamma_k^{grav,(2)} + \mathcal{R}_k)$ and could thus provide a decoupling scale k_{dec} for the flow of the scale-dependent effective action Γ_k^{grav} . We note that the Ricci scalar R enters the modified inverse propagator as well and therefore it could also provide a decoupling scale. However, as we will see below, a scale setting $k^2 \sim R$ satisfies the contracted Bianchi identities only if the RG improvement is performed at the level of the action.

In cosmology one can further limit the form of the effective metric to a Friedmann-Robertson-Walker (FRW) spacetime. Since $T_{\mu\nu}$ is assumed to be separately conserved, the energy density ρ obeys the standard (i.e., classical) conservation equations and thus $\rho(a) = \rho_0(a(t)/a_0)^{-3(1+w)}$ (note that the explicit time dependence of ρ is determined by the form of the scale factor, and therefore it could differ from the classical case). In this case

$$k \propto a(t)^{-\frac{3}{4}(1+w)}. \quad (13)$$

On the other hand, if $T_{\mu\nu}$ is not separately conserved, the Bianchi identities

$$(8\pi G'_k T_{\mu\nu} - \Lambda'_k g_{\mu\nu}) \nabla^\mu k(x) + 8\pi G_k \nabla^\mu T_{\mu\nu} = 0 \quad (14)$$

do not add any additional constraint on the form of the cutoff function $k(x)$: the latter equation provides a generalized conservation equation allowing for an energy flow between the gravitational and matter degrees of freedom.

- CASE II: *RG improvement at the level of the action*

If the RG improvement is performed at the level of the action, the field equations contain an additional contribution encoded in the gravitational energy-momentum tensor $\Delta t_{\mu\nu}$. Its variation reads [89]

$$\begin{aligned} \nabla^\mu \Delta t_{\mu\nu} &= G'_k (G_k^{-1} \Delta t_{\mu\nu}) \nabla^\mu k(x) \\ &\quad + G_k \nabla^\mu \underbrace{[(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) G_k^{-1}]}_{R_{\mu\nu}(G_k^{-1})'} \nabla^\mu k(x) \\ &= G'_k G_k^{-1} \left\{ (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 8\pi G_k T_{\mu\nu} + \Lambda_k g_{\mu\nu}) \right. \\ &\quad \left. - R_{\mu\nu} \right\} \nabla^\mu k(x). \end{aligned} \quad (15)$$

The contracted Bianchi identities (8), together with the assumption that the energy-momentum tensor is separately conserved, thus yield the condition

$$\begin{aligned} \nabla^\mu G_{\mu\nu} = & (8\pi G'_k T_{\mu\nu} - \Lambda'_k g_{\mu\nu}) \nabla^\mu k(x) + 8\pi G_k \underbrace{\nabla^\mu T_{\mu\nu}}_0 \\ & + G'_k G_k^{-1} \left\{ (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 8\pi G_k T_{\mu\nu} + \Lambda_k g_{\mu\nu}) \right. \\ & \left. - R_{\mu\nu} \right\} \nabla^\mu k(x) = 0. \end{aligned} \quad (16)$$

Note that in this case, due to the presence of $\Delta t_{\mu\nu}$, the contribution from the energy-momentum tensor $T_{\mu\nu}$ cancels out. A scaling relation of the form $k^4 \sim \rho$ [87, 90], Equation (12), is only valid under the assumption that $\Delta t_{\mu\nu}$ is negligible. We are thus left with the condition

$$\nabla^\mu G_{\mu\nu} = \{G'_k G_k^{-1} (-\frac{1}{2} R + \Lambda_k) - \Lambda'_k\} g_{\mu\nu} \nabla^\mu k(x) \equiv 0. \quad (17)$$

Diffeomorphism invariance thus requires [86, 88, 89]

$$G'_k R = 2(G'_k \Lambda_k - \Lambda'_k G_k). \quad (18)$$

In the proximity to the NGFP, the couplings scale as in (11) and the constraint (18) gives

$$k^2 = \frac{R}{4\lambda_*}. \quad (19)$$

This condition should also hold in the more general case of $f_k(R)$ theories, if the running of the gravitational couplings is approximated with the corresponding fixed-point scaling [88, 91].

It is worth noting that the replacement $k^2 \sim R$ in the scale-dependent action (6) generates an effective $f(R)$ action, whose analytical expression is determined by the running of the gravitational couplings [48, 88, 92]. This fact is typically used to study effective inflationary models in ASG, as it will be discussed in section 4.

The RG improvement at the level of the field equations and at the level of the action lead to effective field equations which differ by a gravitational energy-momentum tensor $\Delta t_{\mu\nu}$. Based on the idea behind the decoupling mechanism, performing the RG improvement at the level of the action would seem to be more natural. However, the possibility of choosing between different forms of RG improvement is considered as a source of ambiguity. The identification of the decoupling scale k_{dec} is a second possible source of ambiguity: in the case of gravity, there are multiple scales that could potentially act as a decoupling scale for the flow of the scale-dependent gravitational effective action. However, if the matter energy-momentum tensor is covariantly conserved, the form of the cutoff function $k(x)$ is constrained by the contracted Bianchi identities.

Phenomenological implications of ASG have been explored in the literature by means of RG-improved cosmological and astrophysical models. Although these models are not expected to provide precise and quantitative predictions, they are expected

to capture qualitative features of the modifications of Einstein gravity induced by ASG.

Keeping strengths and limitations of the RG-improvement procedure in mind, in the next sections we will review some of the main phenomenological implications of ASG based on models of RG-improved cosmology.

3. RG-IMPROVED COSMOLOGIES

It is an old idea that the gravitational couplings could depend on the cosmic time and that this time-dependence could have implications in cosmology [93]. In the context of RG-improved cosmologies this time-dependence arises from the RG running of the gravitational couplings. In this section we will review some of the main cosmological implications of ASG [54–57, 94]. Each subsection focuses on the cosmological implications obtained in different regimes and/or under assumptions.

3.1. Early-Universe Cosmology: The NGFP Regime

We first focus on the RG-improved cosmological dynamics in the NGFP regime, following the analysis in Bonanno and Reuter [55]. The starting point is the assumption that the effective metric, solution to the fully quantum equation of motion, is a homogeneous and isotropic FRW universe,

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]. \quad (20)$$

The matter degrees of freedom are encoded in a perfect fluid with energy-momentum tensor $T_\mu^\nu = \text{diag}(-\rho, p, p, p)$ and equation of state $p = w\rho$. In this setup quantum-gravitational fluctuations can only modify the effective dynamics of the scale factor $a(t)$. Performing an RG improvement at the level of the fields equations (in the Einstein-Hilbert truncation) yields the modified Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{\Lambda[k(x)]}{3} + \frac{8\pi G[k(x)]}{3} \rho. \quad (21)$$

In this subsection we focus on the case where the energy-momentum tensor is separately conserved, as originally assumed in Bonanno and Reuter [55]. In this case no flow of energy between the gravitational and matter sector is possible [the case in which $T_{\mu\nu}$ is not covariantly conserved has been studied in [57] and will be discussed in section 3.2]. The conditions $\nabla_\mu G^{\mu\nu} = 0$ and $\nabla_\mu T^{\mu\nu} = 0$ thus result in the “standard” conservation equation, $\dot{\rho} = -3H\rho(1+w)$, and in the consistency condition (9). The latter can be split in the following equations

$$\partial_t \Lambda + 8\pi\rho \partial_t G = 0, \quad (22)$$

$$h^{\mu\nu} (\Lambda_{; \nu} - 8\pi p G_{; \nu}) = 0, \quad (23)$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor onto the tangent 3-space orthogonal to the 4-velocity u^μ of an observer

comoving with the cosmological fluid. Provided that G and Λ do not vary along the hypersurfaces orthogonal to u^μ , i.e., $k = k(t)$, the latter relation is identically satisfied. The cosmological evolution of the universe is then (over-) determined by the system of equations [55]

$$\begin{aligned}\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} &= \frac{\Lambda(t)}{3} + \frac{8\pi G(t)}{3}\rho, \\ \dot{\rho} &= -3H\rho(1+w), \\ \dot{\Lambda}(t) + 8\pi\rho\dot{G}(t) &= 0.\end{aligned}\quad (24)$$

The consistency condition (22) can thus be used to constrain the form of the cutoff function $k(t)$. Assuming that in the early universe the growth of the scale factor follows a power law, Equation (22) is consistent with a scale dependence of the form $k = \xi_t t^{-1}$, where ξ_t is a positive constant [55]. In particular, the consistency condition (22) fixes the value of the free parameter ξ_t in terms of fixed-point quantities. As it will become clear soon, the scale-setting $k \sim t^{-1}$ employed in Bonanno and Reuter [55] is also compatible with the one in Equation (12), that was derived and discussed in subsequent studies [87, 90].

In what follows we will only consider the case of a spatially flat universe, $K = 0$. Approximating $G(k)$ and $\Lambda(k)$ with their fixed-point scaling, Equation (11), with $k(t) = \xi_t t^{-1}$, the cosmological system (24) can be solved analytically. The modified Friedmann and conservation equations admit a family of solutions where the scale factor $a(t)$ and the density $\rho(t)$ scale as power laws. Their dependence on the constant ξ_t can then be eliminated by imposing the consistency condition (22), which leads to the relation

$$\xi_t = \sqrt{\frac{8}{3(1+w)^2\lambda_*}}. \quad (25)$$

Using this expression for ξ_t , the family of cosmological solutions associated with the RG-improved system (24) reads [55]

$$\begin{aligned}a(t) &= \left(\frac{9}{64}(1+w)^4 g_* \lambda_* M\right)^{\frac{1}{3(1+w)}} t^{\frac{4}{3(1+w)}}, \\ \rho(t) &= \frac{8}{9\pi(1+w)^4 g_* \lambda_*} t^{-4}, \\ G(t) &= \left(\frac{3}{8}(1+w)^2 g_* \lambda_*\right) t^2, \\ \Lambda(t) &= \left(\frac{3}{8}(1+w)^2\right)^{-1} t^{-2},\end{aligned}\quad (26)$$

where M is an integration constant. These solutions depend on fixed-point quantities through the combination $(\lambda_* g_*)$, which is known to be scheme independent [95, 96]. Moreover, the solutions do not depend on the infrared values of the gravitational couplings, reflecting the universal behavior of the RG flow at the NGFP. Note that, provided that the scale factor follows a power law scaling (i.e., $w \neq -1$), the cutoff identification $k = \xi_t t^{-1}$ is equivalent to the scale settings $k = \xi_h H(t)$ (even at a classical level), $k = \xi_\rho \rho^{1/4}$ (due to the modified power law of the scale factor, and in accordance with

the discussion in section 2) and, in a RG-improved radiation-dominated era, to $k \sim \xi_a a^{-1}$, with

$$\begin{aligned}\xi_t^2 &= \frac{8}{3(1+w)^2\lambda_*}, & \xi_h^2 &= \frac{3}{2\lambda_*}, \\ \xi_\rho^2 &= \frac{8\pi g_*}{\lambda_*}, & \xi_a^2 &= \frac{g_* M}{\lambda_*}.\end{aligned}\quad (27)$$

This magic can only occur in the proximity of a critical fixed point, where physical quantities should vary as power laws of a unique scale. Away from the NGFP, the complete solution to the cosmological system (24) can only be obtained numerically. This has been done in Reuter and Saueressig [56]. In this case the cutoff function $k(t)$ is obtained by solving the full beta functions for the gravitational couplings and the consistency condition (22) numerically. In particular, in Reuter and Saueressig [56] it is shown that the dynamical cutoff $k(t)$ is well approximated by the Hubble constant, $k \sim H(t)$, for any value of the cosmic time t .

The fixed-point scaling of gravitational couplings modifies the power-law scaling of the scale factor, Equation (26), so that also the causal structure of the spacetime is modified at early times. At it can be easily seen, provided that $w \leq 1/3$, there is no particle horizon [55]: quantum effects enlarge the extension of the light-cones such that events occurring at the decoupling era are causally influenced by all points belonging to the hypersurface $t = 0$ [55]. Nonetheless, if $w = 1/3$ the deceleration parameter is zero, i.e., no inflation occurs in a RG-improved radiation-dominated epoch. This problem can be overcome by relaxing the assumptions and/or improving the approximations made in Bonanno and Reuter [55]: a period of inflation can occur in RG-improved cosmologies if the gravitational and matter degrees of freedom can exchange energy [57] (this will be discussed in section 3.2) and/or when the gravitational effective energy-momentum tensor $\Delta t_{\mu\nu}$ discussed in section 2 is taken into account, i.e., when the RG improvement is performed at the level of the action [60, 61, 92, 97] (this point will be discussed in detail in section 4).

The gravitational effective energy-momentum tensor $\Delta t_{\mu\nu}$ is also crucial to make the cosmological evolution non-singular: in the setting introduced above, the RG-improved cosmological evolution (26) is still singular, as scale factor vanishes at $t = 0$. The $\Delta t_{\mu\nu}$ -term in the effective field Equations (7) might mimic the effect of higher-order operators in the gravitational effective action [60, 61, 92, 97] and, as discussed in Lehnert and Stelle [98], higher-order operators could be crucial to explain the early-universe evolution and its initial conditions. As it will be discussed in section 3.3, starting from the Einstein-Hilbert truncation and performing the RG improvement at the level of the action gives rise to additional terms in the modified Friedmann equations [corresponding to the additional $\Delta t_{\mu\nu}$ -term in the effective field Equations (7)] which allow for a non-singular cosmological evolution for any value of the spatial curvature K [54].

3.2. Entropy Production

The results reviewed in the previous subsection are based on the assumption that the matter energy-momentum tensor

$T_\mu^\nu = \text{diag}(-\rho, p, p, p)$ is covariantly conserved. In a series of works [57, 59, 99], the possibility of an energy flow between the gravitational and matter sectors have been considered and its implications have been explored in detail. The main result is that, under certain assumptions, this energy flow could provide an explanation for the production of entropy during the primordial evolution of the universe.

If the matter energy-momentum tensor $T_\mu^\nu = \text{diag}(-\rho, p, p, p)$ is not separately conserved, the continuity equation arising from the contracted Bianchi identity reads

$$\dot{\rho} + 3H(\rho + p) = -\frac{\dot{\Lambda} + 8\pi\rho\dot{G}}{8\pi G} \quad (28)$$

and can be written in the form

$$\frac{d}{dt}(\rho a^3) + p \frac{d}{dt}(a^3) = \mathcal{P}(t) \equiv -\frac{\dot{\Lambda} + 8\pi\rho\dot{G}}{8\pi G} a^3. \quad (29)$$

Identifying $U = \rho a^3$ as the energy encapsulated in the proper volume $V = a^3$, the latter equation assumes the form of the first law of thermodynamics

$$dU + p dV = \mathcal{P}(t)dt \equiv TdS. \quad (30)$$

In classical cosmology $\mathcal{P}(t) = 0$ and the evolution of the universe is regarded as an adiabatic process. The introduction of time-varying gravitational couplings entails instead a variation of the entropy generated by the continuous energy flow between the gravitational and matter sectors,

$$\frac{dS}{dt} = T^{-1}\mathcal{P}(t). \quad (31)$$

Specifically, the production of entropy during the expansion of the universe requires $(\dot{\Lambda} + 8\pi\rho\dot{G}) \leq 0$. As $k(t)$ decreases with the cosmic time t , it follows that $\dot{\Lambda} \leq 0$ and $\dot{G} \geq 0$. Entropy production thus requires the speed of variation of the running cosmological constant to overcome that of the Newton coupling, such that $-\dot{\Lambda}/\dot{G} \geq 8\pi\rho$. Assuming that the universe is initially dominated by radiation and that the primordial evolution is an approximately adiabatic process, $\mathcal{P}(t) \approx 0$, the standard equilibrium conditions relating the thermodynamic variables (ρ, V, T) can still be used. In this case $\rho(T) \propto T^4$ and therefore $S(t) \propto a^3 \rho^{3/4} + \text{const}$. The precise behavior of $S(t)$ can be obtained by solving the RG-improved Friedmann and continuity equations. This requires to set the scaling relation $k = k(t)$. Since the cosmological evolution is assumed to be approximately adiabatic, the consistency condition (22) should be approximately verified and, based on the results of Reuter and Saueressig [56], a cutoff function of the form $k(t) = \xi_h H(t)$ could still be employed.

Focusing on the “NGFP era,” where $G = g_* k^{-2}$ and $\Lambda = \lambda_* k^2$, the scale setting $k(t) \sim \xi_h H(t)$ leads to an effective cosmological evolution where the scale factor varies as a power law $a(t) \propto t^\alpha$, with

$$\alpha = \frac{2}{(3 + 3w)(1 - \Omega_\Lambda^*)} \quad (32)$$

and $\Omega_\Lambda^* = \xi_h^2 \lambda_*/3$. The corresponding deceleration parameter is $q = \alpha^{-1} - 1$. Imposing the consistency condition (22) fixes $\Omega_\Lambda^* = 1/2$ and gives back the solution obtained in Bonanno and Reuter [55] and discussed in section 3.1: allowing for an energy flow between the gravitational and matter degrees of freedom, removes this additional constraint and leads to a family of cosmological solutions, each characterized by a fixed value of Ω_Λ^* . However, since $\mathcal{P}(t) \approx 0$ is assumed, the value of Ω_Λ^* should not differ too much from that obtained by imposing the consistency condition, i.e., $\Omega_\Lambda^* \sim 1/2$. As the value of λ_* derived from FRG computations is of order $\mathcal{O}(1)$, this also implies that $\xi_h = \mathcal{O}(1)$. The transition to the classical FRW cosmology thus occurs when $k(t_{\text{tr}}) = \xi_h H(t_{\text{tr}}) \sim M_{\text{Pl}}$ which, using the fact that $\xi_h \sim 1$, implies that the parameter α sets the ratio between the transition time t_{tr} and the Planck time, $t_{\text{tr}} = \alpha t_{\text{Pl}}$. We thus learn that if $\alpha > 1$ the transition to the classical regime occurs before the Planck time $t_{\text{tr}} > t_{\text{Pl}}$.

In a RG-improved radiation-dominated epoch, the production of entropy is given by the power law $\mathcal{P}(t) \propto (\alpha - 1)t^{3\alpha-4}$, so that

$$S(t) \propto (t^{3(\alpha-1)} + \text{const.}). \quad (33)$$

In particular, the condition $dS \geq 0$ is met if $\alpha \geq 1$. The case $\alpha = 1$ ($\Omega_\Lambda^* = 1/2$) describes a universe where the gravitational and matter degrees of freedom are decoupled or, equivalently, $\mathcal{P}(t) = 0$. If instead $\alpha > 1$, the variation of Λ dominates over the variation of G , resulting in a net entropy production during the Planck era (NGFP regime). Specifically, assuming $S(0) = 0$, within this model the entropy production can be entirely explained by the variation of Λ .

We highlight that the condition $\alpha > 1$, necessary to generate entropy in the early-universe expansion, is the same condition needed in order to produce a period of power-law inflation. In a radiation-dominated epoch this condition is satisfied for $\Omega_\Lambda^* > 1/2$. If $\alpha = 1$ [$\Omega_\Lambda^* = 1/2$, case analyzed in [55]] no inflation occurs but, as in the case $\alpha > 1$, no particle horizon exists. Therefore, within this simplified model, if $\Omega_\Lambda^* \gtrsim 1/2$ (or, equivalently, $\alpha \gtrsim 1$) the RG running of the gravitational couplings during the Planck epoch can explain the production of entropy and, at the same time, provide a period of power-law inflation.

3.3. Cosmological Singularities and Bouncing Cosmologies in ASG

The existence of an ultraviolet-attractive NGFP entails a weakening of the gravitational interaction at high energies. It is then natural to ask whether this weakening can lead to non-singular cosmologies. While a definite answer in the context of ASG is still out of reach, the mechanism underlying a possible singularity resolution might be captured by a simple model embedding the running of the gravitational couplings in the spacetime dynamics [54].

Following the discussion in section 2, the running of the gravitational couplings in the Einstein-Hilbert action generates an additional term $\Delta t_{\mu\nu}$ in the modified field equations. As we have seen, neglecting this term and introducing the running

couplings at the level of the field equations leads to a family of RG-improved cosmologies admitting a period of power-law inflation and explaining the entropy production in terms of the energy flow between the gravitational and matter sectors [57]. These cosmologies are however singular. As shown explicitly in Bonanno et al. [54], taking into account the correction $\Delta t_{\mu\nu}$ to the effective field Equations (7) might modify this conclusion.

It is assumed that the universe is homogeneous and isotropic and that the energy-momentum tensor is covariantly conserved. Introducing the running of the gravitational couplings at the level of the (Einstein-Hilbert) action yields the modified Friedmann equation [54]

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \left(\frac{\Lambda}{3} + \frac{8\pi G}{3} \rho \right) - \eta_N \frac{\dot{a}^2}{a^2}, \quad (34)$$

where $\eta_N = -\frac{\partial \log G(a)}{\partial \log a}$. The key difference between this model [54] and the one analyzed in the previous subsections [55, 56] lies in the presence of an additional term, $\Delta = -\eta_N H^2$, in the modified Friedmann equation. Since $\eta_N \rightarrow 0$ as the RG flow approaches the perturbative regime ($a(t) \gg L_{Pl}$), Δ vanishes in this limit. Going back in time, η_G varies from its classical value $\eta_N = 0$ to the fixed-point value $\eta_N = -2$ (reached when $k \rightarrow \infty$). In Bonanno et al. [54] a scaling $k^2 \propto R \sim a^{-2}$ was assumed. This approximation is valid if the scale factor undergoes a period of exponential growth at early times—an assumption that can be verified *a posteriori*. Replacing the approximate relations [100]

$$\begin{aligned} G(a) &\simeq G_0 (1 + G_0 g_*^{-1} a^{-2})^{-1}, \\ \Lambda(a) &\simeq \Lambda_0 + \lambda_* a^{-2}, \end{aligned} \quad (35)$$

in Equation (34), where (G_0, Λ_0) are the low-energy values of the gravitational couplings, and assuming that the universe is initially dominated by radiation, it can be easily seen [54] that the field Equations (34) admit non-singular cosmological solutions with minimum radius

$$\begin{aligned} a_b^2 &= -\frac{G_0 \Lambda_0 + g_* (\lambda_* - 3K)}{2g_* \Lambda_0} \\ &\pm \sqrt{\left(\frac{G_0 \Lambda_0 - g_* (\lambda_* - 3K)}{2g_* \Lambda_0} \right)^2 - \frac{8\pi M G_0}{\Lambda_0}}, \end{aligned} \quad (36)$$

where M is an integration constant. Depending on the values of the fixed-point parameters and on the spatial curvature K , both a bouncing cosmology or an emergent universe scenario could in principle be realized [54]. This happens if

$$M \leq \frac{\Lambda_0}{8\pi G_0} \left(\frac{G_0 \Lambda_0 - g_* (\lambda_* - 3K)}{2g_* \Lambda_0} \right)^2 \quad (37)$$

and $a_b^2 > 0$. In this case the universe undergoes a period of inflation at early times, where the scale factor grows exponentially [101]. Otherwise, if a_b is not real, the universe is singular and a period of exponential growth of the scale factor is not possible (unless other degrees of freedom are introduced). However this would invalidate the initial assumption that $k^2 \propto$

$R \sim a^{-2}$ and a separate analysis would be required. This model thus shows how the gravitational antiscreening, encoded in the RG running of the gravitational couplings and in the presence of additional terms in the effective Friedmann equation, could lead to non-singular cosmologies and a period of exponential growth of the universe at early times.

4. INFLATION IN ASYMPTOTICALLY SAFE GRAVITY

4.1. The Idea Behind “Asymptotically Safe Inflation”

Primordial quantum fluctuations occurring in the pre-inflationary epoch have left indelible imprints, which we measure today in the form of tiny temperature anisotropies, $\delta T/T \sim 10^{-5}$, in the CMB radiation: according to the standard cosmological model, the inhomogeneities in the CMB can be traced back to the primordial quantum fluctuations in the pre-inflationary era. These fluctuations were subsequently amplified and smoothed out by the exponential growth of the universe, thus resulting in small density fluctuations at the last scattering surface. The distribution of temperature anisotropies in the sky could thus give us indirect information on the physics of the very early universe.

In momentum space, the power spectra of scalar and tensorial perturbations are written as follows

$$\mathcal{P}_s(k) \simeq A_s \left(\frac{k}{k_*} \right)^{n_s-1}, \quad \mathcal{P}_t(k) \simeq A_t \left(\frac{k}{k_*} \right)^{n_t}, \quad (38)$$

where $k = |\mathbf{k}|$ is the norm of the 3-momentum $\mathbf{k}$ and $k_* \sim 0.05 \text{Mpc}^{-1}$ is a reference scale. The spectral index n_s and the tensor-to-scalar ratio $r \equiv A_t/A_s$ can be obtained from observational data. In particular, the most recent observations to date [102] constrain the spectral index to be $n_s = 0.9649 \pm 0.0042$ at 68% confidence level, and limit the tensor-to-scalar ratio to values $r < 0.064$. Note that although the scalar power spectrum is almost scale invariant, perfect scale invariance (corresponding to $n_s = 1$) is excluded.

The existence of a NGFP in the RG flow of gravity could provide a natural and intuitive explanation for the nearly-scale invariance of the power spectrum of temperature fluctuations in the CMB. Close to the NGFP, the effective background graviton propagator behaves as $G(p) \sim 1/p^{d-2-\eta_N}$ [103]. In $d = 4$, the asymptotic-safety condition requires the anomalous dimension of the Newton coupling to approach the value $\eta_N = -2$ in the ultraviolet limit. In this case the background graviton propagator in coordinate space scales as $\mathcal{G}(x, y) \sim \log|x - y|^2$ at the NGFP [103]. Assuming that the temperature fluctuations are entirely due to the amplification of the quantum fluctuation of the spacetime geometry during inflation and that these fluctuations are generated during the Planck era, the corresponding density fluctuations $\delta\rho$ are characterized by a two-point correlation function [57, 103, 104]

$$\xi(\mathbf{x}) = \langle \delta\hat{\rho}(\mathbf{x} + \mathbf{y}) \delta\hat{\rho}(\mathbf{y}) \rangle \propto \langle \delta\mathbf{R}(\mathbf{x} + \mathbf{y}, t) \delta\mathbf{R}(\mathbf{y}, t) \rangle \sim |\mathbf{x}|^{-4}, \quad (39)$$

where $\delta\hat{\rho} = \delta\rho/\bar{\rho}$ is the fractional density fluctuation field and $\delta\mathbf{R}(\mathbf{y}, t)$ stands for the fluctuation of the scalar curvature [or any component of the Riemann or Einstein tensor [55]], induced by a metric fluctuation. The power spectrum in momentum space is given by the 3-dimensional Fourier transform

$$|\delta_k|^2 = V \int d^3x \xi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \quad (40)$$

The spectral index n_s defines the power-law scaling of the power spectrum, $|\delta_k|^2 \propto |\mathbf{k}|^{n_s}$. Thus the scaling $\xi(\mathbf{x}) \sim |\mathbf{x}|^{-4}$ gives rise to a perfectly scale invariant power spectrum, with $n_s = 1$ [57, 94, 103, 104]. The exact scale invariance of the power spectrum reflects the exact scale invariance of the theory at the NGFP. It is thereby possible that the nearly-scale-invariance of the scalar power spectrum $\mathcal{P}_s(k)$ is due to the nearly-scale invariant flow of RG trajectories in the proximity of the NGFP. This observation [57, 94, 103, 104] was the starting point for a number of studies looking for the existence of (unstable) deSitter solutions in ASG [58, 97, 105], giving rise to a sufficiently long period of “NGFP-driven inflation” [57]. The inflationary scenario arising from this mechanism is sometimes called “Asymptotically Safe Inflation.”

4.2. Starobinsky Model and RG-Running in Quadratic Gravity

Among all proposed inflationary models [106], the Starobinsky model is certainly one of the most appealing: it is a zero-parameters model and is compatible with the current observational data [102]. The Starobinsky model relies on the inclusion of an R^2 -term in the gravitational action. This is the minimal modification of Einstein gravity needed to produce inflation. From the point of view of ASG, focusing on an $f(R)$ -truncation, the quadratic gravity Lagrangian

$$\mathcal{L} = \frac{1}{16\pi G} (R - 2\Lambda - BR^2) \quad (41)$$

should comprise all relevant couplings of the theory (with respect to the NGFP): according to the studies of the renormalization group flow of $f(R)$ theories (in pure gravity), the NGFP comes with three relevant directions—those associated with the couplings (G, Λ, B) [26, 34, 80, 81]. The latter are the only free parameters of the theory: every RG trajectory is uniquely identified by the infrared values of the scale-dependent couplings $(G(k), \Lambda(k), B(k))$. A key question is whether there exists an RG trajectory matching the infrared values of these couplings. Moreover, it is interesting to understand whether Starobinsky inflation can be realized naturally in the context of ASG. For this to happen the sign of the coupling B is crucial, as the Starobinsky model requires B to be negative. In addition, it is a key requirement that classical Einstein gravity is recovered at low energies. Studying the RG flow of the couplings $(G(k), \Lambda(k), B(k))$ in the quadratic truncation (41), it has been shown [107] that there exists an RG trajectory such that the

observational constraints

$$\begin{aligned} B(k_{\text{infl}} \sim H_{\text{infl}} = 10^{22} \text{eV}) &= -1.7 \cdot 10^{-46} \text{eV}^{-2}, \\ G(k_{\text{lab}} \sim 10^{-5} \text{eV}) &= 6.7 \cdot 10^{-57} \text{eV}^{-2}, \\ \Lambda(k_{\text{Hubble}} \sim H_0 = 10^{-33} \text{eV}) &= 4 \cdot 10^{-66} \text{eV}^2, \end{aligned} \quad (42)$$

are all fulfilled. We refer the reader to the original paper [107] for the details of the computation. The coupling $B(k)$ is initially ($k \rightarrow \infty$) positive, but it turns negative along the RG flow: the transition scale is $k \sim 10^{23} \text{GeV}$ (well above the Planck scale), so that at inflationary scales the action (41) matches that of the Starobinsky model [107]. Below the Planck scale, the couplings $B(k)$ and $G(k)$ vary by many orders of magnitude in a very short RG-time, $t = \log k$. Their observed constant values are thus reached at inflationary scales. The cosmological constant instead keeps running even after the end of inflation: at inflationary scales its magnitude is $\sim 4 \cdot 10^{30} \text{eV}^2$, while its observed constant value is only reached at $k \sim 10^{-2} \text{eV}$ [95, 107].

4.3. Constraints From Planck Data in Gravity-Matter Systems

In this subsection we review the results in Bonanno et al. [61] and Platania [91]. As the initial conditions for inflation are placed at trans-Planckian scales and since the effective action at inflationary scales depends on how the RG-trajectory realized by Nature emerges from the NGFP, the Planck data on CMB anisotropies can in principle put constraints on the universality properties of the gravitational RG flow. The latter are encoded in the critical exponents θ_i governing the scaling of the gravitational couplings in the vicinity of the NGFP. In turn, the specific values of the critical exponents depend on the number of scalar, Dirac and vectors fields in the theory [32, 81, 83]. The observational constraints on the spectral index n_s and tensor-to-scalar ratio r could then be used to put constraints on the primordial matter content of the universe. Introducing the running of the gravitational couplings at the level of the (Einstein-Hilbert) action [48, 92] provides a simple toy model to understand whether and how this mechanism is realized [61, 91].

We restrict ourselves to the Einstein-Hilbert truncation, where the scaling of the gravitational couplings about the NGFP is that given in Equation (5). Following the discussion in section 2, close to the NGFP the consistency condition (16) imposes the scaling relation $k^2 = \xi R$, with $\xi = \frac{1}{4\lambda_*}$. The RG running (5) thus yields an effective gravitational action of the form [91]

$$S_{\text{grav}}^{\text{eff}} = \int d^4x \sqrt{-g} \left\{ \frac{R^2}{128\pi g_* \lambda_*} + f_{RG}(R) \right\}, \quad (43)$$

where $f_{RG}(R)$ is the part of the action generated as the RG trajectories flow away from the NGFP

$$f_{RG}(R) = b_1 R^{\frac{4-\theta_1-\theta_2}{2}} + b_2 R^{\frac{4-\theta_1}{2}} + b_3 R^{\frac{4-\theta_2}{2}} + b_4 R^{2-\theta_1} + b_5 R^{2-\theta_2}, \quad (44)$$

with the coefficients b_i being defined by

$$\begin{aligned} b_1 &= \frac{c_1 c_2 (e_1^2 e_2^2 + e_1^2 e_2^2) (4\lambda_* M_{Pl}^2)^{\frac{\theta_1 + \theta_2}{2}}}{128\pi (g_* \lambda_*)^2}, \\ b_2 &= \frac{c_1 (e_1^2 \lambda_* - e_1^2 g_* - 2e_1^2 \lambda_*) (4\lambda_* M_{Pl}^2)^{\frac{\theta_1}{2}}}{128\pi (g_* \lambda_*)^2}, \\ b_3 &= \frac{c_2 (e_2^2 \lambda_* - e_2^2 g_* - 2e_2^2 \lambda_*) (4\lambda_* M_{Pl}^2)^{\frac{\theta_2}{2}}}{128\pi (g_* \lambda_*)^2}, \\ b_4 &= \frac{c_1^2 (e_1^2 e_2^2) (4\lambda_* M_{Pl}^2)^{\theta_1}}{128\pi (g_* \lambda_*)^2}, \\ b_5 &= \frac{c_2^2 (e_1^2 e_2^2) (4\lambda_* M_{Pl}^2)^{\theta_2}}{128\pi (g_* \lambda_*)^2}. \end{aligned} \quad (45)$$

Here M_{Pl} is the reduced Planck mass, while the integration constants c_i , the critical exponents θ_i and the eigenvectors $\mathbf{e}_i$ are those introduced in section 2.2 (cf. Equation 5). The action

$$S_{grav}^* = \int d^4x \sqrt{-g} \frac{R^2}{128\pi g_* \lambda_*} \quad (46)$$

is the fixed-point action [88, 91]. This is compatible with the results in Benedetti and Caravelli [108], Dietz and Morris [109], and Demmel et al. [27] where, using the FRG Equation (4) to study the RG flow of $f_k(R)$ -gravity, it has been shown that the fixed-point Lagrangian is $\mathcal{L}_* = f_*(R) \propto R^2$. Additional R^n -operators are generated along the RG flow. The effective action in (43) is thus expected to capture key features of the gravitational RG flow: at the fixed point the action is S_{grav}^* . Lowering the RG scale k down toward the infrared, additional operators are generated and $\mathcal{L}_{grav} = \mathcal{L}_* + \delta\mathcal{L}_{RG}$. The set of operators appearing in $\delta\mathcal{L}_{RG}$ depends on how the RG trajectories emerge from the NGFP. In the simplified model (43), this information lies in the critical exponents θ_i . In what follows we will explore the consequences of this fact in inflationary cosmology [61, 91].

Provided that $f_{RG}^{(2)}(R) \neq -\frac{1}{64\pi g_* \lambda_*}$, the gravitational action (43) is conformally equivalent to Einstein gravity, minimally coupled with a scalar field ϕ

$$S_{grav} = \int d^4x \sqrt{-g_E} \left(\frac{R_E}{16\pi G_0} + \frac{1}{2} g_E^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \quad (47)$$

where the subscript “E” indicates that these quantities are computed using the metric $g_{\mu\nu}^E = e^{\sqrt{2/3}\phi/M_{Pl}} g_{\mu\nu}$ in the Einstein frame, and $V(\phi) = U(\varphi(\phi)) \varphi(\phi)^{-2}$, with

$$\varphi = 16\pi G_0 \left(\frac{R}{64\pi g_* \lambda_*} + f_{RG}'(R) \right) = e^{\sqrt{2/3}\phi/M_{Pl}}, \quad (48)$$

$$U(\varphi) = \frac{R[\varphi]^2}{128\pi g_* \lambda_*} - f_{RG}(R[\varphi]) + R[\varphi] f_{RG}'(R[\varphi]). \quad (49)$$

A period of exponential grow of the scale factor occurs if the dynamics of the scalar field ϕ is dominated by its potential energy $V(\phi)$. This happens under the slow-roll conditions

$$\epsilon(\phi) \equiv \frac{1}{2\kappa} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 \ll 1, \quad \eta(\phi) \equiv \frac{1}{\kappa} \left(\frac{V''(\phi)}{V(\phi)} \right) \ll 1. \quad (50)$$

The violation of the slow-roll conditions, encoded in the equation $\epsilon(\phi_f) = 1$, defines the value of the field at the end of inflation, $\phi_f \equiv \phi(t_f)$. The initial condition $\phi_i \equiv \phi(t_i)$ is then obtained by fixing the number of e-folds

$$N(\phi_i) = \int_{\phi_f}^{\phi_i} \frac{V(\phi)}{V'(\phi)} d\phi \quad (51)$$

before the end of inflation. In the slow-roll approximation, the spectral index and tensor-to-scalar ratio characterizing the scalar power spectrum $\mathcal{P}_s(k)$ in Equation (38) can be easily computed by means of the following relations [110]

$$n_s = 1 - 6\epsilon(\phi_i) + 2\eta(\phi_i), \quad r = 16\epsilon(\phi_i). \quad (52)$$

Moreover, every inflationary model has to be “normalized” [106], i.e., the inflation mass has to be fixed by requiring that the amplitude A_s of the scalar power spectrum (38) is

$$A_s = \frac{V(\phi_i)}{24\pi^2 M_{Pl}^4 \epsilon(\phi_i)} \simeq 2.2 \cdot 10^{-19}. \quad (53)$$

At the NGFP ($k \rightarrow \infty$) $f_{RG}(R) = 0$ and the effective action (43) reduces to the fixed-point action S_{grav}^* . In the Einstein frame, the corresponding fixed-point scalar potential is constant

$$V_*(\phi) = 8\pi g_* \lambda_* M_{Pl}^4, \quad (54)$$

and therefore it would generate an exactly scale-invariant power spectrum, with $n_s = 1$. This is compatible with the discussion made in section 4.1 and based on the scaling of the background graviton propagator at the NGFP [57, 94, 103, 104]. The mass scale associated with the scalar degree of freedom ϕ can be read off from the potential $V_*(\phi)$ and reads

$$m^2 = 8\pi \left(\frac{4}{3} \lambda_* g_* \right) M_{Pl}^2. \quad (55)$$

This mass depends on fixed-point quantities only via the universal product $(\lambda_* g_*)$ [96], as expected from the universality properties of the theory at the NGFP.

In the model (43) the departure from the exact scale invariance is due to the departure of the RG flow from the NGFP. Lowering the RG scale down toward the infrared, the gravitational Lagrangian is modified by the operators in $\delta\mathcal{L} = f_{RG}(R)$ and, in the Einstein frame, this corresponds to a variation of the scalar potential $V(\phi)$,

$$V_* \rightarrow V(\phi) = V_* + \delta V(\phi). \quad (56)$$

Its form is determined by the critical exponents θ_i , which are real numbers in the case of the most commonly studied gravity-matter systems [32, 81, 83]. The asymptotic-safety condition requires the real part of the critical exponents to be positive, $\text{Re}(\theta_i) > 0$. As we are interested in the case of gravity-matter systems, we will only focus on the case where the critical exponents are real. It is assumed, in a first approximation, that

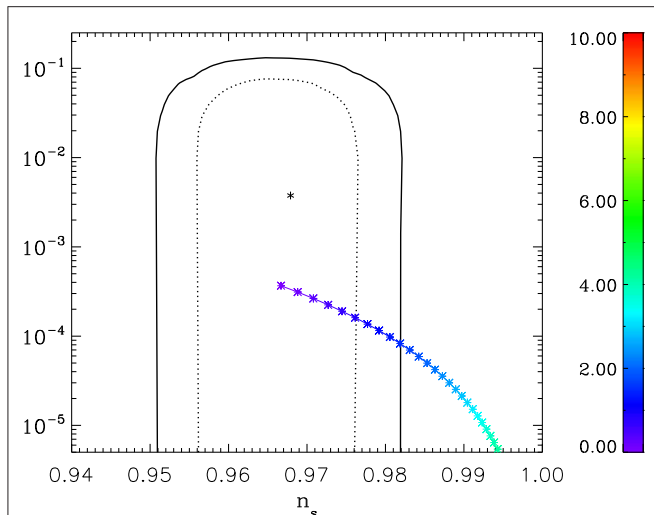


FIGURE 1 | Spectral index and tensor-to-scalar ratio induced by the family of theories in Equation (57) as a function of the power index p for $N = 60$ e-folds and $b = 1$. Dashed and solid lines correspond to the 1σ and 2σ confidence levels on the values of (n_s, r) extracted from the Planck data [102]. The Starobinsky model, denoted with a star symbol, is also shown for comparison. Only theories with $p \leq 1$ are reasonably within the 2σ confidence-level line.

the energy-density of the inflation field ϕ dominates. Under this assumption, the other matter fields do not contribute to the inflationary dynamics [61].

It is crucial to note that if all critical exponents are $\theta_i > 4$, the effective Lagrangian $\mathcal{L}_{\text{eff}}$ reads

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_* + b R^{-p}, \quad p > 0, \quad (57)$$

where R^{-p} is the dominant correction in $f_{RG}(R)$ and b is the corresponding coefficient. Contributions of the form R^{-p} are suppressed when R is large, so that the deviation from the exact scale invariance would be negligible (see **Figure 1**). Moreover in this case the R -operator is not generated by the flow. In the case $\theta_i = 4$, the model (43) gives rise to an inflationary scenario compatible with the Planck data only under specific conditions [91].

The agreement with the Planck data thus requires that at least one of the critical exponents is $\theta_i < 4$. This condition is realized, e.g., when gravity is minimally-coupled to the fields of the Standard Model, at least in the approximation where these fields are free [32, 81, 83]. Within the simple model reviewed here, matter models making all gravitational critical exponents $\theta_i > 4$ would not be compatible with observational data. In this sense, the Planck data on the CMB anisotropies could be used to constrain the primordial matter content of the universe [61].

The scalar potential $V(\phi)$ is shown in **Figure 2** for various values of $\theta_1 = \theta_2$. All functions $V(\phi)$ approach the same constant value V_* for $\phi \gg M_{Pl}$. In fact, as soon as $\theta_i \neq 0$, the coupling of the R^2 -term in Equation (43) is not modified by the presence of the additional operators in $\delta\mathcal{L} = f_{RG}(R)$. Therefore, the height of the plateau and the inflation mass are those in Equations (54) and (55). This is an artifact of the simplified

model in Bonanno et al. [61]: the coupling to the R^2 operator is a running quantity and therefore also the value of the scalar potential at $\phi \gg M_{Pl}$ should vary along the flow. In other words, in a more elaborate model accounting for the running of the coupling B in Equation (41), the family of effective potentials $V(\phi)$ should be characterized by a plateau with $V_{\text{plateau}} \neq V_*$: this decoupling would allow to set the initial conditions for inflation at Planckian scales and, at the same time, to reproduce the correct amplitude of scalar perturbations at the horizon exit [91].

As the RG flow moves away from the NGFP, the scalar potential $V(\phi)$ is dynamically modified such that $V'(\phi)$ is generally non-zero. The scalar field ϕ thus acquires a RG-running-induced kinetic energy $\dot{\phi}_i \sim -V'(\phi_i)/3H(t_i)$. This provides the initial conditions for the subsequent evolution of the scale factor $a(t)$, according to the modified Friedmann equations. Depending on the RG-induced variation of the scalar potential $\delta V(\phi)$, the dynamics of the scalar field ϕ can trigger a period of slow-roll inflation. For instance, the case $\theta_1 = \theta_2 = 2$ gives the scalar potential

$$V(\phi) = e^{-2\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}} \left\{ \frac{3}{4} \left(1 - e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}} \right)^2 m^2 + \Lambda_{\text{eff}} \right\} M_{Pl}^2, \quad (58)$$

i.e., a Starobinsky-like potential in the presence of an effective cosmological constant $\Lambda_{\text{eff}} = -(b_1 + b_4 + b_5)M_{Pl}^2$ (see **Figure 2**). As it is well-known, this model leads to cosmic parameters

$$n_s \simeq 1 - \frac{2}{N} + \mathcal{O}(N^{-3}), \quad r \simeq \frac{12}{N^2} + \mathcal{O}(N^{-3}), \quad (59)$$

in good agreement with the current observational data. In the case $\theta_1 \approx 2$ and $\theta_2 \approx 4$, realized when gravity is minimally coupled with the (free) matter fields of the Standard Model [32, 81], the action (43) differs from the one of the Starobinsky model by subleading terms of the form R^{-p} , with $p > 0$, which are suppressed for large R . The inflationary dynamics, as shown in Bonanno et al. [61], is thus very similar to that of the Starobinsky model (case $\theta_1 = \theta_2 = 2$). In the next subsection we will see how this scenario is modified when the RG-improved effective action is obtained by starting from the quadratic gravity action (41).

4.4. Comparison With the Planck Data in RG-Improved Quadratic Gravity

In Bonanno and Platania [60, 111], a class of inflationary models arising from the RG improvement of quadratic gravity (without matter) has been investigated [see also [48, 92]]. According to the studies of the non-perturbative RG flow of truncated $f(R)$ -theories without matter [104, 112], the NGFP is attractive with respect to three relevant directions, those associated with the dimensionless couplings $(g_k, \lambda_k, \beta_k)$, with $\beta_k = B(k)/G(k)$ [26, 34, 80, 81]. The question motivating the studies in Bonanno and Platania [60, 111] is whether the scale dependence of all relevant gravitational couplings can modify the classical Starobinsky model and if the RG-improved model is compatible with the Planck data. The ansatz for the gravitational action is

$$\mathcal{L}_k = \frac{1}{16\pi g_k} (R - 2\lambda_k k^2) - \beta_k R^2. \quad (60)$$

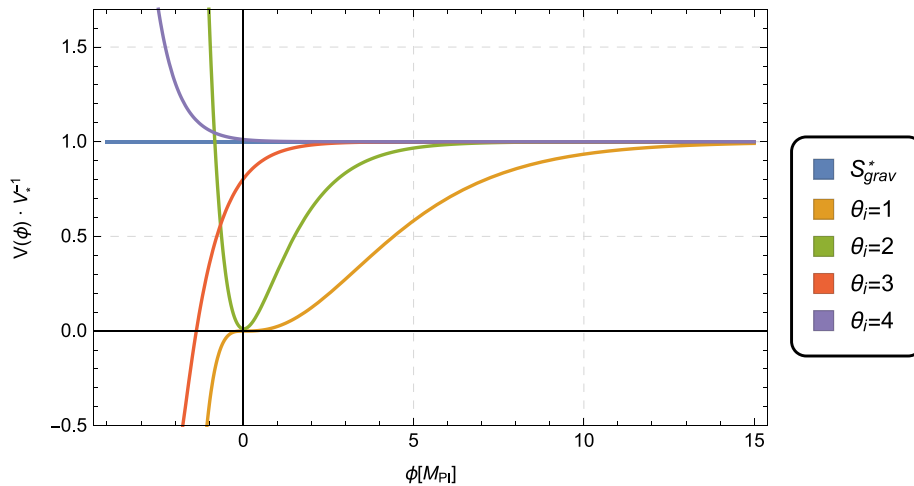


FIGURE 2 | Inflationary potentials $V(\phi)$ produced by the conformal transformation of the action (43) for various values of the critical exponents $\theta_1 = \theta_2$ [91]. The fixed-point potential $V_* = 8\pi G \cdot \lambda \cdot M_{Pl}^4$ associated with the fixed-point action S_{grav}^* , Equation (46), is also shown for comparison. It corresponds to an exactly scale invariant scalar power spectrum, $n_s = 1$. When the RG flow departs from the NGFP, additional operators are generated by the flow. These operators break the perfect scale invariance realized at the NGFP and destabilize the fixed-point potential, $V_* \rightarrow V(\phi) = V_* + \delta V(\phi)$, such that $V'(\phi) \neq 0$ at $\phi \sim M_{Pl}$. The scalar field ϕ thus acquires a RG-induced kinetic energy, $\dot{\phi} \sim -V'(\phi)/3H(t_i)$. The subsequent dynamics depends crucially on the critical exponents θ_i . In particular, the case $\theta_1 = \theta_2 = 2$ reproduces the well-known Starobinsky model.

In order to derive an analytical form for the inflationary potential in the Einstein frame, the running of the gravitational couplings is approximated by [60, 111]

$$\begin{aligned} g_k &= \frac{(c_1 \mu^{-2}) k^2}{1 + g_*^{-1} (k^2 - \mu^2) (c_1 \mu^{-2})} \\ \beta_k &= \beta_* + b_0 \left(\frac{k^2}{\mu^2} \right)^{-1/2}, \\ \lambda_k &= c_2 k^{-2}, \end{aligned} \quad (61)$$

where μ is a reference scale and the three parameters (b_0, c_1, c_2), corresponding to the three relevant directions of the theory, identify the RG trajectories terminating at the NGFP in the ultraviolet limit. These are free parameters of the theory and must be fixed by comparing the results with observations. Using the cutoff $k^2 = \xi R$ [60, 111], the RG-improved effective action reads

$$S_{grav}^{\text{eff}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left\{ R - 2\tilde{\Lambda} + \frac{R^2}{6m^2} + \tilde{\alpha} R^{3/2} \right\}, \quad (62)$$

with

$$\begin{aligned} \kappa &= \frac{48\pi^2 c_1}{6\pi\mu^2 - 23(\mu^2 + 2\xi c_2)c_1}, & m^2 &= \frac{8\pi^2}{\kappa(23\xi - 96\pi^2\beta_*)}, \\ \tilde{\Lambda} &= \frac{\mu^2(6\pi - 23c_1)c_2}{6\pi\mu^2 - 23(\mu^2 + 2\xi c_2)c_1}, & \tilde{\alpha} &= -2b_0\kappa (\xi\mu^{-2})^{-\frac{1}{2}}. \end{aligned} \quad (63)$$

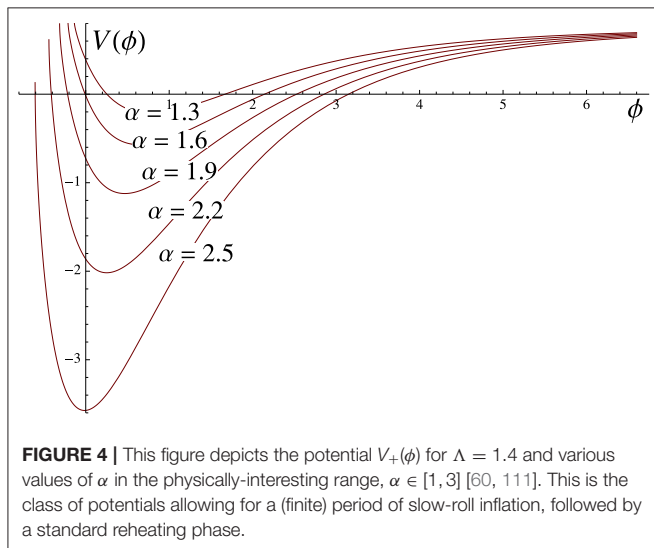
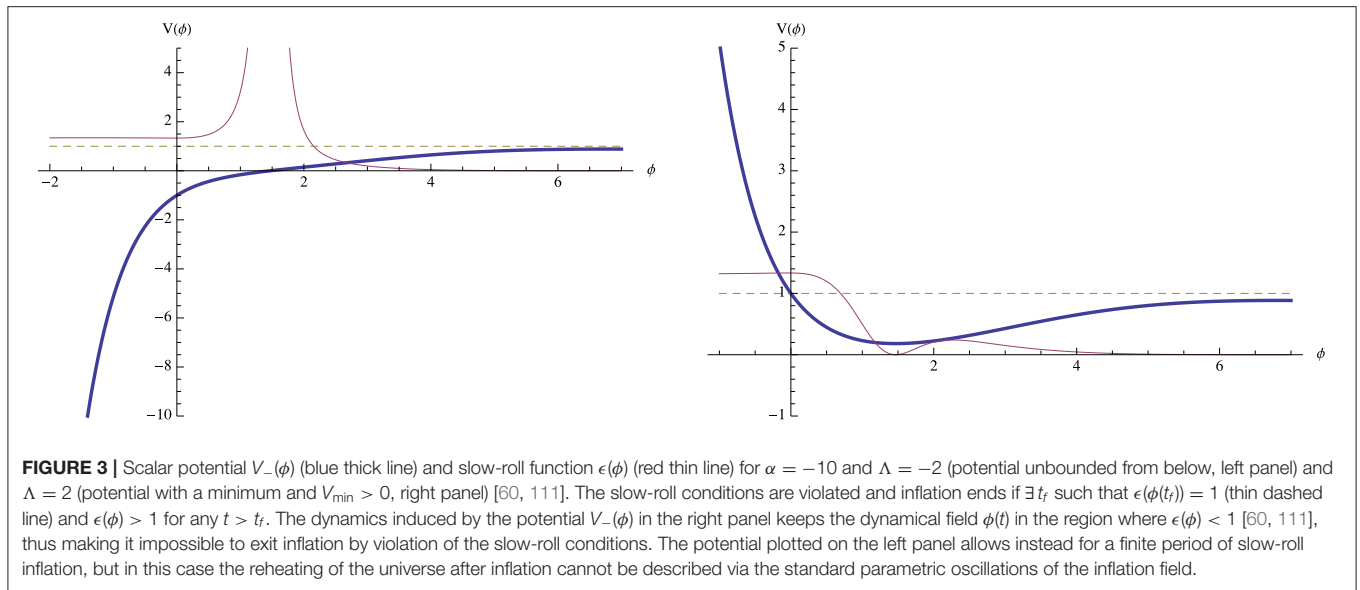
The inflationary scenario generated in this model can be studied in the Einstein frame, where the $f(R)$ action (62) can be written

as in Equation (47). In this case the scalar potential $V(\phi)$ reads [60, 111]

$$\begin{aligned} V_{\pm}(\phi) &= \frac{m^2 e^{-2\sqrt{\frac{2\kappa}{3}}\phi}}{256\kappa} \left\{ 192 \left(e^{\sqrt{\frac{2\kappa}{3}}\phi} - 1 \right)^2 - 3\alpha^4 + 128\Lambda \right. \\ &\quad \left. - \sqrt{32}\alpha \left[\left(\alpha^2 + 8e^{\sqrt{\frac{2\kappa}{3}}\phi} - 8 \right) \pm \alpha \sqrt{\alpha^2 + 16e^{\sqrt{\frac{2\kappa}{3}}\phi} - 16} \right]^{\frac{3}{2}} \right. \\ &\quad \left. - 3\alpha^2 \left(\alpha^2 + 16e^{\sqrt{\frac{2\kappa}{3}}\phi} - 16 \right) \mp 6\alpha^3 \sqrt{\alpha^2 + 16e^{\sqrt{\frac{2\kappa}{3}}\phi} - 16} \right\}, \end{aligned} \quad (64)$$

with the dimensionless couplings Λ and α given by $\Lambda = m^{-2}\tilde{\Lambda}$ and $\alpha = 3\sqrt{3}m\tilde{\alpha}$. The existence of two solutions is due to the presence of the additional $R^{3/2}$ -term in the effective action, and the standard Starobinsky model is recovered by setting $\alpha = \Lambda = 0$. Both functions $V_{\pm}(\phi)$ define a two-parameters family of potentials, parameterized by the couple (α, Λ) . The common feature of these potentials is the existence of a plateau for large positive values of the field ϕ , with $V_{\text{plateau}} = \frac{3m^2}{4\kappa}$. Note that the inclusion of the running of the coupling β_k now allows for an effective potential with $V_{\text{plateau}} \neq V_*$, as mentioned in the previous section.

In order to fulfill the slow-roll conditions, the dynamical evolution of the inflation field must start from a quasi-deSitter state at $V(\phi_i) \sim V_{\text{plateau}}$, and then proceed toward $\phi \ll M_{Pl}$. The inflationary dynamics depends on the values of (α, Λ) . For any (α, Λ) , the potential $V_{\pm}(\phi)$ can either develop a minimum (Figure 3, right panel) or be unbounded from below (Figure 3, left panel). A standard reheating phase is only possible in the first case. In addition, in these models a “graceful exit” from inflation by violation of the slow-roll conditions is only possible when



$V_{\min} \leq 0$. The case $V_{\min} > 0$ leads instead to eternal inflation, as shown in the right panel of **Figure 3**.

We now focus on the class of inflationary potentials providing a well defined exit from inflation by violation of the slow-roll conditions, followed by a phase of parametric oscillations of the inflation field [60, 111]. These conditions are realized by the class of potentials $V_+(\phi)$, with $\alpha \in [1, 3]$ and $\Lambda \in [0, 1.5]$ [60, 111]. The corresponding potential is shown in **Figure 4** for $\Lambda = 1.4$ and various values of α .

As already mentioned, the constants (α, Λ) parameterize the deviations from the Starobinsky model due to the RG running of the gravitational couplings. It is thereby interesting to understand whether these modifications can affect the form of the power spectrum of temperature fluctuations in the CMB, and if the values of the spectral index n_s and tensor-to-scalar ratio r are

TABLE 1 | Values of the spectral index n_s and tensor-to-scalar ratio r obtained from the RG-improved model (62) for different values of (α, Λ) and number of e-folds N [111].

Cases	α	$N = 50$		$N = 55$		$N = 60$	
		n_s	r	n_s	r	n_s	r
0	1.0	0.965	0.0069	0.968	0.0058	0.971	0.0050
	1.8	0.966	0.0074	0.969	0.0063	0.972	0.0055
	2.6	0.967	0.0076	0.969	0.0065	0.972	0.0056
1	1.0	0.965	0.0070	0.968	0.0059	0.971	0.0051
	1.8	0.966	0.0074	0.969	0.0063	0.972	0.0055
	2.6	0.967	0.0076	0.969	0.0065	0.972	0.0056

The range of values obtained for the spectral index, $n_s \in [0.965, 0.972]$, is in agreement with the one obtained by the Planck Collaboration, $n_s = 0.968 \pm 0.006$, and the tensor-to-scalar ratio is always compatible with their upper limit, $r < 0.11$.

modified by this running. The results are summarized in **Table 1** [111]. The range of values for the spectral index is $n_s \in [0.965, 0.972]$, in agreement with the value extracted from the Planck data, $n_s = 0.968 \pm 0.006$. The tensor-to-scalar ratio is always compatible with their upper limit, but it is slightly higher than the one predicted within the Starobinsky model.

5. DISCUSSION

The phenomenological consequences of Asymptotically Safe Gravity (ASG) are typically investigated within models that take the running of the gravitational couplings into account. Based on the decoupling mechanism [72], it is expected that these models can provide a qualitative, yet simple and intuitive, understanding of the effective modifications of General Relativity induced by quantum gravity in the asymptotic-safety approach. Nonetheless, the derivation of these “renormalization group (RG) improved” models is not free of ambiguities. Bearing in mind strengths

and limitations of this approach, the scope of this review was to provide an overview of the main cosmological implications of ASG derived from models of RG-improved cosmology.

ASG is based on the existence of an interacting fixed point which is attained by the gravitational RG flow in the ultraviolet limit. The scale invariance of gravity at high energies and the consequent gravitational antiscreening can be regarded as the hallmarks of ASG. In particular the antiscreening character of gravity, rendering the gravitational interaction weaker at high energies, could lead to non-singular cosmological solutions: the classical singularity could be replaced by a bounce or an emergent universe [54]. The entropy production during inflation might be attributed to an energy flow from the gravitational to the matter degrees of freedom, which causes the primordial evolution of the universe to be approximately (but not exactly) adiabatic [57]. Moreover, the existence of a regime where gravity is approximately scale invariant (fixed-point regime and departure of the RG flow from it) provides a simple and natural interpretation for the nearly-scale-invariance of the power spectrum of temperature fluctuations in the Cosmic Microwave Background (CMB) radiation [57]. In the case of pure gravity, the spectral index and tensor-to-scalar ratio evaluated from models of “RG-improved” inflation are in agreement with the Planck data on CMB anisotropies [60]. When matter is minimally coupled to gravity, the universality properties of the gravitational RG flow are modified and this modification depends on the number and type (scalar, Dirac, vector, etc.) of matter fields: the observational data could thus be used to constrain the matter content of the theory in the early universe [61]. Finally, the running of the R^2 -coupling could also provide a mechanism to set the initial condition for inflation at trans-planckian scales, while being able to reproduce the amplitude of the scalar power spectrum at the horizon exit [91]. If this mechanism is realized, it could provide a solution to the “unlikeness problem” [113] of inflationary cosmology. A definite answer requires however more elaborate and extended studies, going beyond the simple models reviewed here.

Most of the results listed above have been obtained by including the running of the gravitational couplings within the Einstein-Hilbert truncation. As argued in Lehnert et al. [98], fourth-derivatives operators are crucial for the understanding of the early-universe evolution. The inclusion of these operators in the models of RG-improved cosmologies could then be important to determine the phenomenological implications of the gravitational antiscreening.

One of the main problems of models of “RG-improved cosmologies” is the identification of the physical cutoff acting as a decoupling scale [72] for the RG flow of the effective average action [65]. The symmetries of theory play an important role,

as they could provide a guideline for this scale-setting [87, 88, 90, 114, 115]. As shown in Reuter and Weyer [86], Reuter and Weyer [82], Babic et al. [87], Domazet and Stefancic [88], and Koch et al. [89], the contracted Bianchi identities typically lead to a “consistency condition” which can be used to determine the form of the cutoff scale. Close to a fixed point, the RG flow should be universal (no scheme or regulator dependence, at least for the flow of the full—not truncated—effective action) and all physical scales collapse into one: in this case the scale-setting is essentially unique. Away from the fixed point, the physical cutoff is dictated by the consistency condition. However, this condition requires the running of the gravitational couplings as an external input: since the RG flow obtained from the Wetterich equation [65] depends explicitly on the choice of the regulator, this dependence is inherited by the physical cutoff scale and it disappears only in the proximity of the fixed point (provided that no truncation of the effective action is employed).

While it is expected that the RG-improved models capture the qualitative features of the quantum modifications of General Relativity according to ASG, quantitative results require the knowledge of the fully-quantum gravitational effective action. This is expected to be non-local, due to the resummation of quantum fluctuations on all scales. Progress in this direction has been made in Codello and Jain [116], where the leading-order, quadratic part of the effective action has been derived within the framework of effective field theory, in Codello et al. [117], by studying the flow of the non-local part of the one-loop effective action, and, more recently, in Knorr et al. [69], where the beta functions for a specific non-local action have been computed using the functional renormalization group. Future developments of these programs could provide indications on the form of the gravitational effective action. On the phenomenological side, this could allow to derive more quantitative results on the implications of ASG in astrophysics and cosmology.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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On Characterizing the Quantum Geometry Underlying Asymptotic Safety

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The asymptotic safety program builds on a high-energy completion of gravity based on the Reuter fixed point, a non-trivial fixed point of the gravitational renormalization group flow. At this fixed point the canonical mass-dimension of coupling constants is balanced by anomalous dimensions induced by quantum fluctuations such that the theory enjoys quantum scale invariance in the ultraviolet. The crucial role played by the quantum fluctuations suggests that the geometry associated with the fixed point exhibits non-manifold like properties. In this work, we continue the characterization of this geometry employing the composite operator formalism based on the effective average action. Explicitly, we give a relation between the anomalous dimensions of geometric operators on a background d -sphere and the stability matrix encoding the linearized renormalization group flow in the vicinity of the fixed point. The eigenvalue spectrum of the stability matrix is analyzed in detail and we identify a “perturbative regime” where the spectral properties are governed by canonical power counting. Our results recover the feature that quantum gravity fluctuations turn the (classically marginal) R^2 -operator into a relevant one. Moreover, we find strong indications that higher-order curvature terms present in the two-point function play a crucial role in guaranteeing the predictive power of the Reuter fixed point.

Keywords: quantum gravity, asymptotic safety, renormalization group, quantum geometry, scaling dimension

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1. INTRODUCTION

General relativity taught us to think of gravity in terms of geometric properties of spacetime. The motion of freely falling particles is determined by the spacetime metric $g_{\mu\nu}$ which, in turn, is determined dynamically from Einstein's equations. It is then an intriguing question what replaces the concept of a spacetime manifold once gravity is promoted to a quantum theory. Typically, the resulting geometric structure is referred to as “quantum geometry” where the precise meaning of the term varies among different quantum gravity programs.

An approach toward a unified picture of the quantum gravity landscape could then build on identifying distinguished properties which characterize the underlying quantum geometry and lend themselves to a comparison between different programs. While this line of research is still in its infancy, a first step in this direction, building on the concept of generalized dimensions, has been very fruitful. In particular, the spectral dimension d_s , measuring the return probability of a diffusing particle in the quantum geometry, has been computed in a wide range of programs including Causal Dynamical Triangulations [1], Asymptotic Safety [2–5], Loop Quantum Gravity [6], string theory

[7], causal set theory [8–10], the Wheeler-DeWitt equation [11], non-commutative geometry [12–14], and Hořava-Lifshitz gravity [15] [see [16, 17] for reviews]. A striking insight originating from this comparison is that, at microscopic distances, $d_s = 2$ rather universally. The interpretation of d_s as the dimension of a theories momentum space, forwarded in Amelino-Camelia et al. [18], then suggests that the dimensional reduction of the momentum space may be a universal feature of any viable theory of quantum gravity.

Following the suggestion [19]¹, a refined picture of quantum geometry could use the (anomalous) scaling dimension associated with geometric operators, comprising, e.g., spacetime volumes, integrated spacetime curvatures, and geodesic distances. Within the asymptotic safety program [22, 23], also reviewed in Percacci [24], Litim [25], Reuter and Saueressig [26], Ashtekar et al. [27], and Eichhorn [28], these quantities have been studied based on the composite operator formalism [19, 29–32]. This formalism allows to determine the anomalous scaling dimension of geometric operators based on an approximation of the quantum-corrected graviton propagator². For the Reuter fixed point in four dimensions the quantum corrections to the scaling of four-volumes $V_{d=4} \sim L^{4-\gamma_0}$ were determined in Pagani and Reuter [19]. The result $\gamma_0 = 3.986$ lent itself to the interpretation that “spacetime could be much more empty than expected.” Recently, Houthoff et al. [32] generalized this computation by determining the anomalous scaling dimensions associated with an infinite class of geometric operators

$$\mathcal{O}_n \equiv \int d^d x \sqrt{g} R^n, \quad n = 0, 1, 2, \dots \in \mathbb{N} \quad (1)$$

where R denotes the Ricci scalar constructed from $g_{\mu\nu}$. While it was possible to extract analytic expressions for all γ_n , it also became apparent that the single-operator approximation underlying the computation comes with systematic uncertainties. In parallel, the anomalous scaling properties of subvolumes and geodesic distances resulting from the renormalization group fixed points underlying Stelle gravity and Weyl gravity have recently been computed in Becker et al. [31]. In combination, the results show that the scaling of geometric quantities carries information about the renormalization group fixed point providing the high-energy completion of the theory.

The purpose of present work is two-fold: Firstly, we extend the analysis [32] beyond the single-operator approximation and compute the complete matrix of anomalous dimensions associated with the class (1). This information allows to access the spectrum of the scaling matrix. We expect that the data linked to the scaling dimensions of the geometrical operators gives a refined characterization of the quantum spacetime underlying the Reuter fixed point. Our results are closely related but complementary to the ones obtained from solving the Wetterich equation [34–37] for effective average actions of $f(R)$ -type [38–60]. The comparison between the two complementary

computations indicates that one indeed needs to go beyond the single-operator approximation in order to reconcile the results. Secondly, our work gives information on the gauge-dependence of the anomalous dimensions associated with the operators (1). In this light, the value $\gamma_0 = 3.986$ found in Pagani and Reuter [19] may be rather extreme and quantum corrections to the scaling of volumes could be less drastic.

The rest of this work is organized as follows. Section 2 introduces the composite operator formalism and the propagators entering in our computation. The generating functional determining the matrix of anomalous dimensions is computed in section 3. The link to the stability matrix governing the gravitational renormalization group flow in the vicinity of the Reuter fixed point is made in section 4.1 and the spectral properties of the matrix are analyzed in section 4.2. Section 5 contains our concluding remarks and comments on the possibility of developing a geometric picture of Asymptotic Safety from random geometry. The technical details underlying our computation have been relegated to three appendices: **Appendix A** reviews the technical background for evaluating operator traces using the early-time expansion of the heat-kernel, **Appendix B** derives the beta functions governing the renormalization group flow of gravity in the Einstein-Hilbert truncation employing geometric gauge [61, 62], and **Appendix C** lists the two-point functions entering into the computation.

2. COMPUTATIONAL FRAMEWORK AND SETUP

Functional renormalization group methods provide a powerful tool for investigating the appearance of quantum scale invariance and its phenomenological consequences [63]. In particular, the Wetterich equation [34–37],

$$k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} k \partial_k \mathcal{R}_k \right], \quad (2)$$

plays a key role in studying the renormalization group (RG) flow of gravity and gravity-matter systems based on explicit computations. It realizes the idea of Wilson’s modern viewpoint on renormalization in the sense that it captures the RG flow of a theory generated by integrating out quantum fluctuations shell-by-shell in momentum space. Concretely, Equation (2) encodes the change of the effective average action Γ_k when integrating out quantum fluctuations with momentum p close to the coarse graining scale k . The flow of Γ_k is then sourced by the right-hand side where $\Gamma_k^{(2)}$ denotes the second variation of Γ_k with respect to the fluctuation fields, the regulator $\mathcal{R}_k$ provides a k -dependent mass term for quantum fluctuations with momentum $p^2 \lesssim k^2$, and Tr includes a sum over all fluctuation fields and an integral over loop-momenta. Lowering k “unsuppresses” further fluctuations which are then integrated out and change the value of the effective couplings contained in Γ_k . For later convenience, we then also introduce the “RG-time” $t \equiv \ln(k/k_0)$ with k_0 an arbitrary reference scale.

In practice, the Wetterich equation allows to extract non-perturbative information about a theories RG flow by restricting

¹For related ideas advocated in the context of two-dimensional gravity (see [20, 21]).

²Recently, the formalism has been generalized to the computation of operator product expansions [33].

Γ_k to a subset of all possible interaction monomials and subsequently solving Equation (2) on this subspace. For gravity and gravity-matter systems such computations get technically involved rather quickly. Thus, it is interesting to have an alternative equation for studying the scaling properties of sets of operators $\mathcal{O}_n$, $n = 1, \dots, N$, which are not included in Γ_k . Within the effective average action framework such an equation is provided by the composite operator equation [19, 64–66]. As a starting point, the operators $\mathcal{O}_n$ are promoted to scale-dependent quantities by multiplying with a k -dependent matrix $Z_{nm}(k)$

$$\mathcal{O}_n(k) \equiv \sum_m^N Z_{nm}(k) \mathcal{O}_m. \quad (3)$$

The analogy of Z_{nm} to a wave-function renormalization then suggests to introduce the matrix of anomalous dimensions γ whose components are given by

$$\gamma_{nm} \equiv (Z^{-1} \partial_t Z)_{nm}. \quad (4)$$

Following the derivation [19], the γ_{nm} can be computed from the composite operator equation

$$\sum_{m=1}^N \gamma_{nm} \mathcal{O}_m = -\frac{1}{2} \text{Tr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \mathcal{O}_n^{(2)} \left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right], \quad (5)$$

where $\mathcal{O}_n^{(2)}$ denotes the second functional derivative of $\mathcal{O}_n$ with respect to the fluctuation fields. For the geometric operators (1) the evaluation of γ has so far focused on the diagonal matrix elements γ_{nn} [c.f. [19, 32]]. The goal of the present work is to extend this analysis and, for the first time, study the eigenvalues of γ_{ij} associated with the operators (1).

3. COMPUTING THE MATRIX OF ANOMALOUS DIMENSIONS

The computation of γ_{nm} requires two inputs. First, one needs to specify the set of operators $\mathcal{O}_n$. In the present work, these will be given by the geometric operators (1). Secondly, one needs to specify the gravitational propagators $\Gamma_k^{(2)}$. These will be derived from Γ_k approximated by the Euclidean Einstein-Hilbert (EH) action

$$\Gamma_k^{\text{EH}}[g] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} (2\Lambda_k - R) \quad (6)$$

supplemented by a suitable choice for the gauge-fixing action (54). In practice, we obtain $\Gamma_k^{(2)}$ from the background field method, performing a linear split of the spacetime metric $g_{\mu\nu}$ into a background metric $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (7)$$

In order to simplify the subsequent computation, we then chose the background metric as the metric on the d -sphere, so that the background curvature satisfies

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{\bar{R}}{d(d-1)} [\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}], \quad \bar{R}_{\mu\nu} = \frac{\bar{R}}{d} \bar{g}_{\mu\nu}, \quad \bar{D}_\mu \bar{R} = 0. \quad (8)$$

Moreover, we carry out a transverse-traceless (TT) decomposition of the metric fluctuations [67]

$$h_{\mu\nu} = h_{\mu\nu}^T + \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu + \left(\bar{D}_\mu \bar{D}_\nu - \frac{1}{d} \bar{g}_{\mu\nu} \bar{D}^2 \right) \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h, \quad (9)$$

where the component fields are subject to the differential constraints

$$\bar{g}^{\mu\nu} h_{\mu\nu}^T = 0, \quad \bar{D}^\mu h_{\mu\nu}^T = 0, \quad \bar{D}_\mu \xi^\mu = 0, \quad \bar{g}^{\mu\nu} h_{\mu\nu} = h. \quad (10)$$

The Jacobians associated with the decomposition (9) are taken into account by a subsequent field redefinition

$$\sqrt{2} \left[\Delta - \frac{1}{d} \bar{R} \right]^{1/2} \xi_\mu \mapsto \xi_\mu, \quad \left[\frac{d-1}{d} \Delta^2 - \frac{1}{d} \bar{R} \Delta \right]^{1/2} \sigma \mapsto \sigma, \quad (11)$$

and it is understood that in the sequel all propagators and the matrix elements $\mathcal{O}_i^{(2)}$ are the ones associated with the rescaled fields. In combination with the background (8), this decomposition ensures that the differential operators appearing within the trace combine into Laplacians $\Delta \equiv -\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$ constructed from the background metric [61].

We then specify the gauge-fixing introduced in Equation (54) to geometric gauge, setting $\rho = 0$ and subsequently evoking the Landau limit $\alpha \rightarrow 0$. Substituting the general form of the matrix elements listed in Table 2 into the right-hand side of (5) and tracing the α -dependence one finds that the contributions of the transverse vector fluctuations ξ_μ and the scalar σ drop out from the composite operator equation. As a consequence, the anomalous dimensions are only sourced by the transverse-traceless and conformal fluctuations. The relevant matrix elements are then readily taken from Table 2. They read

$$\begin{aligned} \mathcal{O}_n^{(2)}|_{h^T h^T} &= -\frac{1}{2} \bar{R}^{n-1} \left[n\Delta - \left(\frac{2n(d-2)}{d(d-1)} - 1 \right) \bar{R} \right], \\ \mathcal{O}_n^{(2)}|_{hh} &= \frac{n(n-1)(d-1)^2}{d^2} \bar{R}^{n-2} \Delta^2 \\ &\quad + \frac{n(d^2 - (4n-1)(d-1) - 1)}{2d^2} \bar{R}^{n-1} \Delta \\ &\quad + \left(\frac{d-2}{4d} - \frac{n(d-n-1)}{d^2} \right) \bar{R}^n, \end{aligned} \quad (12)$$

together with

$$\begin{aligned} \Gamma_k^{(2)}|_{h^T h^T} &= \frac{1}{32\pi G_k} [\Delta - 2\Lambda_k + C_T \bar{R}], \\ \Gamma_k^{(2)}|_{hh} &= -\frac{(d-1)(d-2)}{32\pi G_k d^2} \left[\Delta - \frac{d}{d-1} \Lambda_k + C_S \bar{R} \right], \end{aligned} \quad (13)$$

where

$$C_T \equiv \frac{d^2 - 3d + 4}{d(d-1)}, \quad C_S \equiv \frac{d-4}{2(d-1)}. \quad (14)$$

Finally, the matrix entries for the regulator $\mathcal{R}_k$ are obtained from the substitution rule (59), which corresponds to a Type I regularization scheme in the nomenclature introduced in [40]

$$\mathcal{R}_k|_{h^T h^T} = \frac{1}{32\pi G_k} R_k, \quad \mathcal{R}_k|_{hh} = -\frac{(d-1)(d-2)}{32\pi G_k d^2} R_k. \quad (15)$$

Here $R_k(\Delta) = k^2 r(\Delta/k^2)$ is a scalar regulator function which later on will be specified to the Litim regulator (51).

Substituting the expressions (12)–(15) into the composite operator Equation (5) then yields

$$\sum_{m=0}^N \gamma_{nm} \mathcal{O}_m = -\frac{1}{2} \left(\text{Tr}_T [W_T(n; \Delta)] + \text{Tr}_S [W_S(n; \Delta)] \right), \quad (16)$$

where the subscripts T and S indicate that the trace is over transverse-traceless (T) and scalar (S) fluctuations, respectively. The explicit form of the operator-valued functions W_T and W_S is

$$\begin{aligned} W_T(n; \Delta) &\equiv 16\pi G_k [P_k - 2\Lambda_k + C_T \bar{R}]^{-2} \bar{R}^{n-1} \\ &\quad \left[-n\Delta + \left(\frac{2n(d-2)}{d(d-1)} - 1 \right) \bar{R} \right] (\partial_t R_k - \eta_N R_k), \\ W_S(n; \Delta) &\equiv -\frac{16\pi G_k}{(d-2)} \left[P_k - \frac{d}{d-1} \Lambda_k + C_S \bar{R} \right]^{-2} \bar{R}^{n-2} \\ &\quad \left[2n(n-1)(d-1) \Delta^2 + n(d+2-4n) \bar{R} \Delta \right. \\ &\quad \left. + \frac{d^2 - 2d(2n+1) + 4n(n+1)}{2(d-1)} \bar{R}^2 \right] (\partial_t R_k - \eta_N R_k). \end{aligned} \quad (17)$$

Equation (16) should then be read as a series expansion in $\bar{R}$ at the origin where the matrix entries γ_{nm} are obtained by matching powers of $\bar{R}$ on the left- and right-hand side. We then define the infinite family of generating functionals, $\Gamma_n(\bar{R})$ with $n \geq 0 \in \mathbb{N}$, via

$$\int d^d x \sqrt{g} \Gamma_n(\bar{R}) \equiv -\frac{1}{2} \left(\text{Tr}_T [W_T(n; \Delta)] + \text{Tr}_S [W_S(n; \Delta)] \right). \quad (18)$$

The structure of the traces appearing in the definition (18) ensures that $\Gamma_n(\bar{R})$ is regular at $\bar{R} = 0$ and can be expressed as a Taylor series expansion. Equating the left-hand sides of Equations (16) and (18) one then has

$$\gamma_{nm} = \frac{1}{2\pi i} \oint_C \bar{R}^{-(m+1)} \Gamma_n(\bar{R}), \quad n, m \geq 0, \quad (19)$$

where C is an infinitesimal curve encircling the origin with counterclockwise orientation.

Before delving into the explicit evaluation of the traces, the following structural remark is in order. Inspecting (17), one observes that the right-hand side associated with the n th row starts at order $\bar{R}^{n-2}$ since $\mathcal{O}_n^{(2)}$ will always contribute at least $n-2$

powers of the background curvature. This entails that the matrix of anomalous dimensions has the following triangular form

$$\gamma = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{03} & \gamma_{04} & \gamma_{05} & \gamma_{06} & \cdots \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \cdots \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} & \cdots \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & \gamma_{35} & \gamma_{36} & \cdots \\ 0 & 0 & \gamma_{42} & \gamma_{43} & \gamma_{44} & \gamma_{45} & \gamma_{46} & \cdots \\ 0 & 0 & 0 & \gamma_{53} & \gamma_{54} & \gamma_{55} & \gamma_{56} & \cdots \end{bmatrix}. \quad (20)$$

This structure originates solely from the properties of the operators $\mathcal{O}_n$ and is independent of the gauge choice or regularization procedure.

The explicit values of the matrix entries (16) are readily computed employing the heat-kernel techniques reviewed in **Appendix A**. In practice, we will truncated the heat-kernel expansion at order R^2 , setting the coefficients $a_n, n \geq 3$ to zero. This is in the spirit of the “paramagnetic approximation” suggested in [68], that the curvature terms relevant for asymptotic safety originate from the curvature terms contained in the propagators. For the matrix entries γ_{nm} this entails that all entries on the diagonal and below (marked in black) are computed exactly while contributions to the terms above the diagonal (marked in blue) will receive additional contributions from higher-orders in the heat-kernel. In particular all entries γ_{nm} with $m \geq n+3$ are generated solely from expanding the curvature terms proportional to C_T and C_S in the transverse-traceless and scalar propagators.

Evaluating (16) based on these approximations then results in an infinite family of generating functionals $\Gamma_n(\bar{R}), n \geq 0 \in \mathbb{N}$:

$$\begin{aligned} \Gamma_n(\bar{R}) &= \frac{16\pi g}{(4\pi)^{d/2}} \left[c_1^T q_{d/2+1}^2(w_T^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right)^{-1} + c_2^T q_{d/2}^2(w_T^{\bar{R}}) \right. \\ &\quad + c_3^T q_{d/2-1}^2(w_T^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right) + c_4^T q_{d/2-2}^2(w_T^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right)^2 \\ &\quad + c_1^S q_{d/2+2}^2(w_S^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right)^{-2} + c_2^S q_{d/2+1}^2(w_S^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right)^{-1} \\ &\quad + c_3^S q_{d/2}^2(w_S^{\bar{R}}) + c_4^S q_{d/2-1}^2(w_S^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right) \\ &\quad \left. + c_5^S q_{d/2-2}^2(w_S^{\bar{R}}) \left(\frac{\bar{R}}{k^2} \right)^2 \right]. \end{aligned} \quad (21)$$

Here we introduced the dimensionless couplings

$$g_k = k^{d-2} G_k, \quad \lambda_k = \Lambda_k k^{-2}, \quad (22)$$

and the anomalous dimension of Newton’s coupling $\eta_N \equiv (G_k)^{-1} \partial_t G_k$. The threshold functions $q_n^p(w)$ are defined in Equation (46) and their arguments in the transverse-traceless and scalar sector are

$$\begin{aligned} w_T &= -2\lambda, \quad w_S \equiv -\frac{d}{d-1} \lambda, \\ w_T^{\bar{R}} &= w_T + C_T \bar{R}/k^2, \quad w_S^{\bar{R}} \equiv w_S + C_S \bar{R}/k^2. \end{aligned} \quad (23)$$

TABLE 1 | Heat-kernel coefficients a_n^i for scalars (S), transverse vectors (TV), and transverse-traceless symmetric tensors (T) on a background d -sphere [69].

	S	TV	T
a_0^i	1	$d - 1$	$\frac{(d-2)(d+1)}{2}$
a_1^i	$\frac{1}{6}$	$\frac{(d+2)(d-3)}{6d}$	$\frac{(d+1)(d+2)(d-5+3\delta_{d,2})}{12(d-1)}$
a_2^i	$\frac{5d^2-7d+6}{360d(d-1)}$	—	$\frac{(d+1)(5d^4-22d^3-83d^2-392d-228+1440\delta_{d,2}+3240\delta_{d,4})}{720d(d-1)^2}$

The terms proportional to $\delta_{d,2}$ and $\delta_{d,4}$ are linked to zero modes of the decomposition (9) on the 2- and 4-sphere. The dash — indicates that the corresponding coefficient is not entering into the present computation.

The coefficients c_k^i depend on d and n . In the tensor sector they are given by

$$\begin{aligned} c_1^T &= \frac{1}{2} n d a_0^T, \\ c_2^T &= \frac{1}{2} n(d-2) a_1^T - \left(\frac{2n(d-2)}{d(d-1)} - 1 \right) a_0^T, \\ c_3^T &= \frac{1}{2} n(d-4) a_2^T - \left(\frac{2n(d-2)}{d(d-1)} - 1 \right) a_1^T, \\ c_4^T &= - \left(\frac{2n(d-2)}{d(d-1)} - 1 \right) a_2^T. \end{aligned} \quad (24)$$

Their counterparts in the scalar sector read

$$\begin{aligned} c_1^S &= \frac{n(n-1)(d-1)d(d+2)}{2(d-2)} a_0^S, \\ c_2^S &= \frac{1}{2} n(n-1)d(d-1) a_1^S + \frac{nd(d+2-4n)}{2(d-2)} a_0^S, \\ c_3^S &= \frac{1}{2} n(n-1)(d-1)(d-4) a_2^S + \frac{1}{2} n(d+2-4n) a_1^S \\ &\quad + \frac{d^2-2d(2n+1)+4n(n+1)}{2(d-1)(d-2)} a_0^S, \\ c_4^S &= \frac{n(d-4)(d+2-4n)}{2(d-2)} a_2^S + \frac{d^2-2d(2n+1)+4n(n+1)}{2(d-1)(d-2)} a_1^S, \\ c_5^S &= \frac{d^2-2d(2n+1)+4n(n+1)}{2(d-1)(d-2)} a_2^S. \end{aligned} \quad (25)$$

Finally, the a_n^i are the heat-kernel coefficients listed in **Table 1**.

Evaluating (19) for the explicit generating functional (21) then yields the entries of the matrix γ . For instance, the two lines of entries below the diagonal, $\gamma_{n,n-2}$, $n \geq 2$, and $\gamma_{n,n-1}$, $n \geq 1$, obtained in this way are

$$\begin{aligned} \gamma_{n,n-2} &= \frac{16\pi g}{(4\pi)^{d/2}} \frac{n(n-1)(d-1)d(d+2)}{2(d-2)} k^4 q_{d/2+2}^2(w_S), \\ \gamma_{n,n-1} &= \frac{16\pi g}{(4\pi)^{d/2}} n d k^2 \left[\frac{1}{4} (d-2)(d+1) q_{d/2+1}^2(w_T) \right. \\ &\quad + \frac{1}{12} (n-1)(d-1) q_{d/2+1}^2(w_S) \\ &\quad + \frac{d+2-4n}{2(d-2)} q_{d/2+1}^2(w_S) \\ &\quad \left. - \frac{(n-1)(d-4)(d+2)}{2(d-2)} q_{d/2+2}^3(w_S) \right]. \end{aligned} \quad (26)$$

Equation (21) together with the relation (19) constitutes the main result of this work. They give completely analytic expressions for all entries of the anomalous dimension matrix γ .

At this stage, a few remarks are in order.

- (1) The entries of the anomalous dimension matrix carry a specific k -dependence: $\gamma_{nm} \propto (k^2)^{n-m}$. This can be understood by noticing that the matrix γ acts on operators $\mathcal{O}_m$ with different canonical mass dimensions. The k -dependence then guarantees that the eigenvalues of γ are independent of k .
- (2) The entries $\gamma_{n,n-2}$ are solely generated from the scalar contributions, i.e., the transverse-traceless fluctuations do not enter into these matrix elements. Technically, this feature is associated with the Hessians $\mathcal{O}_n^{(2)}$ (cf. **Table 2**): the matrix elements in the scalar sector start at $\bar{R}^{n-2}$ while the transverse-traceless sector starts at $\bar{R}^{n-1}$.
- (3) Notably, $d = 4$ is special. In this case the entries above the diagonal, γ_{nm} with $m \geq n+3$ are generated from the transverse-traceless sector only. All contributions from the scalar sector are proportional to at least one power of C_S and thus vanish if $d = 4$.
- (4) The matrix γ is a function of the (dimensionless) couplings entering the Einstein-Hilbert action. Thus γ assigns a set of anomalous dimensions to every point in the g - λ -plane. Since γ is proportional to g , the magnitude of the anomalous dimensions becomes small if $g \ll 1$. In particular, γ vanishes at the Gaussian fixed point $g_* = \lambda_* = 0$ where one recovers the classical scaling of the geometric operators.

4. SCALING ANALYSIS FOR THE REUTER FIXED POINT

Starting from the general result (19), we now proceed and discuss its implications for the quantum geometry associated with Asymptotic Safety.

4.1. Relating the Scaling of Geometric Operators and the RG Flow

By construction, the matrix γ assigns anomalous scaling dimensions to any point in the g - λ plane. In order to characterize the quantum geometry related to Asymptotic Safety, we study the properties of this matrix at the Reuter fixed point found in **Appendix B** [cf. Equation (64)]

$$\begin{aligned} d=3: \quad & g_* = 0.198, \quad \lambda_* = 0.042, \quad \lambda_* g_*^2 = 1.65 \times 10^{-3}, \\ d=4: \quad & g_* = 0.911, \quad \lambda_* = 0.160, \quad \lambda_* g_* = 0.146. \end{aligned} \quad (27)$$

From the definition of the beta function $\partial_t u_n = \beta_{u_n}(u_i)$ and the fact that at a fixed point $\beta_{u_n}(u_i^*) = 0$, it follows that the properties of the RG flow in the vicinity of the fixed point are encoded in the stability matrix $\mathbf{B} = [B_{nm}]$,

$$\partial_t u_n(k) = \sum_m B_{nm}(u_m(k) - u_m^*), \quad B_{nm} \equiv \left. \frac{\partial \beta_{u_n}}{\partial u_m} \right|_{u=u^*}. \quad (28)$$

Let us denote the eigenvalues of $\mathbf{B}$ by λ_n so that $\text{spec}(\mathbf{B}) = \{\lambda_n\}$. Equation (28) then entails that eigendirections corresponding

to eigenvalues with a negative (positive) real part attract (repel) the RG flow when k is increased, i.e., they correspond to UV-relevant (UV-irrelevant) directions. The number of UV-relevant directions then gives the number of free parameters which are not fixed by the asymptotic safety condition: along these directions the RG flow automatically approaches the Reuter fixed point as $k \rightarrow \infty$.

Formally, one can then derive a relation between γ and the stability matrix $\mathbf{B}$ [32, 70],

$$B_{nm} = -d_n \delta_{nm} + \gamma_{nm}, \quad (29)$$

where $d_n = d - 2n$ is the canonical scaling dimension of the operator $\mathcal{O}_n$. This relation is remarkable in the following sense: The construction of the (approximate) fixed point solution (27) is based on the two operators $\mathcal{O}_0$ and $\mathcal{O}_1$, comprising the Einstein-Hilbert truncation. The relation (29) then shows that *the matrix of anomalous dimensions carries information about the stability properties of the Reuter fixed point beyond the set of operators which are considered when solving the Wetterich equation to locate the fixed point*. We illustrate this idea by studying the spectrum of B_{nm} obtained at the fixed points (27). Before embarking on this discussion, the following cautious remark is in order though. While the composite operator formalism may allow to obtain information on the stability properties of a fixed point beyond the approximation used for the propagators, it is also conceivable that the formalism becomes unreliable for eigenvalues λ_n with $n \geq N_{\max}^3$. Heuristically, this is suggested by the following argument: when studying fixed point solutions in the $f(R)$ -approximation the propagators include powers of $\bar{R}$ beyond the linear terms captured by the Einstein-Hilbert action. These terms give rise to additional contributions in the generating functional (19) which may become increasingly important in assessing the spectrum of $\mathbf{B}$ for eigenvalues with increasing numbers of n . This picture is also suggested by our results in section 4.2⁴.

This said, we now investigate the properties of the stability matrix (29). Here we will resort to the following frameworks:

- I** The spectrum of $\mathbf{B}$ generated by the full generating functional (21) including the contribution of zero-modes in the heat-kernel for $d = 4$.
- II** In the conformally reduced approximation [71]. In this case, the contribution of the tensor fluctuations is set to zero by

³Most conservatively, one may expect that the composite operator formalism allows a qualitatively reliable determination of the stability properties of operators containing two additional spacetime-derivatives on top of the terms included in the propagators. This picture is readily confirmed by comparing the spectrum of $\mathbf{B}$ obtained from the composite operator equation with the solution of the Wetterich equation for actions of $f(R)$ -type.

⁴A second effect which could lead to a stabilization of the spectrum of $\mathbf{B}$ at the Reuter fixed points could come from improving the truncation of the early-time expansion of the heat-kernel. There are two reasons to expect that these contributions will not play a relevant role though. Firstly, $f(R)$ -type solutions of the Wetterich equation [38–40, 58], where the spectrum of $\mathbf{B}$ has been shown to be stable under the inclusion of further operators, essentially use the same truncation of the heat-kernel. Secondly, the structure of the heat-kernel expansion shows that the higher-order terms are highly suppressed compared to the ones included in our computation [40].

hand, so that γ contains the contribution from the scalar trace in (16) only.

The latter choice is motivated by the observation that this framework gives rise to the $\text{spec}(\mathbf{B})$ which is the most robust under increasing the size of the matrix $\mathbf{B}$. Clearly, one could easily envision other approximations which could be applied to the general result (21). Examples include the exclusion of the zero-mode terms appearing in $d = 4$ or the “sparse approximation” where only two lines above and below the diagonal are non-trivial, i.e., the entries in the upper-triangular sector which are solely created by expanding the curvature terms contained in the gravitational propagators are eliminated. In order to understand the working (and limitations) of the conformal operator formalism, the frameworks **I** and **II** are sufficient though. We checked by explicit computations that the exclusion of zero-modes or evaluating the spectrum of $\mathbf{B}$ in the sparse approximation leads to the same qualitative picture.

4.2. Spectral Properties of the Stability Matrix

We first give the diagonal entries γ_{nn} within framework **I**. This corresponds to the “single-operator approximation” of the composite operator formalism employed in Pagani and Reuter [19], and Houthoff et al. [32]. At the fixed points (27) one finds

$$\begin{aligned} d = 3: \quad \gamma_{nn}^* &= 0.653 - 0.872n - 0.029n^2, \\ d = 4: \quad \gamma_{nn}^* &= 2.299 - 3.765n. \end{aligned} \quad (30)$$

These relations exhibit two remarkable features. Firstly, the structure of $\mathcal{O}_n^{(2)}$ (cf. **Table 2**) entails that the entries of γ are second order polynomials in n . It is then remarkable that the diagonal entries essentially follow a linear scaling law up to $n \approx 30$ ($d = 3$) or even exactly ($d = 4$). Secondly, Equation (30) entails that the diagonal entries of the stability matrix $\mathbf{B}$ are *always negative*. Thus the single-operator approximation predicts that all eigendirections of the Reuter fixed point in the $f(R)$ -space are UV-attractive. It was noted in Houthoff et al. [32] that this is actually in tension with results obtained from solving the Wetterich equation on the same space. On this basis, it is expected that the off-diagonal entries in γ play a crucial role in determining the spectrum of $\mathbf{B}$.

We now discuss the properties of the stability matrix evaluated at the Reuter fixed points (27) generated from the functional (21). In practice, we truncate $\mathbf{B}$ to square-matrices of size N choosing $N = 100$ if not stated otherwise. The eigenvalues $\lambda_n \in \text{spec}(\mathbf{B})$ satisfying

$$\mathbf{B}V_n = \lambda_n V_n, \quad n = 1, \dots, N, \quad (31)$$

with V_n denoting the right-eigenvectors of $\mathbf{B}$, are readily found numerically. Since $\mathbf{B}$ is not symmetric there is no a priori reason that the λ_n are real or that the left- and right-eigenvectors of $\mathbf{B}$ agree.

The structure of $\mathbf{B}$ then entails that there is always one eigenvalue which is independent of the matrix size. For

TABLE 2 | Components of the Hessians entering the right-hand side of the composite operator equation (5) and the Wetterich equation evaluated for the Einstein-Hilbert truncation.

Operator	Value of the matrix element
$\mathcal{O}_n^{(2)} _{h^T h^T}$	$-\frac{1}{2} \bar{R}^{n-1} \left[n\Delta - \left(\frac{2n(d-2)}{d(d-1)} - 1 \right) \bar{R} \right]$
$\mathcal{O}_n^{(2)} _{\xi\xi}$	$\frac{2n-d}{2d} \bar{R}^n$
$\mathcal{O}_n^{(2)} _{hh}$	$\frac{n(n-1)(d-1)^2}{d^2} \bar{R}^{n-2} \Delta^2 + \frac{n(d^2-(4n-1)(d-1)-1)}{2d^2} \bar{R}^{n-1} \Delta + \left(\frac{d-2}{4d} - \frac{n(d-n-1)}{d^2} \right) \bar{R}^n$
$\mathcal{O}_n^{(2)} _{\sigma\sigma}$	$\frac{n(n-1)(d-1)}{d} \bar{R}^{n-2} \Delta^2 + \frac{n(d-2n)}{2d} \bar{R}^{n-1} \Delta - \frac{d-2n}{2d} \bar{R}^n$
$\mathcal{O}_n^{(2)} _{\sigma h}$	$\left[\frac{n(n-1)(d-1)}{d} \bar{R}^{n-2} \Delta + \frac{n(d-2n)}{2d} \bar{R}^{n-1} \right] \left[\frac{d-1}{d} \Delta^2 - \frac{1}{d} \bar{R} \Delta \right]^{1/2}$
$\Gamma_k^{(2)} _{h^T h^T}$	$\frac{1}{32\pi G_k} [\Delta - 2\Lambda_k + C_T \bar{R}]$
$\Gamma_k^{(2)} _{\xi\xi}$	$\frac{1}{32\pi G_k \alpha} [\Delta - \frac{1}{d} \bar{R} + \alpha (C_V \bar{R} - 2\Lambda_k)]$
$\Gamma_k^{(2)} _{hh}$	$\frac{1}{32\pi G_k \alpha d^2} [2\rho^2 \Delta - \alpha(d-1)(d-2) (\Delta - \frac{d}{d-1} \Lambda_k + C_S \bar{R})]$
$\Gamma_k^{(2)} _{\sigma\sigma}$	$\frac{1}{32\pi G_k \alpha d} [2(d-1)\Delta - 2\bar{R} - \alpha((d-2)\Delta + 2d\Lambda_k + C_S \bar{R})]$
$\Gamma_k^{(2)} _{\sigma h}$	$\frac{2\rho+(d-2)\alpha}{32\pi G_k \alpha d} [\Delta (\frac{d-1}{d} \Delta - \frac{1}{d} \bar{R})]^{1/2}$
$\Gamma_k^{(2)} _{\hat{C}^T \hat{C}^T}$	$[\Delta - \frac{1}{d} \bar{R}]$
$\Gamma_k^{(2)} _{\bar{\eta}\eta}$	$\frac{2}{d} [(d-1-\rho)\Delta - \bar{R}]$
$\mathcal{R}_k _{h^T h^T}$	$\frac{1}{32\pi G_k} R_k$
$\mathcal{R}_k _{\xi\xi}$	$\frac{1}{32\pi G_k \alpha} R_k$
$\mathcal{R}_k _{hh}$	$\frac{1}{32\pi G_k \alpha d^2} [2\rho^2 - \alpha(d-1)(d-2)] R_k$
$\mathcal{R}_k _{\sigma\sigma}$	$\frac{1}{32\pi G_k \alpha d} [2(d-1) - \alpha(d-2)] R_k$
$\mathcal{R}_k _{\sigma h}$	$\frac{2\rho+(d-2)\alpha}{32\pi G_k \alpha d} \left\{ [P_k (\frac{d-1}{d} P_k - \frac{1}{d} \bar{R})]^{1/2} - [\Delta (\frac{d-1}{d} \Delta - \frac{1}{d} \bar{R})]^{1/2} \right\}$
$\mathcal{R}_k _{\hat{C}^T \hat{C}^T}$	R_k
$\mathcal{R}_k _{\bar{\eta}\eta}$	$\frac{2}{d} [d-1-\rho] R_k$

The fluctuations are expressed by the component fields (9) and (56) followed by the field redefinitions (11) and (57). The matrix elements are labeled by the fluctuation fields, i.e., $\mathcal{O}_n^{(2)}|_{h^T h^T}$ results from taking two functional derivatives of $\mathcal{O}_n$ with respect to h^T . The off-diagonal terms are symmetric, e.g., $\mathcal{O}_n^{(2)}|_{h\sigma} = \mathcal{O}_n^{(2)}|_{\sigma h}$.

framework I its value is given by

$$\begin{aligned} d=3: & \quad \lambda_1^I = -2.347, \\ d=4: & \quad \lambda_1^I = -1.701. \end{aligned} \quad (32)$$

The eigenvector V_1 associated with these eigenvalues is aligned with the volume operator $\mathcal{O}_0$ entailing that $V = \int d^d x \sqrt{g}$ is actually an eigenoperator of Equation (16). In the conformally reduced approximation (framework II) in $d=3$, the independence of λ_n on the matrix size N extends to a second eigenvalue

$$d=3: \quad \lambda_1^{II} = -2.828, \quad \lambda_2^{II} = -0.967. \quad (33)$$

The normalized eigenvector associated with λ_1^{II} is again given by the volume operator $\mathcal{O}_0$ while the one associated with λ_2^{II} is almost aligned with $\mathcal{O}_1$, i.e., $V_2 = (0.07, 0.995, \dots)^T$ for $N=5$.

The properties of $\text{spec}(\mathbf{B})$ beyond these universal eigenvalues obtained from the framework I in $d=4$ and $d=3$ as well as in the conformally reduced approximation in $d=3$ (framework II) are shown in Figures 1–3, respectively. The left diagrams show the real part, $\text{Re}(\lambda_n)$ of the stability matrices of size $N=25$ (left line, green dots), $N=50$ (middle line, orange dots), and $N=100$ (right line, blue dots). The lines

clearly illustrate that increasing N adds additional eigenvalues coming with both increasingly positive and increasingly negative real parts. This feature is shared by all frameworks discussed above. The middle diagrams illustrate the location of $\text{spec}(\mathbf{B})$ for $N=100$ in the complex plane. While the patterns are quite distinct, they share the existence of nodes where complex eigenvalues are created which then move out into the complex plane along distinguished lines. The right diagrams trace the first two negative eigenvalues as a function of the matrix size N . In all cases, the structure of $\mathbf{B}$ implies that the first eigenvalue is independent of N while the other parts of the spectrum exhibit an N -dependence. As illustrated in Figures 1, 2, the eigenvalues λ_n , $n \geq 2$ follow intriguing periodicity patterns. The average over the second and third eigenvalues found in the matrices of size up to $N=100$ (for $\bar{\lambda}_2$) and $N=20$ (for $\bar{\lambda}_3$, excluding values where a complex eigenvalue has appeared in the interval spanned by λ_1 and λ_3) are⁵

$$\begin{aligned} d=3: & \quad \lambda_2^I = -2.35 \pm 0.08 & \lambda_3^I = -0.61 \pm 0.40 \\ d=4: & \quad \lambda_2^I = -2.86 \pm 0.61 & \lambda_3^I = -6.36 \pm 2.04 \end{aligned} \quad (34)$$

Carefully analyzing the N -dependence of $\text{spec}(\mathbf{B})$ reveals that there is a close relation between the distribution of eigenvalues in the complex plane (middle diagrams) and the oscillations of λ_2 visible in the left diagrams: the oscillations are linked to the appearance of new complex pairs of eigenvalues. Focusing on the four-dimensional case where this feature is most prominent, one finds that singling out the values of λ_2 just before the occurrence of the new pair of complex eigenvalues in $\text{spec}(\mathbf{B})$ essentially selects the $\lambda_2(N)$ constituting the maxima in the oscillations. The resulting subset of eigenvalues is displayed in the inset shown in Figure 1 and is significantly more stable than the full set. The statistical analysis shows that in this case

$$d=4: \quad \bar{\lambda}_2^{\text{subset}} = -2.61 \pm 0.39, \quad (35)$$

so that the fluctuations are reduced by a factor two as compared to the full set 34.

At this stage, it is interesting to compare the averages 34 to the eigenvalue spectrum obtained from the smallest non-trivial stability matrix $\mathbf{B}$ with size $N=3$:

$$\begin{aligned} d=3: & \quad \lambda_2^I = -2.35, & \lambda_2^I = -1.26, & \lambda_3^I = -0.20, \\ d=4: & \quad \lambda_2^I = -1.70, & \lambda_2^I = -2.74, & \lambda_3^I = -5.95. \end{aligned} \quad (36)$$

Thus we conclude that small values of N already give a good estimate of the (averaged) spectrum of $\mathbf{B}$.

We close this section with a general remark on the structure of $\text{spec}(\mathbf{B})$. The stability matrix is not tied to the Reuter fixed point but well-defined on the entire g - λ -plane: the generating functional (21) assigns an infinite tower of eigenvalues to each

⁵Our errors are purely statistical, giving the standard deviation based on the data set of eigenvalues. An estimate of the systematic errors is highly non-trivial and will not be attempted in this work.

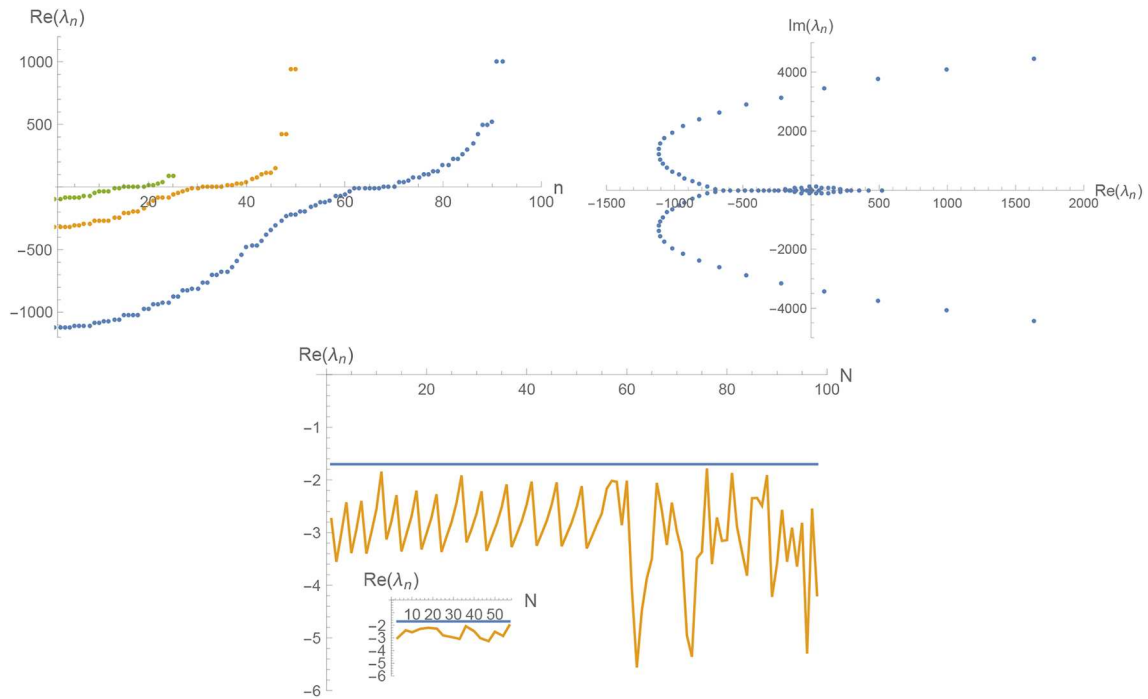


FIGURE 1 | $\text{Spec}(B)$ in $d = 4$ dimensions obtained within framework I. The top-left diagram displays the real parts $\text{Re}(\lambda_n)$ of the eigenvalues found for the stability matrices of sizes $N = 25$ (left line, green dots), $N = 50$ (middle line, orange dots), and $N = 100$ (right line, blue dots). The top-right diagram shows the location of the eigenvalues λ_n ($N = 100$) in the complex plane. The bottom diagram traces the value of the first two relevant eigenvalues as a function of the matrix size N .

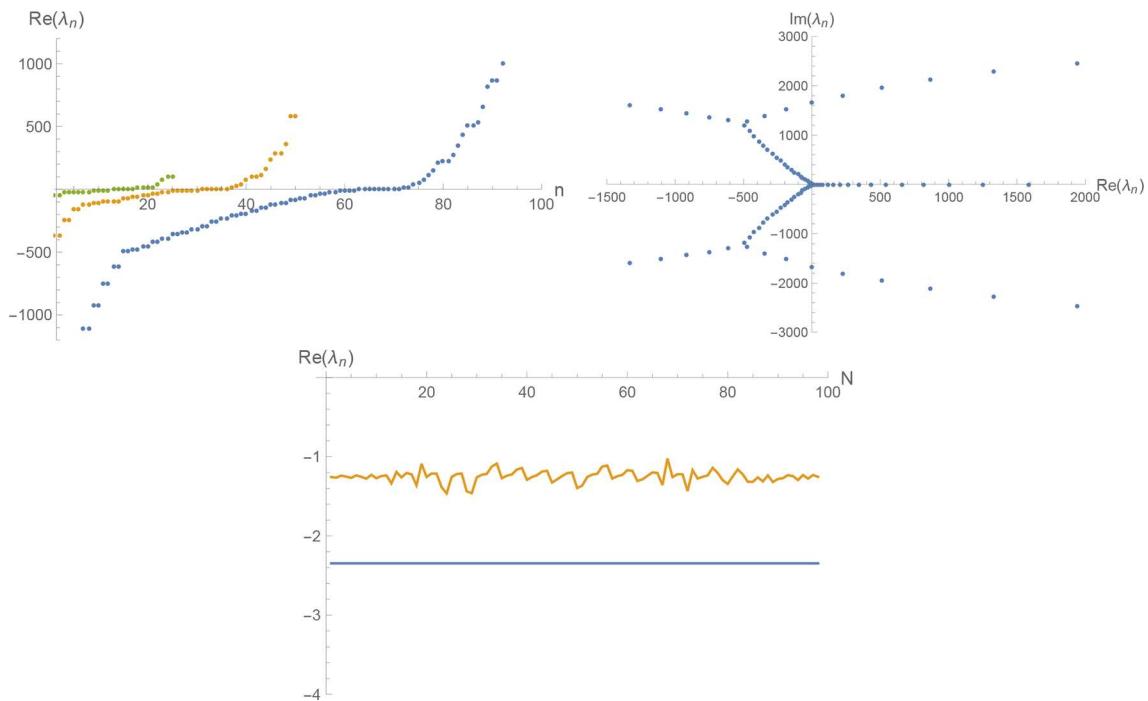
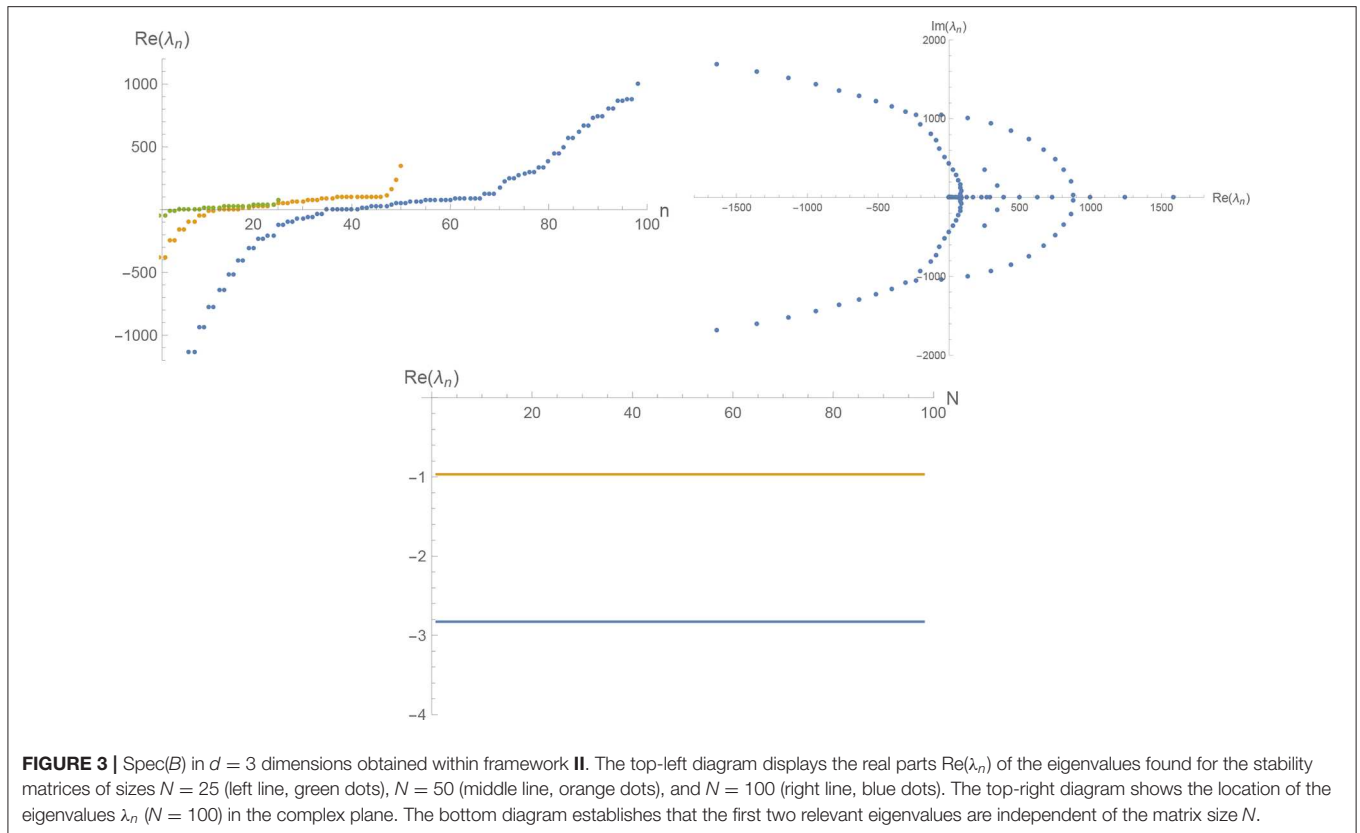


FIGURE 2 | $\text{Spec}(B)$ in $d = 3$ dimensions obtained within framework I. The top-left diagram displays the real parts $\text{Re}(\lambda_n)$ of the eigenvalues found for the stability matrices of sizes $N = 25$ (left line, green dots), $N = 50$ (middle line, orange dots), and $N = 100$ (right line, blue dots). The top-right diagram shows the location of the eigenvalues λ_n ($N = 100$) in the complex plane. The bottom diagram traces the value of the first two relevant eigenvalues as a function of the matrix size N .



point in this plane. At the Gaussian fixed point, $(\lambda_*, g_*) = (0, 0)$, $\gamma = 0$ and $\text{spec}(\mathbf{B})$ follows from classical power counting. The strength of the quantum corrections to $\text{spec}(\mathbf{B})$ is then controlled by the values of g and λ . In particular, there is a region in the vicinity of the Gaussian fixed point where these corrections are small. This motivates defining “perturbative domains” $\mathcal{P}$ by the condition that $\text{spec}(\mathbf{B})$ is dominated by its classical part. Concretely, we define

$$d = 3: \quad \begin{cases} \mathcal{P}_2 = \{(\lambda, g) | \text{spec}(\mathbf{B}) \text{ has 2 UV-relevant eigenvalues} \} \\ \mathcal{P}_3 = \{(\lambda, g) | \text{spec}(\mathbf{B}) \text{ has 3 UV-relevant eigenvalues} \} \end{cases}$$

$$d = 4: \quad \mathcal{P}_3 = \{(\lambda, g) | \text{spec}(\mathbf{B}) \text{ has 3 UV-relevant eigenvalues} \}. \quad (37)$$

Loosely speaking, the definitions of these domains corresponds to imposing that the quantum corrections are not strong enough to turn more than one classically UV-marginal ($d = 4$) or UV-irrelevant ($d = 3$) eigendirection into a relevant one.

Figure 4 illustrates the shape of the domains $\mathcal{P}$ obtained from the spectrum of the stability matrices with $N = 10$ (framework I) in $d = 3$ (left panel) and $d = 4$ (right panel). In $d = 3$ the regions $\mathcal{P}_2$ and $\mathcal{P}_3$ are shaded in blue and orange, respectively while in $d = 4$ $\mathcal{P}_3$ is shaded blue. At the boundary of these regions a new complex pair of eigenvalues with negative real part appears in the spectrum which then violates the definitions (37). Within the present computation the Reuter fixed points (27) are located

outside of $\mathcal{P}_3$ which is consistent with the eigenvalue spectra shown in **Figures 1, 2**.

The boundary of the domains $\mathcal{P}_3$ is very well-described by the parametric curves

$$b = \frac{g}{(1 - 2\lambda)^p} \quad (38)$$

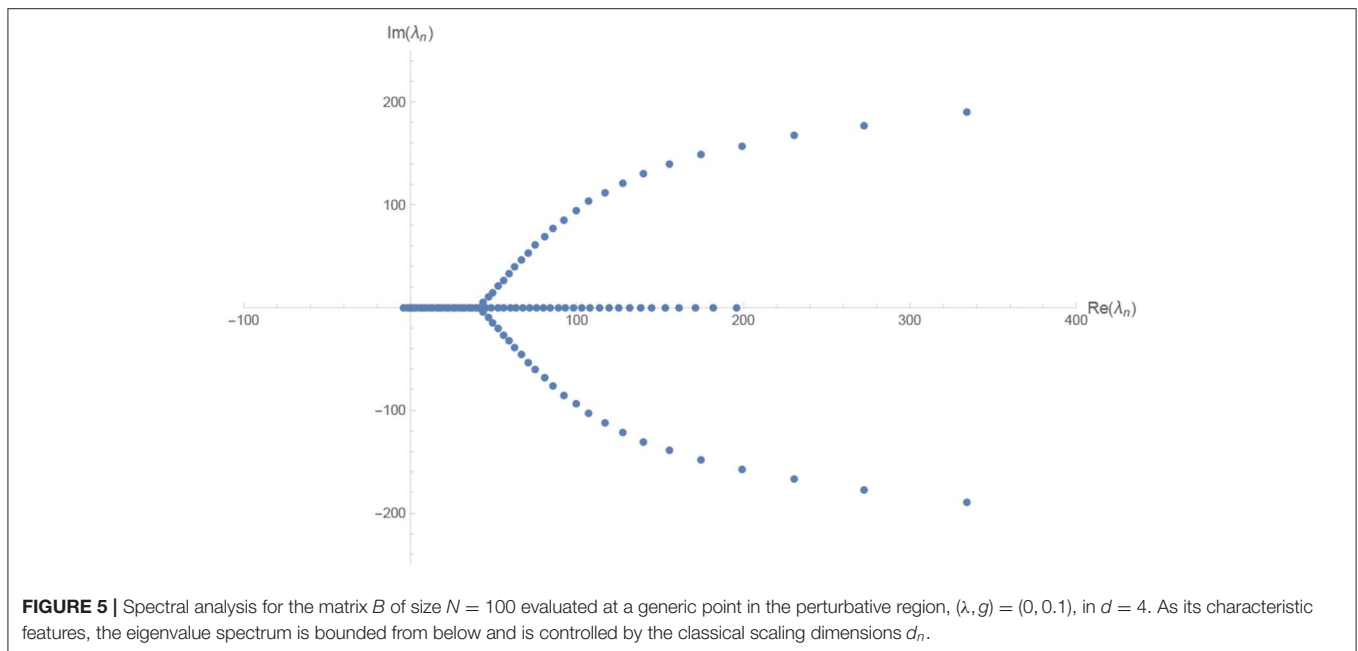
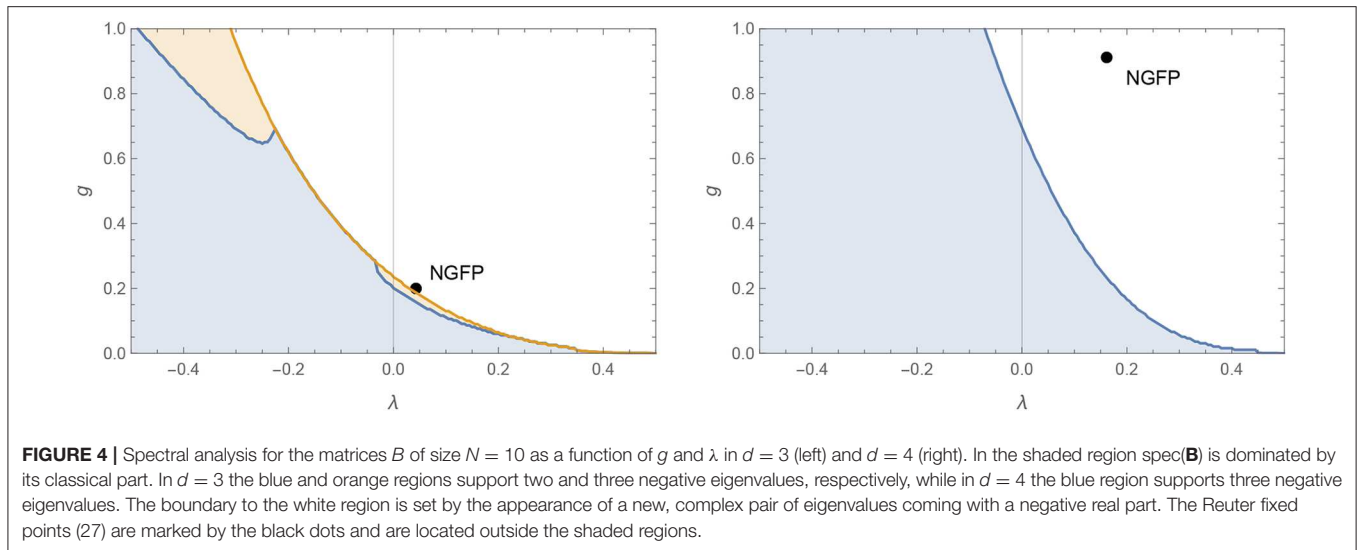
with the best-fit parameters

$$\begin{aligned} d = 3: \quad & b = 0.24, \quad p = 2.94, \\ d = 4: \quad & b = 0.69, \quad p = 2.77. \end{aligned} \quad (39)$$

Following the ideas [72, 73], advocated in the context of gravity-matter systems, it is suggestive to interpret the right-hand side of (38) as the “effective strength of the gravitational fluctuations”. The values b then correspond to the critical value of the effective gravitational coupling g_{eff} which separates perturbative from non-perturbative behavior. Comparing the eigenvalue distributions for the Reuter fixed points shown in **Figures 1–3** to a typical spectrum obtained in the perturbative region (cf. **Figure 5**), it is clear that this phase transition is easily visible in the scaling properties of the operators $\mathcal{O}_n$.

5. CONCLUSIONS AND OUTLOOK

In this work, we applied to composite operator formalism to construct a completely analytic expression for the matrix γ



encoding the anomalous scaling dimensions of the geometrical operators $\mathcal{O}_n \equiv \int d^d x \sqrt{g} R^n$, $n \in \mathbb{N}$, on a background sphere. Our work constitutes the first instance where the composite operator formalism for gravity is extended beyond the single-operator approximation. Within the geometric gauge adopted in our work, the anomalous dimensions originate from the transverse-traceless and trace mode of the gravitational fluctuations. The gauge-modes, corresponding to the vector sector of the transverse-traceless decomposition, decouple. Our derivation made two assumptions: firstly, we assumed that the propagators of the fluctuation fields can be approximated by the (gauge-fixed) Einstein-Hilbert action. Secondly, we assumed that terms appearing in the early-time expansion of the heat-kernel beyond the R^2 -level can be neglected. On

this basis, we derived the generating functional (21) from which the matrix of anomalous dimensions (20) can be generated efficiently.

As illustrated in section 4 the stability matrix $\mathbf{B}$ resulting from the composite operator formalism allows to study the stability properties of the Reuter fixed point. This novel type of analysis provided the following structural insights on Asymptotic Safety:

- (1) The composite operator approach suggests that in $d = 4$ quantum fluctuations turn the classically marginal R^2 -operator into a UV-relevant one. Similarly, the analysis in $d = 3$ dimensions predicts that the classically irrelevant R^2 -coupling becomes UV-relevant.

- (2) The eigenvectors of $\mathbf{B}$ do not coincide with the geometric operators $\mathcal{O}_n$. In general they are given by linear combinations containing an infinite number of terms.
- (3) The non-diagonal terms $\gamma_{nm}, n \neq m$ play a crucial role in determining the spectrum of $\mathbf{B}$. Within the assumptions made in our derivation one furthermore finds that increasing the size of $\mathbf{B}$ creates complex pairs of eigenvalues which wander through the complex plane and lead to new (most likely spurious) UV-relevant directions.

The analysis of the spectrum of the stability matrix as a function of the dimensionless Newton coupling g and cosmological constant λ reveals the existence of a domain where the eigenvalues are dominated by classical power counting. The resulting spectrum is then similar to the one encountered when solving the Wetterich equation in the polynomial $f(R)$ -approximation which determined the eigenvalues of the stability matrix for $N = 6$ [38, 39], $N = 8$ [40], $N = 35$ [44, 47], and lately also $N = 71$ [58]. In particular, Falls et al. [58] reported that for large values of n the real parts of the eigenvalues λ_n follow an almost Gaussian behavior

$$\lambda_n^{f(R)} \approx an - b, \quad a = 2.042 \pm 0.002, \quad b = 2.91 \pm 0.05. \quad (40)$$

where a and b are the best-fit values. As indicated in **Figure 4**, the present computation places the Reuter fixed point outside of this scaling domain, i.e., for sufficiently large matrices one obtains new eigenvalues coming with both positive and negative real parts. This makes it conceivable that the higher-order curvature terms appearing in the propagators of the $f(R)$ -approximation play a crucial role in extending the domain such that it includes the fixed point, thereby guaranteeing its predictive power.

Putting our results into a broader context, we note that, by now, several classes of consistency tests related to the viability of an RG fixed point for the asymptotic safety program have been put forward. These include, e.g., the stability of the eigenvalue spectrum of $\mathbf{B}$ when increasing the set of operators included in Γ_k [38–40, 58], the concept of “apparent convergence” [74], “effective universality” in gravity-matter systems [75], or the “almost perturbative” nature of the fixed point [76]. Our results then provide key insights on how convergence of fixed point properties could organize itself outside the almost perturbative domain.

Arguably, the most intriguing result of our work is the spectral analysis of the stability matrix showing the distributions of its eigenvalues in the complex plane, c.f. the top-right diagrams of **Figures 1–3, 5**. The resulting patterns are reminiscent of the Lee-Yang theory for phase transitions [77]. This suggests two immediate applications. First, the status of Asymptotic Safety makes it conceivable that there are actually an infinite number of Reuter-type fixed points arising from gravity and gravity-matter systems. Understanding the characteristic features of their eigenvalue distributions in terms of nodal points creating complex eigenvalues may then constitute a powerful tool for classifying these fixed points and giving a precise definition to the notion of “gravity-dominated” renormalization group

fixed points in gravity-matter systems. Secondly, tracing the eigenvalues λ_n along their Lee-Yang type orbits in the complex plane could provide a novel tool for testing the convergence of the eigenvalue distribution of $\mathbf{B}$ beyond the realm of a weak effective gravitational coupling (38) where the spectrum is governed by classical power counting. Clearly, it would be interesting to follow up on these points in the future.

As a by-product our analysis also computed the diagonal entries of the anomalous dimension matrix in geometric gauge [cf. Equation (30)]. It is instructive to compare this result to the value of the diagonal entries obtained in harmonic gauge [19, 32]

$$\begin{aligned} d = 3: \quad \gamma_{nn}^* &= 1.591 - 1.505n - 0.118n^2, \\ d = 4: \quad \gamma_{nn}^* &= 3.987 - 4.733n - 0.095n^2. \end{aligned} \quad (41)$$

This identifies two features which are robust under a change of gauge-fixing: in both cases, the values of γ_{nn} up to $n \simeq \mathcal{O}(10)$ follows a linear scaling law: in all cases the coefficients multiplying the quadratic terms are small or even vanishing when adopting geometric gauge in four dimensions. Secondly, the entries in the stability matrix B_{nn} are negative definite for all values n . At the same time, this comparison gives a first idea of the accuracy to which the composite operator formalism in the single-operator approximation is capable to determine the anomalous scaling dimension of the geometric operators: most likely, the results have the status of order-of-magnitude estimates: they should not be interpreted as “precision results” which one should try and reproduce to the given accuracy. Conceptually, it would be interesting to understand (and eliminate) the gauge-dependence of the result. Most likely, this will require imposing on-shell conditions to the master Equation (5) following, e.g., the ideas outlined in Benedetti [78] and Falls [79]. Along a different line, it would be interesting to extend the results for the single-operator approximation obtained in the present work to the case where the gravitational propagators include all terms up to order R^4 in the curvature expansion. This computation would be “complete” in the sense that it includes *all* terms which contribute to the γ_{nn}^* and therefore constitutes the “best” result attainable in the composite operator framework. We leave these points to future work though.

As one of its most intricate features, the composite operator formalism employed in this work could act as a connector between Asymptotic Safety [22, 23] and more geometric approaches to quantum gravity based on causal dynamical triangulations [80, 81] or random geometry. In $d = 2$ dimensions, a natural benchmark would involve a quantitative comparison of scaling properties associated with the geodesic length recently considered in Pagani and Reuter [19], Becker and Pagani [29, 30], Becker et al. [31], and Houthoff et al. [32] and exact computations for random discrete surfaces in the absence of matter fields [21, 82] as well as rigorous and numerical bounds arising from Liouville Gravity in the presence of matter [83, 84]. On the renormalization group side this will involve taking limits akin to Nink and Reuter [85]. Conversely, it is interesting to generalize the two-dimensional constructions to higher dimensions. The connection between the stability matrix $\mathbf{B}$ and the anomalous scaling dimension γ of

geometric operators may then be an interesting link allowing to probe Asymptotic Safety based on geometric constructions of a quantum spacetime.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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APPENDICES

A. HEAT-KERNEL, MELLIN TRANSFORMS, AND THRESHOLD FUNCTIONS

The calculation of γ requires the evaluation of the operator traces appearing on the right-hand side of the composite operator equation (5). This computation can be done effectively by applying the early-time heat-kernel expansion for minimal second-order differential operators $\Delta \equiv -\bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$. Following the ideas advocated in Lauscher and Reuter [69] and Benedetti et al. [61], we carry out a transverse-traceless decomposition of the fluctuation fields. Paired with a maximally symmetric background geometry, this decomposition ensures that all differential operators in the trace arguments organize themselves into Laplacians Δ .

These traces can then be evaluated using the Seeley-deWitt expansion of the heat-kernel on the d -sphere S^d :

$$\text{Tr}_i [e^{-s\Delta}]|_{S^d} \quad (42)$$

$$\simeq \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{\bar{g}} [a_0^i + a_1^i s \bar{R} + a_2^i s^2 \bar{R}^2 + \dots].$$

Here $i = \{S, TV, T\}$ labels the type of field on which the Laplacian acts and the dots represent higher-order curvature terms. The relevant coefficients a_n^i have been computed in Lauscher and Reuter [69] and are listed in **Table 1**. Their derivation manifestly uses the identities (8) in order to simplify the heat-kernel expansion on a general manifold [87].

The expansion (42) is readily generalized to functions of the Laplacian. Introducing the Q -functionals

$$Q_n[W] \equiv \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(z), \quad n > 0, \quad Q_0[W] = W(0), \quad (43)$$

one has [40]

$$\text{Tr}_i [W(\Delta)] = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\bar{g}} [a_0^i Q_{d/2}[W] + a_1^i Q_{d/2-1}[W] \bar{R} + a_2^i Q_{d/2-2}[W] \bar{R}^2 + \dots]. \quad (44)$$

In order to write γ and the beta functions of the Einstein-Hilbert truncation in a compact form, it is convenient to express the Q -functionals in terms of the dimensionless threshold functions [37]

$$\Phi_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{r(z) - z r'(z)}{[z + r(z) + w]^p}, \quad (45)$$

$$\tilde{\Phi}_n^p(w) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \frac{r(z)}{[z + r(z) + w]^p}.$$

Here $r(z)$ is the dimensionless profile function associated with the scalar regulator $R_k(z) = k^2 r(z)$ introduced in Equation (50) and

the prime denotes a derivative with respect to the argument. For later convenience we also define the combination

$$q_n^p(w) \equiv \Phi_n^p(w) - \frac{1}{2} \eta_N \tilde{\Phi}_n^p(w). \quad (46)$$

The arguments of the traces appearing in γ , Equation (16), and the Einstein-Hilbert truncation studied in Appendix B have a canonical form. Defining $P_k \equiv z + R_k(z)$, the identity

$$Q_n [z^q (P_k + wk^2)^{-p} G_k \partial_t (G_k^{-1} R_k)] = 2 \frac{\Gamma(n+q)}{\Gamma(n)} (k^2)^{n+q+1-p} q_{n+q}^p(w) \quad (47)$$

allows to convert the corresponding Q -functionals into the dimensionless threshold functions. For $q = 0$ this reduces to

$$Q_n [(P_k + wk^2)^{-p} \partial_t R_k] = 2 (k^2)^{n+1-p} \Phi_n^p(w), \quad (48)$$

$$Q_n [(P_k + wk^2)^{-p} G_k \partial_t (G_k^{-1} R_k)] = 2 (k^2)^{n+1-p} q_n^p(w).$$

Notably, the second set of identities suffices to derive the beta functions of the Einstein-Hilbert truncation while the evaluation of γ requires the generalization (47).

For maximally symmetric backgrounds the background curvature $\bar{R}$ is covariantly constant. As a consequence, it has the status of a parameter and can be included in the argument of the threshold functions. Expansions in powers of $\bar{R}$ can then be constructed from the recursion relations

$$\frac{d}{dw} \Phi_n^p(w) = -p \Phi_n^{p+1}(w), \quad \frac{d}{dw} q_n^p(w) = -p q_n^{p+1}(w). \quad (49)$$

Throughout the work, we specify the (scalar) regulator

$$R_k(\Delta) = k^2 r(\Delta/k^2), \quad (50)$$

to the Litim regulator [89, 90]. In this case the dimensionless profile function $r(z)$ is given by

$$r(z) = (1-z)\Theta(1-z), \quad (51)$$

with $\Theta(x)$ the unit-step function. For this choice the integrals (45) can be carried out analytically, yielding

$$\Phi_n^{p,\text{Litim}}(w) = \frac{1}{\Gamma(n+1)} \frac{1}{(1+w)^p},$$

$$\tilde{\Phi}_n^{p,\text{Litim}}(w) = \frac{1}{\Gamma(n+2)} \frac{1}{(1+w)^p}. \quad (52)$$

B. THE EINSTEIN-HILBERT TRUNCATION IN GENERAL GAUGE

Structurally, the composite operator equation provides a map from the couplings contained in the Hessian $\Gamma_k^{(2)}$ to the matrix of anomalous dimensions γ . This map is independent of the RG flow entailed by the Wetterich equation. In order to characterize the geometry associated with the Reuter fixed point, the map

has to be evaluated at the location of the fixed point. This appendix then studies the flow of Γ_k in the Einstein-Hilbert truncation supplemented by a general gauge-fixing term. The key result is the position of the Reuter fixed point, Equation (27), which underlies the spectral analysis of section 4. Our analysis essentially follows [61, 62, 86], to which we refer for further details.

The Einstein-Hilbert truncation approximates the effective average action $\Gamma_k[h; \bar{g}]$ by the Einstein-Hilbert action $\Gamma_k^{\text{EH}}[g] = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} (2\Lambda_k - R)$ supplemented by a gauge-fixing functional $\Gamma_k^{\text{gf}}[h; \bar{g}]$ and the corresponding ghost action $S^{\text{ghost}}[h, \bar{C}, C; \bar{g}]$

$$\Gamma_k[h; \bar{g}] \simeq \Gamma_k^{\text{EH}}[g] + \Gamma_k^{\text{gf}}[h; \bar{g}] + S^{\text{ghost}}[h, \bar{C}, C; \bar{g}]. \quad (53)$$

This ansatz contains two scale-dependent couplings, Newton's coupling G_k and the cosmological constant Λ_k . In the present analysis, we work with a generic gauge-fixing term

$$\begin{aligned} \Gamma_k^{\text{gf}}[h; \bar{g}] &= \frac{1}{32\pi G_k \alpha} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu, \\ F_\mu &\equiv \bar{D}^\nu h_{\mu\nu} - \frac{1+\rho}{d} \bar{D}_\mu h, \end{aligned} \quad (54)$$

where α and ρ are free, dimensionless parameters. The harmonic gauge used in Pagani [70] and Houthoff et al. [32] corresponds to $\alpha = 1, \rho = d/2 - 1$ while the present computation significantly simplifies when adopting geometric gauge, setting $\rho = 0$ before evoking the Landau limit $\alpha \rightarrow 0$. The ghost action associated with (54) is

$$\begin{aligned} S^{\text{ghost}}[h, \bar{C}, C; \bar{g}] &= \\ &- \int d^d x \sqrt{\bar{g}} \bar{C}_\mu [\bar{D}^\rho \delta_\nu^\mu D_\rho + \bar{D}^\rho g_{\rho\nu} D^\mu \\ &- \frac{2(1+\rho)}{d} \bar{D}_\mu \bar{g}^{\sigma\rho} g_{\rho\nu} D_\sigma] C^\nu \end{aligned} \quad (55)$$

Following the strategy employed in the gravitational sector, c.f. Equation (9), the fields $\bar{C}_\mu, C^\mu$ are decomposed into their transverse and longitudinal parts

$$C^\mu = C^{\text{T}\mu} + \bar{D}^\mu \eta, \quad \bar{D}_\mu C^{\text{T}\mu} = 0, \quad (56)$$

followed by a rescaling

$$\Delta^{1/2} \eta \mapsto \eta. \quad (57)$$

The part of the ghost action quadratic in the fluctuation fields then becomes

$$\begin{aligned} S^{\text{ghost,quad}} &= \int d^d x \sqrt{\bar{g}} \left\{ \bar{C}_\mu^{\text{T}} \left[\Delta - \frac{1}{d} \bar{R} \right] C^{\text{T}\mu} \right. \\ &\quad \left. + 2\bar{\eta} \left[\frac{d-1-\beta}{d} \Delta - \frac{1}{d} \bar{R} \right] \eta \right\}. \end{aligned} \quad (58)$$

We now proceed by constructing the non-zero entries of the Hessian $\Gamma_k^{(2)}$. These are obtained by expanding Γ_k to

second order in the fluctuation fields, substituting the transverse traceless decomposition (9) and (56), and implementing the field redefinitions (11) and (57). Subsequently taking two functional variations with respect to the fluctuation fields then leads to the matrix elements listed in the middle block of **Table 2**.

The final ingredient entering the right-hand side of the Wetterich equation is the regulator $\mathcal{R}_k$. We generate this matrix from the substitution rule

$$\Delta \mapsto P_k \equiv \Delta + R_k(\Delta), \quad (59)$$

dressing each Laplacian by a scalar regulator $R_k(\Delta)$. The latter then provides a mass for fluctuation modes with momentum $p^2 \lesssim k^2$. In the nomenclature introduced in Codello et al. [40] this corresponds to choosing a type I regulator. The non-zero entries of $\mathcal{R}_k$ generated in this way are listed in the bottom block of **Table 2**.

We now have all the ingredients to compute the beta functions resulting from the Wetterich equation projected onto the Einstein-Hilbert action. Adopting the geometric gauge $\rho = 0, \alpha \rightarrow 0$ used in the main section, all traces appearing in the equation simplify to the Q -functionals evaluated in Equation (48). Defining

$$\partial_t g_k = \beta_g(g, \lambda_k; d) \equiv (d-2+\eta_N)g_k, \quad \partial_t \lambda_k = \beta_\lambda(g, \lambda_k; d) \quad (60)$$

where the anomalous dimension of Newton's coupling is parameterized by Reuter [37]

$$\eta_N(g, \lambda) = \frac{gB_1(\lambda)}{1 - gB_2(\lambda)}, \quad (61)$$

the explicit computation yields

$$\begin{aligned} \beta_\lambda &= (\eta_N - 2)\lambda + \frac{g}{(4\pi)^{d/2-1}} \left[(d-2)(d+1)q_{d/2}^1(w_T) \right. \\ &\quad \left. + 2q_{d/2}^1(w_S) + 2dq_{d/2}^1(0) - 4d\Phi_{d/2}^1(0) \right], \end{aligned} \quad (62)$$

and

$$\begin{aligned} B_1 &= \frac{1}{(4\pi)^{d/2-1}} \left[\frac{(d+1)(d+2)(d-5)}{3(d-1)} \Phi_{d/2-1}^1(w_T) \right. \\ &\quad - \frac{2(d-2)(d+1)(d^2-3d+4)}{d(d-1)} \Phi_{d/2}^2(w_T) \\ &\quad + \frac{2}{3} \Phi_{d/2-1}^1(w_S) - \frac{2(d-4)}{d-1} \Phi_{d/2}^2(w_S) \\ &\quad \left. - \frac{4(d^2-d+1)}{d(d-1)} \Phi_{d/2}^2(0) - \frac{2(d^2-6)}{3d} \Phi_{d/2-1}^1(0) \right], \\ B_2 &= -\frac{1}{2(4\pi)^{d/2-1}} \left[\frac{(d+1)(d+2)(d-5)}{3(d-1)} \tilde{\Phi}_{d/2-1}^1(w_T) \right. \\ &\quad - \frac{2(d-2)(d+1)(d^2-3d+4)}{d(d-1)} \tilde{\Phi}_{d/2}^2(w_T) \\ &\quad + \frac{2}{3} \tilde{\Phi}_{d/2-1}^1(w_S) - \frac{2(d-4)}{d-1} \tilde{\Phi}_{d/2}^2(w_S) \\ &\quad \left. + \frac{4(d^2-d+1)}{d(d-1)} \tilde{\Phi}_{d/2}^2(0) + \frac{2(d^2-6)}{3d} \tilde{\Phi}_{d/2-1}^1(0) \right]. \end{aligned} \quad (63)$$

Here the threshold functions Φ_n^p , $\tilde{\Phi}_n^p$ and $q_n^p(w)$ are defined in Equations (45) and (46) and their arguments w_T and w_S have been introduced in (23).

It is now straightforward to localize the Reuter fixed point by determining the roots of the beta functions (60) numerically. For the Litim regulator (51) this yields

$$\begin{aligned} \text{Reuter fixed point: } d = 3: \quad g_* &= 0.198, \quad \lambda_* = 0.042, \\ \text{Reuter fixed point: } d = 4: \quad g_* &= 0.911, \quad \lambda_* = 0.160. \end{aligned} \quad (64)$$

Analyzing the stability properties of the RG flow in its vicinity, it is found that the fixed point constitutes a UV attractor, with the eigenvalues of the stability matrix given by

$$\begin{aligned} \text{Reuter fixed point: } d = 3: \quad \lambda_{1,2}^{\text{EH}} &= -1.658 \pm 0.546i, \\ \text{Reuter fixed point: } d = 4: \quad \lambda_{1,2}^{\text{EH}} &= -2.132 \pm 2.697i. \end{aligned} \quad (65)$$

These results agree with the ones found in Benedetti et al. [61] at the 10% level. The difference can be traced back to the two distinct regularization procedures employed in the computations, so that the findings are in qualitative agreement. This completes our analysis of the Einstein-Hilbert truncation underlying the scaling analysis in the main part of this work.

C. MATRIX-ELEMENTS OF GEOMETRIC OPERATORS

The expansions of $\mathcal{O}_n$ and Γ_k in the fluctuation fields are readily computed using the xPert extension [88] of xAct. For completeness, the relevant expressions are listed in **Table 2**. The d -dependent coefficients C_i multiplying the curvature terms in $\Gamma_k^{(2)}$ are

$$\begin{aligned} C_T &= \frac{d^2 - 3d + 4}{d(d-1)}, \quad C_V = \frac{d-2}{d}, \quad C_S = \frac{d-4}{2(d-1)}, \\ C_\sigma &= -(d-2). \end{aligned} \quad (66)$$



Renormalization in Quantum Theories of Geometry

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A hallmark of non-perturbative theories of quantum gravity is the absence of a fixed background geometry, and therefore the absence in a Planckian regime of any notion of length or scale that is defined a priori. This has potentially far-reaching consequences for the application of renormalization group methods à la Wilson, which rely on these notions in a crucial way. We review the status quo of attempts in the Causal Dynamical Triangulations (CDT) approach to quantum gravity to find an ultraviolet fixed point associated with the second-order phase transitions observed in the lattice theory. Measurements of the only invariant correlator currently accessible, that of the total spatial three-volume, has not produced any evidence of such a fixed point. A possible explanation for this result is our incomplete and perhaps naïve understanding of what constitutes an appropriate notion of (quantum) length near the Planck scale.

Keywords: quantum gravity, phase transitions, causal dynamical triangulations, lattice field theory, asymptotic safety

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1. INTRODUCTION

The Wilsonian concept of renormalization has been of immense importance for our understanding of quantum field theory and its relation to critical phenomena in statistical mechanics and condensed matter physics. In the context of lattice field theory it has been the guiding principle for approaching a continuum quantum field theory, starting out with a lattice regularization of the theory. Usually we view the ultraviolet (UV) regularization of the quantum field theory as a step on the way to defining the theory. For a given theory there will in general be many ways to introduce such a regularization, some more convenient than others, depending on the calculations one wants to perform. The lattice regularization is usually not the most convenient regularization if one wants to perform analytic calculations, but for some theories it allows one to perform non-perturbative calculations, for instance in the form of Monte Carlo (MC) simulations of the field theories in question. It also allows one to address in a non-perturbative way the question of whether or not a given quantum field theory exists, the simplest example being a ϕ^4 -theory in four dimensions. This is a perturbatively renormalizable quantum field theory, so one can fix the physical mass and the physical coupling constant of the theory, and to any finite order in the coupling constant calculate the correlation functions. However, this does not imply that the theory really exists in the limit where the UV cut-off is taken to zero, since the perturbative expansion is only an asymptotic expansion. The lattice field formulation of the ϕ^4 -theory provides us with a tool to go beyond perturbation theory, and (as will be discussed below) the result is that the ϕ^4 -quantum field theory does not exist in four spacetime dimensions. In a similar vain, lattice field theory seems to confirm the existence of the quantum version of non-Abelian gauge theories.

The lattice field theories address the question of existence of certain quantum field theories using the Wilsonian picture: if the continuum quantum field theory exists as a limit of the lattice field theory when the cut-off is removed (the lattice spacing goes to zero), there exists a UV fixed point of the renormalization group. One can approach such a fixed point in the following way: choose observables which define the physical coupling constants of the theory and measure them for a certain choice of the *bare* coupling constants used to define the lattice theory. Then change the lattice spacing by a factor 1/2 and find the new bare coupling constants which leave the observables unchanged¹. Continue halving the lattice spacing and in this way create a flow of the bare coupling constants. The bare coupling constants will then flow to a UV fixed point (if it exists).

The next question is which observables to choose. In the case of a ϕ^4 -theory this is simple (and we will make a choice below). In the case of non-Abelian gauge theories it is already somewhat more difficult, since observables should be gauge-invariant, while the theory is usually not formulated in terms of gauge-invariant variables. In MC simulations of the quantum field theory it is important to choose such gauge-invariant observables, since in quantum field theories the quantum fluctuations are dominated by UV fluctuations. If one uses the path integral (as one does in MC simulations), it implies that a typical field configuration is almost nothing but UV fluctuations. This is true also for scalar theories like a ϕ^4 -theory, but since the field variables in gauge theories are not gauge-invariant, most of these fluctuations are even more unphysical “noise.” However, this noise will cancel when calculating expectation values of gauge-invariant observables. If we next move to quantum theories of geometry, in particular attempts to quantize General Relativity (GR), the choice of “gauge-invariant” observables becomes even more tricky. Gauge invariance in this context is usually replaced by diffeomorphism invariance, and there are few invariant local observables. However, it is even more important that the concept of “distance” now becomes field-dependent. For a given geometry the distance between two points depends on the geometry. Therefore, if we integrate over geometries in the path integral, it becomes unclear how to think about a quantum correlation between fields as function of a distance. In particular, since distance, or scale, is paramount in the Wilsonian theory of critical phenomena, a new challenge arises in this program when we quantize geometries. This is what we want to discuss in this article.

In section 2, we review the standard Wilsonian picture for a ϕ^4 -theory in four flat spacetime dimensions, emphasizing how to find a UV fixed point in the *bare* coupling constant space of the theory. In section 3, we discuss how to use the Wilsonian picture for the theory of quantum geometry denoted Causal Dynamical

Triangulations (CDT), which has been suggested as a theory of quantum gravity. Section 4 discusses some examples where “quantum distances” appear in correlation functions, whether these distances are observables and to what extent the “fractal structure” of quantum geometry can be observed. Finally, section 5 contains a discussion.

2. APPROACHING A UV FIXED POINT

Let us consider a ϕ^4 -field theory on a four-dimensional hypercubic lattice with periodic boundary conditions. We assume that the lattice has L_1, L_2, L_3 , and L_4 lattice links in the four directions, and that $L_i \gg 1$. The total number of lattice points is $N = L_1 \cdots L_4$. If the lattice spacing is a_0 , the corresponding physical volume is $V = Na_0^4$. Let $n = (n^1, \dots, n^4)$ denote the integer lattice coordinates of the vertices. The corresponding spacetime coordinates will be $x_n = a_0 n$. A scalar field ϕ_0 lives on the lattice vertices and we write $\phi_0(n)$ or $\phi_0(x_n)$. The lattice field theory action is

$$S[\phi_0, m_0, \lambda_0; a_0] = \sum_n a_0^4 \left(\frac{1}{2} \sum_{i=1}^4 \frac{(\phi_0(n+\hat{i}) - \phi_0(n))^2}{a_0^2} + \frac{1}{2} m_0^2 \phi_0^2(n) + \frac{1}{4!} \lambda_0 \phi_0^4(n) \right), \quad (1)$$

where $\hat{i}$ denotes a unit vector in direction i . The action is characterized by two so-called “bare,” dimensionless coupling constants $m_0 a_0$ and λ_0 . A correlation function is defined as

$$\langle \phi_0(n_1) \cdots \phi_0(n_k) \rangle = \frac{\int \prod_n d\phi_0(n) \phi_0(n_1) \cdots \phi_0(n_k) e^{-S[\phi_0, m_0, \lambda_0; a_0]}}{\int \prod_n d\phi_0(n) e^{-S[\phi_0, m_0, \lambda_0; a_0]}}. \quad (2)$$

We obtain the same action if we simultaneously change $a_0 \rightarrow a$, set $a_0 \phi_0 = a \phi$, $m_0 a_0 = m a$ and leave λ_0 unchanged, and we have trivially

$$a_0^k \langle \phi_0(n_1) \cdots \phi_0(n_k) \rangle_{a_0, m_0, \lambda_0} = a^k \langle \phi(n_1) \cdots \phi(n_k) \rangle_{a, m, \lambda_0}. \quad (3)$$

In the theory we also have renormalized coupling constants m_R and λ_R , which are determined by some explicit prescription, allowing us to “measure” them. For instance, m_R can be defined from the exponential fall-off of the two-point function, while λ_R can be defined as the connected four-point function at zero momentum. We thus have $m_R a_0 = 1/\xi$, where ξ is the dimensionless correlation length of the two-point ϕ -correlator, measured in units of the lattice spacing. Similarly, there is an explicit definition of λ_R . Let us state how to measure these quantities on the lattice since we will use the same techniques in the case of gravity. We choose one of the lattice axes as the “time” direction and define the spatial average

$$\Phi_0(n^t) := \sum_{\vec{n}} \phi_0(\vec{n}, n^t), \quad \vec{n} = (n^x, n^y, n^z), \quad (4)$$

¹Using a description like this we assume we are so close to a continuum limit that we can use a continuum language for the observables, rather than referring explicitly to the lattice. In addition, note that a procedure like this will not leave all observables unchanged, but only the physical coupling constants. One could follow another renormalization procedure, where the action contained “all possible coupling constants.” In this space one could follow a path which leaves all observables invariant.

and we have

$$\langle \Phi_0(n_1^t) \Phi_0(n_2^t) \rangle_c = \text{const.} e^{-|n_1^t - n_2^t|/\xi} + \dots, \quad (5)$$

where the subscript c in $\langle \cdot \rangle_c$ is the connected part, and the dots indicate terms falling off faster at large time differences. The exponential decay for large $|n_1^t - n_2^t|$ determines the physical mass $m_R = 1/(a_0\xi)$. Similarly, we can define the susceptibilities

$$\chi_k := \sum_{n_1^t, \dots, n_{k-1}^t} \langle \Phi_0(n_1^t) \cdots \Phi_0(n_{k-1}^t) \Phi_0(0) \rangle_c \quad (6)$$

and the second moment

$$\mu_2 := \sum_{n^t} (n^t)^2 \langle \Phi_0(n^t) \Phi_0(0) \rangle_c. \quad (7)$$

One then obtains² (in the case $\langle \phi_0(n) \rangle = 0$ where there is no symmetry breaking)

$$\lambda_R = -\frac{16\chi_4}{\mu_2^2}. \quad (8)$$

a_0 is a fictitious parameter in the above formulation in the sense that if we make the above-mentioned change from $(a_0, \phi_0, m_0, \lambda_0)$ to (a, ϕ, m, λ_0) we obtain the same ξ and the same λ_R , while m_R changes in a trivial way since ξ is unchanged.

Let us choose a value for λ_R . For given values $(m_0 a_0, \lambda_0)$ of the bare coupling constants we obtain a value $\lambda_R(m_0 a_0, \lambda_0)$. Among these there will be sets $(m_0(s) a_0, \lambda_0(s))$, parameterized by some parameter s , such that $\lambda_R(m_0(s) a_0, \lambda_0(s)) = \lambda_R$. They form a curve in the $(m_0 a_0, \lambda_0)$ -coupling constant space. Note that this curve is unchanged if we change $a_0 \rightarrow a$ and $m_0 \rightarrow m = m_0 a_0/a$ and consider the (ma, λ_0) -coupling constant plane. Moving along this curve, the correlation length $\xi(s)$ will change, so we can exchange our arbitrarily chosen parameter s with ξ . If we reach a point along the curve where $\xi = \infty$, we have reached a second-order phase transition point in the $(m_0 a_0, \lambda_0)$ -coupling constant plane. This point can now serve as a UV fixed point for the ϕ^4 -theory, since we are free to insist that m_R is constant along the curve provided that we redefine a such that $m_R a(\xi) = 1/\xi$. This will define $a(\xi)$ as a function of ξ , and – since we are free to define the lattice theory with $a(\xi)$ instead of a_0 – if we at the same time make a trivial rescaling of m_0 to $m(\xi) = m_0 a_0/a(\xi)$, we will in this redefined theory obtain the same ξ and λ_R . Thus, it can be viewed as a rescaling of the lattice to smaller a while keeping the continuum physics (i.e., m_R and λ_R) constant. In particular, the correlation length in real spacetime is kept fixed since $|x|_{\text{corr}} \equiv \xi a(\xi) = 1/m_R$. In the limit $\xi \rightarrow \infty$ the lattice spacing goes to zero and we have our continuum quantum field theory with the cut-off removed.

The approach to this assumed UV fixed point is governed by the so-called β -function³, which relates the change in λ_0 to the

change in $a(\xi) = 1/(m_R \xi)$ as we move along the trajectory of constant m_R, λ_R ,

$$-a \frac{d\lambda_0}{da} \Big|_{m_R, \lambda_R} = \xi \frac{d\lambda_0}{d\xi} \Big|_{m_R, \lambda_R} = \beta(\lambda_0). \quad (9)$$

Denote the fixed point by λ_0^* , and assume⁴ that $\lambda_0^* \neq 0$. Since $\lambda_0(\xi)$ stops changing when $\xi \rightarrow \infty$, we have $\beta(\lambda_0^*) = 0$ and expanding the β -function to first order one finds

$$\lambda_0(\xi) = \lambda_0^* + \text{const.} \xi^{\beta'(\lambda_0^*)}, \quad \beta'(\lambda_0) = \frac{d\beta}{d\lambda_0}. \quad (10)$$

It is seen from (10) that the existence of a UV fixed point implies that $\beta'(\lambda_0^*) < 0$.

In a theory like ϕ^4 in four dimensions it is not clear that there exists a UV fixed point. The non-existence of such a fixed point will show up in the following way: no matter which value of λ_R we choose, following the curve of constant λ_R in the $(m_0 a_0, \lambda_0)$ -coupling constant plane, the correlation length ξ will never diverge along the curve. This implies that there is no continuum limit of the theory with a finite value of the renormalized coupling constant. This seems to be the case for the ϕ^4 -theory in four dimensions [2]. It does not mean that there are no points in the $(m_0 a_0, \lambda_0)$ -coupling constant plane with infinite correlation length. In fact, there is an entire curve of such points where the lattice model undergoes a second-order phase transition between the broken and unbroken symmetry⁵ $\phi \rightarrow -\phi$. However, these points are not related to a UV fixed point, but are related to an infrared fixed point of the theory. They cannot be reached on a path of constant λ_R physics and they cannot be used to define an interacting quantum ϕ^4 -field theory in the limit where the lattice spacing goes to zero.

It will be convenient for us to reformulate the above coupling constant flow in terms of so-called finite-size scaling. For a regular hypercubic lattice in d dimensions with lattice spacing a , the physical volume is $V_d = N_d a^d$, where N_d is the total number of hypercubes. To make sure that V_d can be viewed as constant along a trajectory of the kind described above, with m_R and λ_R kept fixed, we keep the ratio between the linear size $L = N_d^{1/d}$ of the lattice and the correlation length ξ fixed. In terms of the renormalized mass m_R and the lattice spacing $a(\xi)$, the ratio can also be written as

$$\frac{\xi^d}{N_d} = \frac{1}{(a(\xi) m_R)^d N_d} = \frac{1}{m_R^d V_d}. \quad (11)$$

Thus, if we are moving along a trajectory of constant m_R and λ_R in the bare $(m_0 a_0, \lambda_0)$ -coupling constant plane and change N_d according to (11), the finite continuum volume stays fixed. Assuming that there is a UV fixed point, such that $a(\xi) \rightarrow 0$, we see that N_d goes to infinity even if V_d stays finite, and furthermore, again from (11), that the dependence on the

²For a detailed discussion see the book [1].

³A priori the β -function is a function of λ_0 and $m_0 a_0$, but one can show that close to the fixed point one can ignore the $m_0 a_0$ -dependence.

⁴If $\lambda_0^* = 0$, we have a so-called Gaussian fixed point and the formula (10) has to be modified slightly.

⁵In the parametrization chosen here, symmetry breaking can occur when we also allow negative values of the bare coupling constant m_0^2 in (1).

correlation length ξ in (10) can be substituted by a dependence on the linear size $N_d^{1/d}$ in lattice units of the spacetime, leading to

$$\lambda_0(N_d) = \lambda_0^* + \text{const.} N_d^{\beta'(\lambda_0^*)/d}. \quad (12)$$

As we saw above, the absence of a UV fixed point could be deduced by the absence of a divergent correlation length along a trajectory of constant physics in the $(m_0 a_0, \lambda_0)$ -plane (i.e., a trajectory with constant m_R, λ_R). In the finite-size scaling scenario this is restated as N_d not going to infinity along any such curve of constant physics.

We have outlined in this section in some detail how to define and follow lines of constant physics in the ϕ^4 -lattice scalar field theory, because we want to apply the same technique to understand the UV behavior of lattice theories of quantum gravity. The most important lesson is that one is automatically led to UV fixed points (if they exist), if one follows trajectories of constant continuum physics.

3. CDT

3.1. The Lattice Gravity Program

Causal Dynamical Triangulations (CDT) represent an attempt to formulate a lattice theory of quantum gravity (for reviews see [3, 4]). The spirit is precisely that of lattice field theory: one has a continuum field theory with a classical action, and defines formally a quantum theory via the path integral. However, the formal path integral needs to be regularized and one way to do this is to use a lattice regularization, where the length of the lattice links provides the UV cut-off. The idea is then to search for a UV fixed point where the lattice spacing a can be taken to zero while continuum physics is kept fixed, following the same philosophy as outlined above for the ϕ^4 -theory. Immediately a number of issues arise. (1) Given the continuum, classical theory, what is a good lattice regularization of this theory? (2) The classical Einstein-Hilbert action is perturbatively non-renormalizable. The situation is thus somewhat different from the ϕ^4 -theory in four dimensions. The latter exists as a perturbative theory in m_R, λ_R , the mass and the coupling constant, and it makes sense to ask whether there exists a non-perturbatively defined quantum field theory, independent of a cut-off for given physical values m_R, λ_R . For a classical action which is non-renormalizable it is not clear that the correct way to search for a UV-complete theory is to keep a lattice version of the classical action in the lattice path integral and then search for UV fixed points. (3) What are the physical observables in quantum gravity, and how does one stay on a path of constant physics when changing the lattice spacing in the search for a UV fixed point? Let us discuss these points in turn.

- (1) The so-called Regge prescription [5] provides a way to assign local curvature to piecewise linear geometries defined by a (d -dimensional) triangulation and the resulting Regge action is a version of the Einstein-Hilbert action to be used for piecewise linear geometries. A convenient feature of the Regge formalism is its coordinate independence.

The geometry of the piecewise linear manifold defined by a triangulation is entirely determined by the lengths of the links and how the d -dimensional simplices are glued together. Regge originally wanted to use this prescription to approximate a given classical geometry with arbitrary precision without using coordinates. In the path integral we will use it in a different way. We restrict ourselves to triangulations where all links have the same length a , and then sum in the path integral over all such triangulations of a given topology, using as our lattice action the Regge action for the triangulations. In this way, a becomes a UV cut-off and the hope is that this class of piecewise linear geometries can be used to approximate any geometry which would be used in the continuum path integral over geometries⁶.

A good analog is the representation of the propagator $G(x, y)$ of a free particle in Euclidean space as the path integral over all paths in R^d from x to y , with the action being the length of the path. This integral can be approximated by the sum over all paths on a hypercubic lattice with lattice links of length a . This set of paths is *dense* in the set of all continuous paths when the distances between paths are measured with the same metric used to define the Wiener measure for the set of continuum paths from x to y (see [6] for a detailed discussion with the geometric perspective relevant here). We call the way of performing the path integral over geometries⁷ described above Dynamical Triangulations (DT) [7–9]. The “proof of principle” that this method works is two-dimensional quantum gravity. Seen from a classical gravitational perspective it is a trivial theory since the Einstein action in two dimensions is just a topological invariant. For a fixed topology the Einstein term does not contribute to the path integral, which implies that the action reduces to the cosmological constant times the spacetime volume. Thus, if we also fix the spacetime volume in the path integral, the action is just a constant and the path integral becomes a sum over all geometries of fixed topology and fixed spacetime volume with constant weight. This integral is still highly non-trivial and “maximally quantum” in the sense that whatever the action is, in the limit $\hbar \rightarrow \infty$ the weight of a configuration in the path integral will be 1. The integral can be performed in the continuum, giving rise to Liouville quantum gravity [10–13]. At the same time one can also sum over the triangulations analytically [14]. One can then verify that in the triangulated case one recovers the continuum result when the lattice spacing vanishes, $a \rightarrow 0$. It is also important to note that the continuum limit of this

⁶The continuum path integral over four-dimensional geometries has not yet been constructed in any mathematically rigorous way, but one expects that the geometries will include many “wild” geometries which are continuous but nowhere differentiable. In this sense the set of piecewise geometries proposed is a set of “nice” geometries.

⁷It should be emphasized that it is a sum over geometries, not a sum over metrics g_{ij} defining a geometry. In a gauge theory this would correspond to a sum over equivalence classes of gauge fields, something one has only been able to dream about.

lattice theory is fully diffeomorphism-invariant in the sense that it is identical with a diffeomorphism-invariant theory⁸.

While DT works beautifully in two-dimensional spacetime, the generalizations to higher dimensions [15–17] have not been successful yet. The major obstacle has been the nature of the phase diagram of the lattice theory. The goal was to find a UV fixed point where one can define a continuum theory when removing the cut-off. In our usual understanding this requires a second- or higher-order phase transition. One has found phase transitions in the bare coupling constants, but so far they have been first-order transitions only [18, 19] (see [19–21] for recent attempts to avoid the first-order transitions). This led to the suggestion that one should use a somewhat different ensemble of triangulations, denoted Causal Dynamical Triangulations (CDT) [22–26]. The difference with the DT ensemble is that one restricts the triangulations to have a global time foliation, which can be viewed as a lattice version of the requirement of global hyperbolicity in classical General Relativity. While the DT formalism is inherently Euclidean, one can view the CDT triangulations as originating from triangulations of geometries with Lorentzian signature. The construction is such that one can analytically continue each individual piecewise linear triangulation to Euclidean signature. In addition, the associated Regge action also transforms as one would naïvely expect, namely, as $iS[\text{LG}] = -S[\text{EG}]$, where “LG” is the Lorentzian geometry and “EG” the rotated Euclidean geometry. The path integral is then performed over these Euclidean piecewise linear geometries. It turns out that the phase diagram of CDT is highly non-trivial and possesses phase transition lines of both first and second order [27–33]. We will provide some details below. It should be emphasized, again with the ϕ^4 -example in mind, that the mere existence of a second-order line of phase transitions does not ensure that there is a UV fixed point in the theory.

- (2) There are at least three ways to try to resolve the problem of the non-renormalizability of the Einstein-Hilbert action. One way is to view the theory as an approximation to a larger theory which is renormalizable. The Standard Model of Particle Physics is the prime example of how this works. Phenomenologically, the weak interactions were described by a four-fermion interaction, which is non-renormalizable. However, this is a low-energy effective action, which in the Standard Model is resolved into a gauge theory with massive vector particles (the W and Z particles). Thus, new degrees of freedom were introduced, which made the electroweak theory renormalizable. Similarly, the effective low-energy theory of strong interactions, involving mesons and hadrons, was not renormalizable, and again the introduction of new degrees of freedom (the quarks and gluons) made the theory renormalizable. In the case of gravity, string theory represents such an extension of degrees of freedom,

but one which is much more drastic than the extensions represented by the Standard Model. And importantly, while the extension by the Standard Model was dictated by experiments, no string-theoretic extension of gravity has yet been forced upon us by experiments.

Another way to address the non-renormalizability of the Einstein-Hilbert action is to modify the way we view the quantum theory in the case of gravity. Loop quantum gravity represents such a route. There are still a number of issues that need to be addressed in this approach, in particular, how to obtain ordinary GR in the limit where $\hbar \rightarrow 0$. We will not discuss this approach any further. The lattice regularization of gravity fits naturally into the third framework, called *asymptotic safety* [34]. Here one relies on the existence of a non-perturbative UV fixed point in some quantum field theory, whose bare Lagrangian can contain many other terms in addition to the Einstein-Hilbert term. The UV properties of the theory are defined by this fixed point, which one should be able to approach in such a way that the lattice spacing scales to zero, while keeping a finite number of observables fixed and only adjusting a similar number of bare coupling constants. This is highly non-trivial since using naïve perturbation theory will create an infinite set of new counterterms which cannot be ignored. In the CDT theory we will look for such UV fixed points by enlarging the Einstein-Hilbert action slightly. It would perhaps be preferable to work with a more general action, but there are significant numerical limitations which prevent us from exploring this in a systematic way. On the other hand, invoking Occam’s razor, CDT quantum gravity in its present form is a perfectly viable candidate theory of quantum gravity, without any compelling reasons to generalize it. The use of the renormalization group in the continuum provides strong evidence for the existence of such a UV fixed point [35–39]. However, some truncations are used to obtain these results, whose validity is difficult to assess quantitatively. This provides a strong motivation to search for such a fixed point in lattice quantum gravity, which is an independent way to define quantum gravity non-perturbatively.

- (3) One of the steps in the search for a UV fixed point is to choose a suitable set of physical observables to be kept fixed along the path to the putative UV fixed point. In the case of pure gravity this becomes non-trivial. For the ϕ^4 -theory, one could choose to keep the coupling constants m_R and λ_R fixed, because the correlators of the scalar field can be deduced from observations, and the coupling constants can be expressed in terms of these correlators, as mentioned earlier. In a theory of quantum gravity, the concept of a correlator as a function of the distance between two spacetime points is problematic, since the distance is itself a function of the geometry we are integrating over in the path integral. Thus, the concept of a correlation length becomes non-trivial, and the whole Wilsonian approach to renormalization—based on having a divergent correlation length on the lattice when one approaches the UV fixed point—needs to be clarified. Even the relation between the UV cut-off (the length a of a lattice link) and any actual

⁸No coordinates were introduced at any point in the lattice theory, so agreement with a diffeomorphism-invariant theory means that all coordinate-invariant quantities which can be calculated agree.

physically measurable length is not clear a priori. We will return to this in more detail in section 4.

3.2. Phase Diagram for CDT

In DT and CDT the Regge action for a given piecewise linear geometry appearing in the path integral becomes very simple. In dimensionless units, where the lattice spacing a is set to 1, the DT Regge action for a four-dimensional triangulation T consisting of N_4 four-simplices, glued together to form a four-dimensional closed manifold in such a way that it contains N_0 vertices, is given by⁹

$$S[T] = -\kappa_0 N_0(T) + \kappa_4 N_4(T). \quad (13)$$

In this formula $\kappa_0 \propto a^2/G_0$, where G_0 is the bare gravitational coupling constant, while κ_4 is related to the cosmological coupling constant. Remarkably, no details of the triangulation except for the global quantities N_4 and N_0 appear in Equation (13). In the case of CDT we have a foliation in one direction, which we denote the time direction. The triangulation thus consists of a sequence of three-dimensional time-slices, where each slice has the same fixed three-dimensional topology (typically that of S^3 or T^3). Each of the time-slices is triangulated, constructed by gluing together equilateral tetrahedra. Neighboring time-slices are joined by four-dimensional simplices, which come in two types: (4,1)-simplices with four vertices in one time-slice and one vertex in one of the neighboring time-slices, and (3,2)-simplices, with three vertices in one time-slice and two vertices in one of the neighboring time-slices. The Regge action is slightly more complicated for such a triangulation (see [3] for a detailed discussion) and has the form

$$S[T] = -(\kappa_0 + 6\Delta) N_0(T) + \kappa_4 (N_{4,1}(T) + N_{3,2}(T)) + \Delta N_{4,1}(T), \quad (14)$$

where $N_{4,1}(T)$ and $N_{3,2}(T)$ denote the number of (4,1)- and (3,2)-simplices in the triangulation T . For $\Delta = 0$ one recovers the simpler functional form (13). Here we will view Δ as an additional coupling constant¹⁰, with no immediate continuum interpretation. We thus have the lattice partition function

$$Z(\kappa_0, \Delta, \kappa_4) = \sum_T e^{-S[T]}, \quad (15)$$

and the first task is to find the phase diagram in the coupling constant space. We have three coupling constants, κ_0 , Δ , and κ_4 . κ_4 is multiplying the total number of four-simplices and acts like a cosmological constant. In the numerical simulations it is convenient to keep the volume N_4 of spacetime fixed. One can subsequently perform simulations with different total volumes and study finite-size scaling as a function of the total volume, as already mentioned in the discussion of the ϕ^4 -model. Keeping N_4 fixed implies that we have to fix κ_4 . This

⁹We assume here that N_0 and N_4 are large, since the Euler characteristic of the closed manifold in principle also appears in (13), but is ignored.

¹⁰Originally in CDT, Δ was associated with an asymmetry between the lengths of lattice links in the time direction and in the other directions.

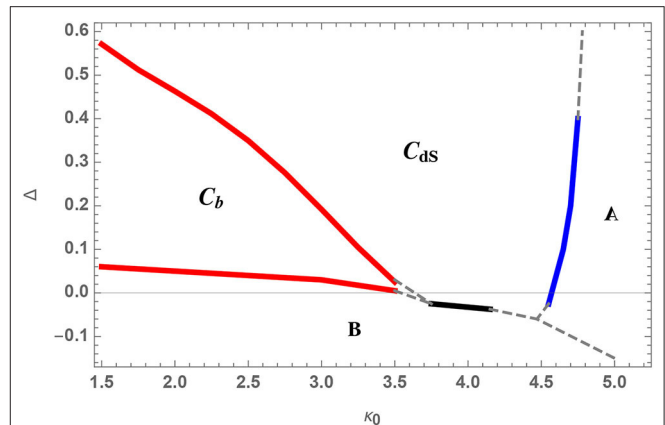


FIGURE 1 | The CDT phase diagram. Phase transition between phase C_{ds} and C_b is (most likely) second order, as is the transition between C_b and B , while the transition between C_{ds} and A is first order. The transition between C_{ds} and B is still under investigation, but preliminary results suggest a first-order transition.

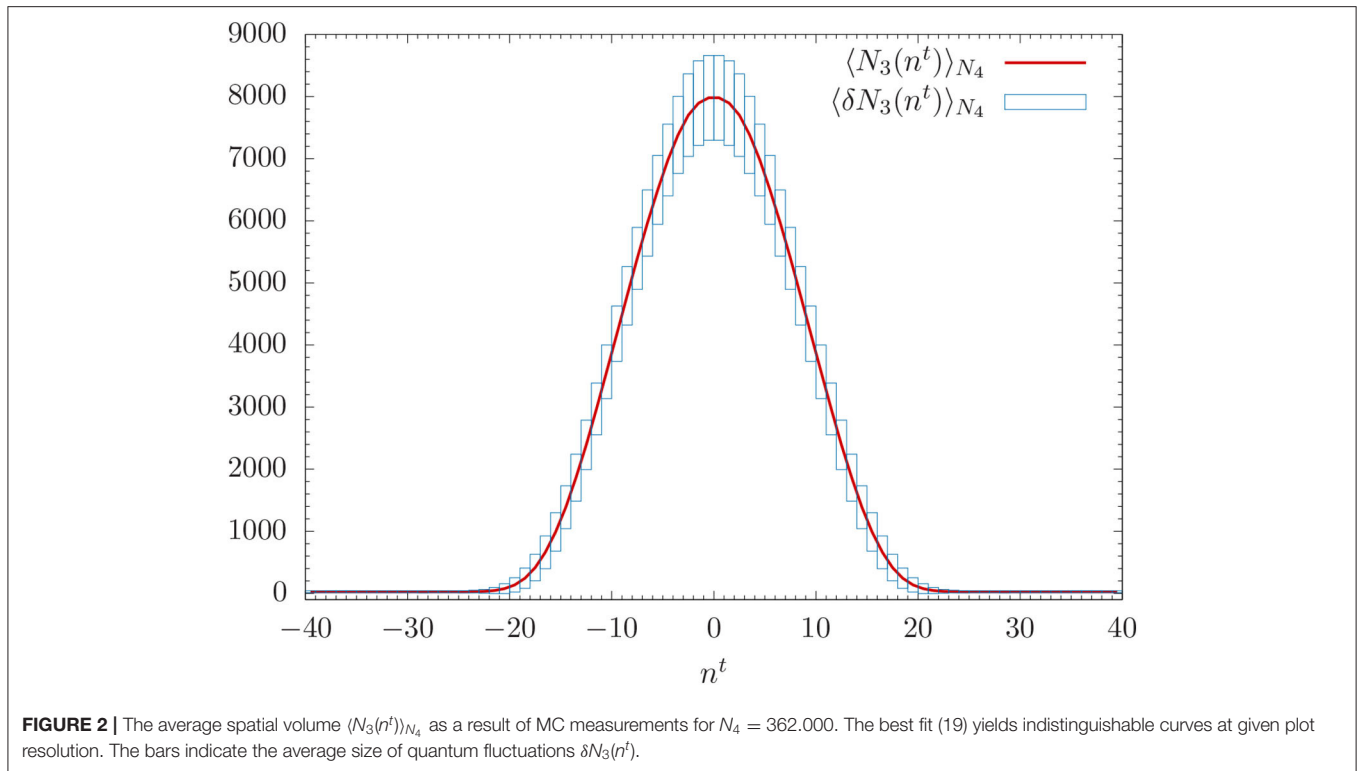
leaves us with two coupling constants, κ_0 and Δ . In **Figure 1** we show the phase diagram of CDT, determined from Monte Carlo simulations. The diagram is surprisingly complicated and part of it is still under investigation. We refer to the original articles for a careful discussion [27–32, 40]. What is important for the present discussion is that in phase C_{ds} in **Figure 1**, which we denote the de Sitter-phase, geometries with continuum-like properties are found. Thus, we would like to start with some bare coupling constants (κ_0, Δ) in that phase, calculate the values of some physical observables, and then follow the path of constant physics by changing the bare coupling constants until we reach a second-order phase transition point on the phase transition line separating the C_{ds} and C_b phases. If it exists (which is not at all granted), this point will then be a UV fixed point.

3.3. Observables and the UV Limit

What kind of observables can we use in CDT in search of a UV fixed point? We have no fields we can associate with lattice vertices or the centers of (sub)-simplices¹¹. However, we have geometric quantities, like the Regge curvature which is assigned to two-dimensional sub-simplices in the four-dimensional triangulation, and we also have the trivial field “1(n),” which assigns the real number 1 to each four-simplex and which turns out to be quite useful¹². At the same time, for any given geometry we can talk about geodesic distances between vertices or (sub)-simplices. This can be transferred to the quantum gravity theory in the path integral formalism, where

¹¹One can in principle associate by hand a coordinate system to each simplex, compute transition functions between the different coordinate systems and assign metric tensor fields g_{ij} to each simplex, but this becomes very cumbersome and has so far not been useful in a DT or CDT context. It would also re-introduce a coordinate dependence which is clearly unwanted.

¹²As observed in [41], if one assumes the existence of a time foliation and expands the general continuum effective action for quantum gravity to quadratic order, one obtains naturally a projection on the constant mode when integrating certain correlators over space, as we will do in (18) and as was done in (4) in flat spacetime. In this sense one is naturally led to 1(n) for such integrated correlators.



one can talk about correlations between some of these quantities when they are separated by a certain geodesic distance. The subtlety lies in the fact that this distance has to be fixed outside the path integral, since we are integrating over geometries that define what we mean by distance. We will return to this point in section 4. Here we will use it in a specific CDT context where the situation is simpler. CDT is special because we have a time foliation, which on the lattice becomes an explicit time coordinate, namely, the n^t labeling of the various time-slices. In this sense the set-up in CDT is precisely the lattice set-up one would use in proper-time gauge in Hořava-Lifshitz gravity (HLG) [42, 43], although the presence of a preferred time in CDT is *not* associated with a breaking of four-dimensional diffeomorphism invariance (see [4] for a related discussion).

Let us introduce the notation $\langle \mathcal{O} \rangle_{N_4}$ for a quantity $\mathcal{O}$. It signifies the average of the quantity $\mathcal{O}$, calculated using the partition function (15), but for fixed discrete four-volume N_4 . (In practice the “calculation” means that we are performing MC simulations.) Now we can define the CDT version of (4) for the trivial field $\phi(n) = 1$:

$$N_3(n^t) = \frac{1}{2} \sum_{\vec{n}(n^t)} (1 + 1). \quad (16)$$

The notation is as follows: each time-slice is assigned a lattice time n^t . On this time-slice each three-simplex (tetrahedron) is assigned a label $\vec{n}(n^t)$ by analogy with the notation for the hypercubic lattice in Equation (4). This notation is only symbolic, since the three-dimensional triangulations are not regular and different time-slices need not contain the same number of

three-simplices $N_3(n^t)$. Each of these three-simplices belongs to precisely two (4,1)-simplices, whose trivial fields “1” are represented in the sum in (16), and we divide by 2 to obtain $N_3(n^t)$. On a regular lattice, this number is of course trivial and fixed, but here it can vary, as mentioned, and becomes a dynamical variable. We now calculate averages and correlation functions like in (5), i.e., we calculate

$$\langle N_3(n^t) \rangle_{N_4} \quad (17)$$

and

$$\langle N_3(n_1^t) N_3(n_2^t) \rangle_c = \langle N_3(n_1^t) N_3(n_2^t) \rangle_{N_4} - \langle N_3(n_1^t) \rangle_{N_4} \langle N_3(n_2^t) \rangle_{N_4}. \quad (18)$$

Figure 2 shows the average of $N_3(n^t)$ over many configurations in the case where the topology of the spatial slices is that of S^3 . It also shows the size of the fluctuations, i.e., it is a plot of $\langle N_3(n^t) \rangle$ and $\delta N_3(n^t) = \sqrt{\langle N_3^2(n^t) \rangle_c}$ from (17) and (18). In the region where $\langle N_3(n^t) \rangle > 100$, the curve in **Figure 2** fits perfectly to the functional form

$$N_3^{cl}(n^t) := \langle N_3(n^t) \rangle = N_4 \frac{3}{4\omega N_4^{1/4}} \cos^3 \left(\frac{n^t}{\omega N_4^{1/4}} \right), \quad (19)$$

where ω depends on κ_0 and Δ . Despite the fact that no background geometry enters into the path integral, a background volume profile appears to emerge. It is identical to a (Euclidean) de Sitter universe volume profile and the configurations created by the MC simulations can be viewed as quantum geometries that

fluctuate around this background. While this is very interesting¹³, our main question here is whether we can use (17) and (18) to follow a path in the bare coupling constant space (κ_0, Δ) toward a UV fixed point in the same way as for the ϕ^4 -theory. More precisely, we want to identify physical observables. Since we can perform the MC simulations for various finite volumes N_4 , we want to use finite-size scaling to identify a possible UV fixed point.

A few starting remarks are in order. We have replaced a real field $\phi(n)$ with $1(n)$ in (16) and (18). Thus we cannot necessarily expect an exponential fall-off and a corresponding correlation length ξ like in (5). However, in the solvable two-dimensional models of both CDT and DT one does find an exponential fall-off related to the field $1(n)$ [45, 46]. This fall-off is related to the cosmological constants of the models, and the “mass” goes to zero with a vanishing cosmological constant. In four-dimensional gravity we expect massless gravitons (and thus maybe no exponential fall-off), but as shown in [41], there are terms in an effective continuum action of quantum gravity, which can lead to such an exponential fall-off, e.g., the non-local term

$$\Gamma^{NL} = -\frac{b^2}{96\pi G} \int d^4x \sqrt{g} R \frac{1}{\Delta_g^2} R, \quad (20)$$

where Δ_g is the scalar Laplacian defined in the geometry related to the metric $g_{ij}(x)$. Expanding the fluctuations to quadratic order around flat spacetime, b will appear as a mass term. We might observe such terms in case of toroidal topology, where the fluctuations we observe seem to be around flat spacetime. If the spatial topology is S^3 , the contributions from a term like (20) would mix with contributions from the cosmological term via the curvature of the background geometry used for S^3 . Thus, there might be a number of sources for an exponential fall-off of the (spatial) volume-volume correlator.

Equation (19) shows that for fixed κ_0 and Δ we have a well-defined scaling with N_4 . The same is true for the volume-volume correlator, where the MC data (for spatial topology S^3) is consistent with the formula

$$\langle N_3(n_1^t) N_3(n_2^t) \rangle_c = \gamma^2 N_4 F\left(\frac{n_1^t}{\omega N_4^{1/4}}, \frac{n_2^t}{\omega N_4^{1/4}}\right), \quad (21)$$

$$\sqrt{\langle N_3^2(n^t) \rangle_c} = \gamma \sqrt{N_4} G\left(\frac{n^t}{\omega N_4^{1/4}}\right). \quad (22)$$

Here γ depends on κ_0 and Δ . F is some scaling function which only depends slightly on κ_0 and Δ , and $G = \sqrt{F}$. Unfortunately, we cannot really use Equation (21) to extract a correlation length ξ independent of N_4 . If any ξ could be associated with the correlator, it would already be “maximal,” i.e., of order $\omega N_4^{1/4}$, the whole average time-length of the universe, without any fine-tuning of the bare coupling constants. A condition like (11) then

becomes empty¹⁴ and we thus have to find other measures to keep continuum physics constant, when taking the lattice spacing to zero.

Figure 2 is for a specific value of N_4 and, as remarked above, we already have a scaling for fixed values of the bare coupling constants κ_0 and Δ . Equations (19), (21), and (22) are these scaling formulas. We see that the height of $\langle N_3(n^t) \rangle$ will grow as $N_4^{3/4}$, while the fluctuations only grow as $N_4^{1/2}$. For fixed (κ_0, Δ) in phase C_{ds} , the fluctuations will thus decrease relative to the volume for $N_4 \rightarrow \infty$. The interpretation of this is that for fixed κ_0 and Δ the limit $N_4 \rightarrow \infty$ is one where $V_4 = N_4 a^4$ goes to infinity while a stays constant.

An attempt to replace the ϕ^4 -observables (m_R, λ_R) with geometrical observables is the following. The *physical* volume of spacetime is $V_4 = N_4 a^4$. Similarly, the volume of a time-slice is $V_3(t) = \langle N_3(n^t) \rangle a^3$, $t = n^t a$. Let us attempt to take a continuum limit where V_4 and $V_3(t)$ are *finite*, while $N_4 \rightarrow \infty$. Such a limit would force $a \rightarrow 0$, which is what we want. How do we ensure that $N_4 \rightarrow \infty$ forces $a \rightarrow 0$? For the scalar field we had the correlation length ξ and m_R which monitored $a(\xi)$. Here we will insist that the relative size of $V_3(t)$ and the quantum fluctuations $\delta V_3(t)$ stay unchanged if we scale $N_4 \rightarrow \infty$. This would be expected if $V_3(t)$ can be interpreted as a physical continuum three-volume in the limit $N_4 \rightarrow \infty$. Thus we require that (for sufficiently large N_4)

$$\frac{\sqrt{\langle (N_3^2(n^t))_c \rangle_{N_4}}}{\langle N_3(n^t) \rangle_{N_4}} = \frac{\delta V_3(t)}{V_3(t)}, \quad \text{independent of } N_4. \quad (23)$$

From (19) and (22) this requirement implies that

$$\omega(\kappa_0, \Delta) \gamma(\kappa_0, \Delta) = \text{const. } N_4^{1/4}. \quad (24)$$

ω and γ are constants independent of N_4 for fixed κ_0 and Δ . Starting out with some (κ_0, Δ) and a four-volume $N_4(0)$ in phase C_{ds} , and then increasing N_4 will force us to change (κ_0, Δ) if (24) is to be fulfilled. Continuing to increase N_4 will trace out a path in the (κ_0, Δ) -coupling constant plane, and the endpoint for $N_4 \rightarrow \infty$ will be a candidate for a UV fixed point.

The coupling constant flow related to (24) was investigated in [47] and the conclusion was like in the ϕ^4 -case. There seems to be no starting point in phase C_{ds} which leads to a curve where $N_4 \rightarrow \infty$. In fact, while both ω and γ change somewhat when changing the coupling constants, their product does not change much. We conclude that this particular renormalization group analysis has not led us to a UV fixed point candidate. But even stronger, Equation (23) expresses the simple requirement that if we have a continuum universe of a certain size, it will have quantum fluctuations of a certain size. However, our model does not meet this requirement when we relate discretized and continuum variables in the most natural and simple-minded way.

There are a number of possible interpretations of this result. Firstly, on the technical side, since the analysis in [47] was made,

¹³The dominant “semiclassical background geometries” seem to depend on the topology of space (as do classical solutions of Einstein’s equations). If we change the topology of space from S^3 to T^3 , the dominant volume profile will be constant. However, the phase diagram is unchanged [40, 44].

¹⁴The situation might be different in the case of toroidal spatial topology, where the time extent of the universe is not dynamically adjusted to the total four-volume N_4 . This is presently under investigation.

we have obtained a better understanding of the phase diagram. At the time of the analysis in [47] phase C_{ds} was assumed to extend all the way down to phase B . Currently the most promising phase transition line for a higher-order transition is the C_{ds} - C_b transition line, and the endpoint of that transition line in particular. We now have a chance to approach this fixed point in an easier way using toroidal spatial topology. This is presently being explored. Secondly, we may be thinking of the quantum universe in the wrong way. In our reasoning we are applying some standard logic related to fluctuations to a macroscopic quantity like the three-volume of the universe. Maybe that is wrong. On the other hand, we have tried to estimate the size of the quantum universes by constructing the effective action for the three-volume, and comparing with mini-superspace expressions. The universes are estimated to have linear sizes not larger than 20 Planck lengths [3] for the N_4 -values we are using. Therefore, a picture like that of **Figure 2** should be correct: for a continuum universe of this size we expect significant quantum fluctuations. Thirdly, although we tried to emulate the flat-space quantum field theoretic way of looking for UV fixed points, we have not (yet) been able to identify a divergent correlation length, which is a crucial ingredient of the Wilsonian approach to quantum field theory and the renormalization group. It is the source of universality and dictates the way one moves from the regularized quantum field theory on the lattice to the continuum quantum field theory. There seems no reason that there should not be massless long-range excitations in a theory of gravity related to a universe like ours. However, it is much less clear what kind of excitations one would observe in a quantum universe of the size of 20 Planck lengths, and to what extent one can talk about scaling the lattice spacing a to zero compared to the Planck length. The estimates referred to above led to a lattice spacing of twice the Planck length. If these estimates can be trusted, our a is far from sub-Planckian. However, it is possible that the global conformal mode of the metric, whose effective behavior we are studying, is not well-suited for extracting a correlation length. In other words, it may not be possible to push the lattice spacing to a sub-Planckian region while maintaining an interpretation that is based on notions which are closely related to classical geometry, like “volume profiles.” The question of whether there is a correlation length in non-perturbative quantum gravity and whether its divergence relates to a UV phase transition therefore leads us to an even more basic question: what is “length” in quantum gravity, when in the path integral one integrates over the geometries that classically define the length? We turn to a discussion of this question in the next section.

4. QUANTUM LENGTH

In ordinary quantum field theory, lengths and distances are defined with respect to a (flat) spacetime metric, which is part of the fixed background structure. One simply has

$$\langle \phi(x)\phi(y) \rangle = f(|x - y|), \quad (25)$$

where $|x - y|$ is the invariant spacetime distance between the spacetime points x and y . When trying to define correlation functions like (25) rigorously, e.g., on the lattice as in (2), one may

have to rescale fields, coupling constants and the lattice spacing in order to obtain a finite continuum result, but the geodesic distance $|x - y|$ in (Euclidean) spacetime is not touched. The situation is similar when we generalize to quantum field theory on a fixed, *curved* background. The analog of the two-point function (25) will still depend on the geodesic distance between x and y , but also on other coordinate-independent quantities involving the fixed spacetime geometry.

Moving on to quantum gravity, the path integral will contain an integration over geometries, in addition to the integration over field configurations. For these geometries, the geodesic distance between x and y will vary, as will the curvature invariants associated with a given geometry. In the absence of any a priori given background geometry, the only way in which a dependence on a distance (or other geometric invariants) could reappear in some propagator would be with respect to some “effective” or “emergent” geometry, generated by the quantum dynamics, and accompanied by quantum fluctuations¹⁵. The propagator should also reflect this to some approximation, depending on the size the geometric fluctuations. Such an “emergence” of a class of dominant geometries is what one observes in the MC simulations of CDT¹⁶ in phase C_{ds} .

For reference, let us examine the situation in two-dimensional quantum gravity, which we have argued is in some sense maximally “quantum.” Suppose we have a universe with the topology of a cylinder, where we fix the lengths of the two boundaries to L and the area (the spacetime volume) to A . For suitable values of L and A there will be a “minimal-area surface” with constant negative curvature between the two boundaries. Could this nice, classical geometry be the one that dominates the path integral, such that the integration over geometries could be approximated by considering only small fluctuations around it? It turns out that the answer is no. However, *if* two-dimensional gravity is coupled to a conformal field theories with a large negative central charge the answer is yes.

Whichever the case may be in four dimensions, some invariant notation of length or distance is clearly needed in the quantum theory to construct any meaningful propagators or n -point functions. Again, two-dimensional quantum gravity may provide some guidance for how to proceed. When discussing a quantum-gravitational generalization of (25), we used coordinates x and y to label spacetime points, while emphasizing the arbitrariness of this choice. In the context of non-perturbative quantum gravity it is more convenient to base the construction of invariant correlators on the notion of distance instead. Thus, we integrate only over geometries where x and y are separated precisely by a geodesic distance D . Equivalently, for a given geometry and a given x , we average in the matter functional integral over all points y which are separated a given distance D from x , and *then* integrate over all geometries. In this way we obtain a correlation function $G_\phi(x, D)$, which

¹⁵One can of course choose a fictitious “background” geometry and expand everything around it. But nothing can depend on this geometry, which implies that distances defined with respect to it will be as fictitious as the geometry itself.

¹⁶To be precise, the emergence of classical behavior refers only to those aspects of geometry that are captured by the behavior in proper time t of the three volume $V(t)$.

explicitly depends on what one could call the *quantum distance* D . Generalizing (2), its definition is

$$G_\phi(x, D) := \frac{\int \mathcal{D}[g_{\mu\nu}] e^{-S[g]} \int \mathcal{D}_g \phi e^{-S[g, \phi]} \int dy \sqrt{g(y)} \phi(x) \phi(y) \delta(D_g(x, y) - D)}{\int \mathcal{D}[g_{\mu\nu}] e^{-S[g]} \int \mathcal{D}_g \phi e^{-S[g, \phi]}}, \quad (26)$$

where $D_g(x, y)$ denotes the geodesic distance between x and y in the geometry with metric $g_{\mu\nu}(x)$. Even in two-dimensional quantum gravity, the expression (26) is too complicated to compute analytically for a scalar field $\phi(x)$. However, for $\phi(x) = 1$ – the “trivial” field we considered for CDT in section 3 – one can in the DT formalism write down a lattice version of (26), solve analytically for the lattice propagator, and take the continuum limit where the lattice spacing goes to zero [45, 48]. After the continuum limit has been taken one finds

$$G_1(x, D) \propto D^3 F(D/V^{1/4}), \quad F(x) = 1 + O(x^4), \quad (27)$$

if one fixes the spacetime volume to be V . Equation (27) shows that the quantum length D is very “quantum,” since it has an anomalous dimension, which moreover is related to the fractal dimension 4 of the quantum spacetime. If we set $\phi(y) = 1$ in (26), the integral over y is the total volume (in this case the total length) of all points at geodesic distance D from x , forming a “spherical shell” $S_x(D)$. For a smooth classical d -dimensional geometry we expect $S_x(D) \propto D^{d-1}$ for D sufficiently small. Here we find instead

$$G_1(x, D) = \langle S_x(D) \rangle \propto D^3 \quad \text{for } D \ll V^{1/4}, \quad (28)$$

which shows that the fractal dimension of two-dimensional Euclidean quantum spacetime is 4. The important point here is that the distance or length has become a quantum operator, which is natural in a theory of quantum geometry. Since the geodesic distance is a very complicated non-local quantity, it is remarkable that the quantum average of this quantity, defined in Equation (26) for $\phi(x) = 1$, has a non-trivial well-defined scaling dimension. However, its noncanonical value implies a nonstandard scaling behavior of the quantum geodesic distance D in the regularized DT-lattice theory for a spacetime volume $V = N_2 a^2$, where N_2 counts the number of triangles in the triangulation. Namely, in a continuum limit where V stays finite and $N_2 \rightarrow \infty$ (and thus $a \rightarrow 0$), D on average involves only a number of links $\propto 1/\sqrt{a}$. This is very different from the generic situation in the ϕ^4 -theory, where linear distance in the continuum limit would scale $\propto 1/a$. In the ϕ^4 -lattice scenario a behavior $\propto 1/\sqrt{a}$ would correspond to zero length in the continuum limit. However, it is possible and *nontrivial* on the DT lattices because of the fractal structure of a generic triangulation.

Another related example where distances become quantum comes from bosonic string theory, although in a string-theoretical context it is usually not presented this way. Bosonic string theory in d dimensions can be viewed as a theory of two-dimensional quantum gravity with coordinates (ω_1, ω_2) on

the world sheet, coupled to d scalar fields $X^i(\omega_1, \omega_2)$, taking values in the target space R^d . Let us consider closed strings, and the so-called tree-amplitude for the two-point function. This is calculated by considering two infinitesimal loops separated by a distance D in target space, summing in the path integral over all surfaces $X^i(\omega_1, \omega_2)$ with cylinder topology in target space, with these loops as boundaries, weighted by the string action. One way to carry out this calculation is to find the classical string solution $X_{\text{cl}}^i(\omega_1, \omega_2)$ with the given boundaries, perform a split

$$X^i = X_{\text{cl}}^i + X_{\text{q}}^i \quad (29)$$

and integrate over the quantum fields X_{q}^i . Just like in standard quantum field theory, this integration will in general require a regularization. In addition, to obtain a finite effective action, X_{cl}^i will need a wave function renormalization. However, the distance D appears as a parameter in X_{cl}^i and the wave function renormalization of X_{cl}^i in reality becomes a renormalization of the distance D in target space, as shown in detail in [49, 50]. Like in the case of pure two-dimensional quantum gravity mentioned above, the need for a renormalization of the distance D can be related to a fractal structure, in this case, the fractal structure of the random surfaces embedded in R^d [49, 50].

The lesson to take away from this discussion is that unless some yardstick emerges alongside a “dominant” geometry in a non-perturbative path integral over geometries, or is provided by hand through suitable boundary conditions, a notion of (quantum) distance must be introduced in the Planckian regime. As the above examples illustrate, such notions can be found, but will typically behave nonclassically or even scale anomalously relative to the volume. Clearly, this needs to be taken into account when constructing and interpreting propagators and other geometric observables, for example, in a renormalization group analysis near a UV fixed point. Whether such a quantum length possesses a large-scale classical limit or can be promoted to a “physical” length needs to be investigated, and is certainly not a given.

5. DISCUSSION

In the asymptotic safety scenario, quantum gravity is defined as an ordinary quantum field theory at a UV fixed point. We have shown here how one can in principle use computer simulations to search for such a fixed point, in close analogy with the search for a UV fixed point in a four-dimensional ϕ^4 -theory. The framework of CDT quantum gravity is well suited to try and verify the findings of the functional renormalization group analysis in the continuum independently. One particular correlation function, that of the spatial volume profile (equivalently, the global conformal mode of the spatial metric), has already been studied extensively, but so far no indication of a UV fixed point has been seen. There could be many reasons for this. Despite the compelling evidence from a body of work in the continuum theory [35–39, 51]¹⁷, such a fixed point may not

¹⁷The calculation reported in [51] seems in particular to be close in spirit to the CDT approach.

exist, and the asymptotic safety scenario not realized as a way to define quantum gravity beyond perturbation theory. Defining trajectories of constant physics near the Planck scale through an observable that describes the global shape of the universe may be a wrong choice. As emphasized in [47], at the very least one would like to repeat the analysis in terms of other, more local observables. A new candidate may be the quantum Ricci curvature [52, 53], currently under investigation. Our assessment that the lattice version of $\delta V_3(t)$ is too small and does not increase sufficiently when we move toward the C_{dS} - C_b phase transition line may be based on our incomplete understanding of how quantum length and volume behave near the Planck scale. Also, maybe we are not using an action which is general enough to localize the UV fixed point? We are using the Regge discretized version of the Einstein-Hilbert action with one additional deformation parameter Δ . From a continuum point of view one could think of adding all kind of higher curvature terms to the action. We have not done that for two reasons. The firstly, in the formalism of CDT there are no simple natural candidates for the higher curvature terms. The geometric Regge prescription only exists for the R -term, and attempts to put in by hand arbitrary *ad hoc* generalizations have not worked (see [54] for old attempts). Secondly, the functional renormalization group analysis sees clear evidence for a fixed point even if one truncates the effective action to contain only the Einstein-Hilbert term. From the lattice perspective the interpretation of this is that one should be able to get quite close to the fixed point by finetuning the two bare coupling constants κ_0 and Δ , even if we might not be able to reach all the way to the fixed point. However, it is disappointing that we have not really seen much sign of an *approach* to a fixed point, as we would have expected from the continuum renormalization group calculations. Another possibility that may be worthwhile exploring is that the quantum-geometric phase transitions in CDT are different from the more conventional Landau-type phase transitions where the Wilsonian renormalization group philosophy works so well. In particular, the C_{dS} - C_b

transition may share traits with the topological phase transitions occurring in condensed matter physics [55, 56]. The transition is associated with the appearance of a localized structure in an otherwise seemingly homogeneous and isotropic universe. It was overlooked for a long time, since the order parameters that exhibit the transition are also of a non-standard nature with a strong topological flavor [33]. In addition, one has observed long auto-correlation times in the MC simulations at the C_{dS} - C_b transition, presumably caused by major rearrangements of the internal connectivity of the triangulations in connection with the symmetry breaking. This is again reminiscent of some features seen in topological phase transitions, some of which also have no clear divergent correlation lengths associated with them. How to relate such transitions to a UV fixed point in quantum gravity is an interesting challenge.

DATA AVAILABILITY STATEMENT

The datasets generated for this study are available on request to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Towards a Unitary, Renormalizable, and Ultraviolet-Complete Quantum Theory of Gravity

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For any fundamental quantum field theory, unitarity, renormalizability, and relativistic invariance are considered to be essential properties. Unitarity is inevitably connected to the probabilistic interpretation of the quantum theory, while renormalizability guarantees its completeness. Relativistic invariance, in turn, is a symmetry that derives from the structure of spacetime. So far, the perturbative attempt to formulate a fundamental local quantum field theory of gravity based on the metric field seems to be in conflict with at least one of these properties. In quantum Hořava gravity, a quantum Lifshitz field theory of gravity characterized by an anisotropic scaling between space and time, unitarity and renormalizability can be retained while Lorentz invariance is sacrificed at high energies and must emerge only as approximate symmetry at low energies. This article reviews various approaches to perturbative quantum gravity, with a particular focus on recent progress in the quantization of Hořava gravity, supporting its theoretical status as a unitary, renormalizable, and ultraviolet-complete quantum theory of gravity.

Keywords: quantum gravity, perturbation theory, renormalization, Lorentz violation, anisotropic scaling
Hořava-Lifshitz gravity

1. INTRODUCTION

The search for a consistent quantum theory of gravity can be dated back almost 90 years to the work of Rosenfeld [1]. Since then, many different approaches have been suggested, each with its own assumptions, predictions (if any), and limitations; see [2] for an overview. Prominent roads to quantum gravity include canonical approaches, such as quantum geometrodynamics [3, 4] and loop quantum gravity [5–8], discrete approaches, such as causal dynamical triangulations [9–11], and unified approaches, such as string theory [12–16].

In this review, I restrict the discussion to *local* field theories, in which gravity is fundamentally described by the *metric field*. For non-local (infinite-derivative) theories of gravity, (see e.g. [17–26]), and for non-metric theories of gravity, (see e.g. [27–32]). In view of the tremendous success of perturbative quantum field theory in different areas of physics, including the standard model of particle physics, it seems natural to quantize gravity within this highly developed and strongly tested unified framework along with the fundamental interactions between the matter fields. For most of this review, I will focus on the *covariant perturbative* approach to quantum gravity. Much of the progress in this direction can be attributed to Bryce S. DeWitt, who pioneered the field and set the standards for most of its developments in the subsequent decades [33–35].

While the direct approach to quantizing general relativity perturbatively is considered a failure because of its non-renormalizability in the strict sense [36, 37], the perturbative quantization and renormalization can be consistently carried out when general relativity is treated as an effective field

theory [38–40]. However, by construction the effective description breaks down at a finite energy scale and therefore does not extend to the arbitrarily high energies required for a fundamental theory of quantum gravity. In this respect, the non-perturbative asymptotic safety program towards quantum gravity might offer a solution in providing a consistent ultraviolet completion [41–44]. A different strategy, which retains the perturbative treatment, is based on the quantization of modifications of general relativity. Quadratic gravity, the extension of the Einstein-Hilbert action by all quadratic curvature invariants, is a perturbatively renormalizable quantum theory of gravity [45]. While the higher derivatives in quadratic gravity improve the ultraviolet behavior, relativistic invariance necessarily implies the inclusion of higher time derivatives, which in turn results in an enlarged particle spectrum, including a massive spin-2 ghost. At the classical level, the presence of the ghost leads to runaway solutions, known as Ostrogradsky instability [46]. At the quantum level, within the usual quantization prescription, the ghost was found to lead to a violation of unitarity [45]. Recent proposals, which involve different quantization prescriptions for the ghost, preserve unitarity but instead lead to a violation of micro-causality [47, 48].

In view of these problems, it has been suggested to explore the consequences of the assumption that Lorentz invariance is not a fundamental symmetry but only emerges as an approximate symmetry at low energies. In this way, higher spatial derivatives can be introduced to tame the ultraviolet divergences, while retaining only second-order time derivatives to avoid the problems associated with the occurrence of higher-derivative ghosts. The breaking of relativistic invariance at a fundamental level is naturally realized in *Lifshitz theories* by an anisotropic scaling between space and time [49, 50].

After a brief overview of various relativistic covariant approaches to quantum gravity, I will review several aspects of the Lifshitz theory of gravity, *Hořava gravity* [50], in $D = 2 + 1$ and $D = 3 + 1$ dimensions, including the consequences of the reduced invariance group of foliation-preserving diffeomorphisms, the geometrical formulation in terms of Arnowitt-Deser-Misner variables, the phenomenological implications of the additional propagating gravitational scalar degree of freedom, and the current status of the experimental constraints. I discuss the quantization of projectable Hořava gravity, a particular version of Hořava gravity in which the lapse function is not a propagating degree of freedom. I will also sketch the proof that projectable Hořava gravity is a perturbatively renormalizable quantum theory of gravity [51, 52] and report recent results on its renormalization group flow [53, 54].

The article is structured as follows. In section 2, I introduce the general formalism for the perturbative quantization of local field theories. In section 3, I summarize the essential properties of general relativity and the major drawback of its perturbative quantization: non-renormalizability. In section 4, I briefly comment on the status of general relativity as an effective field theory. In section 5, I discuss several aspects of the asymptotic safety conjecture in the context of gravity and its status as a possible ultraviolet-complete scenario for a quantum

theory of gravity. In section 6, I review the perturbatively renormalizable theory of quadratic gravity and discuss the ghost problem. In section 7, I present various aspects of the classical theory of Hořava gravity in $D = 2 + 1$ and $D = 3 + 1$ dimensions. In section 8, I discuss the perturbative quantization of projectable Hořava gravity, its perturbative renormalizability, and its status as an ultraviolet-complete theory. Finally, I conclude in section 10 with a short summary and a brief outlook on important further steps towards a unitary, renormalizable, and ultraviolet-complete quantum theory of gravity in $D = 3 + 1$ dimensions.

2. PERTURBATIVE QUANTUM FIELD THEORY: GENERAL FORMALISM

Consider a *local* field theory, which is defined by the action functional S ,

$$S[\phi] = \sum_n \int d^D X c_n \mathcal{O}_n(\phi, \partial). \quad (1)$$

Locality means that the operators $\mathcal{O}_n(\phi, \partial)$ are functions of a *finite* number of derivatives (including no derivative) of the generalized field(s) $\phi^i = \phi^A(x)$ evaluated *at the same point* x . The operators $\mathcal{O}_n$ are restricted by the symmetries of S . The c_n are coupling constants characterizing the strength of the interaction associated with the operator $\mathcal{O}_n$ ¹. The main object in the quantum field theory (QFT) is the *quantum effective action* Γ .

2.1. Perturbation Theory

The starting point for the formal derivation of the Euclidean effective action is the partition function Z , which is defined by the functional integral over the field configurations ϕ^i and is a functional of the external source J_i ,

$$Z[J] := e^{-W[J]} = \int \mathcal{D}\phi e^{-(S[\phi] + J_i \phi^i)}. \quad (2)$$

The mean field ϕ^i is defined as the quantum average in the presence of the source J_i ,

$$\phi^i := \langle \phi^i \rangle_J = \frac{\delta W[J]}{\delta J_i}. \quad (3)$$

The quantum effective action Γ is defined as the functional Legendre transformation of the Schwinger functional W ,

$$\Gamma[\varphi] := W[J] - \varphi^i J_i. \quad (4)$$

Combining (2) and (3) yields the functional integro-differential equation²

$$e^{-\Gamma[\varphi]} = \int \mathcal{D}\phi e^{-\{S[\phi] - (\varphi^i - \phi^i) \Gamma_{,i}[\varphi]\}}. \quad (5)$$

¹I use the ultra-condensed DeWitt notation, in which the generalized index $i = \{A, X\}$ of a generalized field $\phi^i = \phi^A(X)$ encompasses the discrete bundle index A and the continuous spacetime point X . Summation over i implies summation over A as well as integration over X , i.e., $\phi_i \phi^i = \int d^D X \phi_A(X) \phi^A(X)$.

²Post-fix notation with indices separated by a comma is used to denote functional derivatives with respect to the argument, e.g., $\Gamma_{,i} = \delta \Gamma[\varphi] / \delta \varphi^i$.

Equation (5) provides the starting point for the perturbative expansion of Γ (reinserting powers of $\hbar$),

$$\Gamma[\varphi] = S[\varphi] + \hbar\Gamma_1[\varphi] + \hbar^2\Gamma_2[\varphi] + \mathcal{O}(\hbar^3). \quad (6)$$

The diagrammatic representation of the expansion (6) is given in terms of vacuum diagrams in which the number of loops corresponds to the power of $\hbar$ in (6), as shown in **Figure 1**.

In the background field method (BFM), ϕ^i is decomposed into a background field $\bar{\phi}^i$ and a linear perturbation $\delta\phi^i$,

$$\phi^i = \bar{\phi}^i + \delta\phi^i. \quad (7)$$

The first two orders of the expansion (6) correspond to the vacuum diagrams shown in **Figure 1**:

$$\begin{aligned} \Gamma_1 &= \frac{1}{2} \text{Tr} \ln F_{ij}, \\ \Gamma_2 &= \frac{1}{8} G^{ij} S_{ijkl} G^{kl} + \frac{1}{12} S_{ijk} G^{i\ell} G^{jm} G^{kn} S_{\ell mn}. \end{aligned} \quad (8)$$

Here, Tr is the functional trace, F_{ij} is the fluctuation operator, and the Green's function G^{ij} (propagator) is its inverse, such that

$$F_{ij} G^{ik} = -\delta_i^k. \quad (9)$$

The operator F_{ij} , which propagates the linear perturbations $\delta\phi^i$ on the background $\bar{\phi}^i$, is defined as the Hessian of S ,

$$F_{ij}(\bar{V}) := S_{,ij}|_{\phi=\bar{\phi}}. \quad (10)$$

The covariant derivative ∇_μ defines the commutator ("bundle") curvature

$$\mathcal{R}_{\mu\nu}^i{}_j \phi^j := [\nabla_\mu, \nabla_\nu] \phi^i. \quad (11)$$

The effective action is the generating functional of off-shell one-particle-irreducible (1PI) n -point correlation functions

$$\langle \phi_{i_1}, \dots, \phi_{i_n} \rangle = \Gamma_{,i_1, \dots, i_n}. \quad (12)$$

In particular, for $J_i = 0$, the mean field $\varphi^i = \langle \phi \rangle$ is the solution of the quantum effective equations of motion

$$\Gamma_{,i} = 0. \quad (13)$$

Physical observables that derive from the S-matrix of scattering amplitudes are calculated from the off-shell correlation functions (12) via the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [55].

2.2. Gauge Theories

In gauge theories, different field configurations that correspond to the same physical state are related by an infinitesimal gauge transformation

$$\phi_\varepsilon^i := \delta_\varepsilon \phi^i = R_\alpha^i \varepsilon^\alpha. \quad (14)$$

The $R_\alpha^i(\phi)$ are the generators of the gauge transformations, and ε^α is the infinitesimal gauge parameter³. For linearly realized symmetries (considered here), $R_{i,jk}^\alpha = 0$. For gauge algebras that close off-shell, the generators satisfy

$$R_{\alpha,j}^i R_\beta^j - R_{\beta,j}^i R_\alpha^j = R_\gamma^i C_{\alpha\beta}^\gamma. \quad (15)$$

The $C_{\alpha\beta}^\gamma$ are the structure functions (here assumed to be field independent, $C_{\alpha\beta,i}^\gamma = 0$) and satisfy the Jacobi identity

$$C_{\alpha\beta}^\epsilon C_{\epsilon\gamma}^\delta + C_{\gamma\alpha}^\epsilon C_{\epsilon\beta}^\delta + C_{\beta\gamma}^\epsilon C_{\epsilon\alpha}^\delta = 0. \quad (16)$$

Gauge invariance $\delta_\varepsilon S = 0$ of the action (1) implies the Noether identity

$$S_{,i} R_\alpha^i = 0. \quad (17)$$

Differentiation of (17) shows that the fluctuation operator (10) for gauge theories is *degenerate* (on shell $S_{,i} = 0$),

$$F_{ij} R_\alpha^i = 0. \quad (18)$$

The gauge degeneracy $\text{Det}(F_{ij}) = 0$ prevents the construction of the inverse $(F^{-1})^{ij}$, and the associated Green's function G^{ij} does not exist. In order to break the gauge degeneracy, a gauge-breaking action must be added:

$$S_{\text{gb}} = \chi^\alpha O_{\alpha\beta}(\bar{V}) \chi^\beta. \quad (19)$$

The background covariant gauge condition $\chi^\alpha(\bar{\phi}; \delta\phi)$ depends linearly on the difference $\delta\phi^i - \bar{\phi}^i$ between the "quantum field" $\delta\phi^i$, i.e., the variable that is integrated over in the path integral, and the background field $\bar{\phi}^i$. But, like the operator $O_{\alpha\beta}(\bar{\phi}; \bar{V})$, it may have an arbitrary (non-linear) parametric dependence on the background field $\bar{\phi}^i$. In this way, invariance of the effective action under *background gauge transformations* is realized. For the linear split (7), an infinitesimal, linearly realized gauge transformation (14) can be distributed in different ways, in particular by

$$\delta_\varepsilon^Q \bar{\varphi}^i = 0, \quad \delta_\varepsilon^Q \delta\phi^i = R_\alpha^i(\bar{\phi} + \delta\phi) \varepsilon^\alpha,$$

or

$$\delta_\varepsilon^B \bar{\varphi}^i = R_\alpha^i(\bar{\phi}) \varepsilon^\alpha, \quad \delta_\varepsilon^B \delta\phi^i = R_\alpha^i(\delta\phi) \varepsilon^\alpha. \quad (20)$$

While the linearity of the generators ensures that in both cases $\delta_\varepsilon \phi^i = \delta_\varepsilon^Q(\bar{\phi}^i + \delta\phi^i) = \delta_\varepsilon^B(\bar{\phi}^i + \delta\phi^i) = R_\alpha^i(\phi) \varepsilon^\alpha$, the "quantum gauge transformation" δ_ε^Q does not affect the background field $\bar{\phi}^i$ but only the "quantum" field $\delta\phi^i$, whereas for the background gauge transformations δ_ε^B , the transformation (14) is split between the background field and the quantum

³The generalized DeWitt gauge index $\alpha = (a, X)$ is taken from the beginning of the Greek alphabet and not to be confused with indices $\mu, \nu, \dots$ from the tangent bundle.

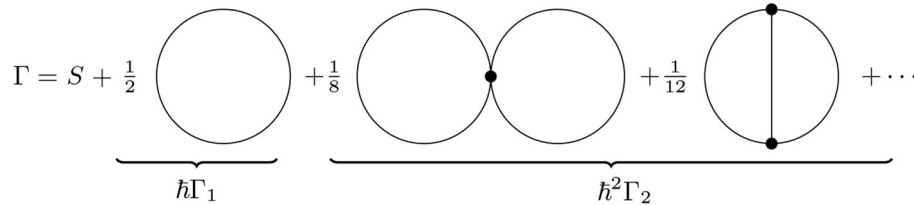


FIGURE 1 | The diagrammatic expansion of the quantum effective action in powers of loops.

field according to (7). The gauge-breaking action (19) must be compensated for by the ghost action

$$S_{\text{gh}} = c_\alpha^* Q_\beta^\alpha c^\beta. \quad (21)$$

The anticommuting independent ghost field c^α and anti-ghost field c_α^* have fermionic statistics. The ghost operator Q_β^α is defined as the variation of the gauge-transformed gauge condition,

$$Q_\beta^\alpha(\bar{\nabla}) := \frac{\delta \chi^\alpha[\phi_\varepsilon^i]}{\delta \varepsilon^\beta}. \quad (22)$$

Summarizing, for gauge theories the partition function (2) generalizes to

$$Z[J] = \text{Det}(O_{\alpha\beta})^{1/2} \int \mathcal{D}[\phi, c, c^*] e^{-\{S_{\text{tot}}[\phi, c, c^*] + J_i \phi^i\}}, \quad (23)$$

with the total action S_{tot} defined as the sum of (1), (19) and (21),

$$S_{\text{tot}}[\phi, c, c^*] = S[\phi] + S_{\text{gb}}[\bar{\phi}; \phi] + S_{\text{gh}}[\bar{\phi}, c, c^*]. \quad (24)$$

In particular, the gauge-fixed fluctuation operator is no longer degenerate and can be inverted:

$$F_{ij}^{\text{gb}}(\bar{\nabla}) := (S + S_{\text{gb}})_{ij} \Big|_{\phi=\bar{\phi}}. \quad (25)$$

The structure of the effective action and the proof of perturbative renormalizability of a local gauge theory are described in more general terms by exploiting the residual non-linearly realized Becchi-Rouet-Stora-Tyutin (BRST) symmetry of the gauge-fixed action [56, 57]. For the application of these methods in the context of general relativity and Yang-Mills theories, (see [58]); for a generalization to non-relativistic theories, (see [52]).

2.3. Functional Traces and the Heat-Kernel Technique

In addition to the abstract formalism presented in section 2, explicit calculations in the perturbative expansion (6) require the evaluation of functional traces, for which the combination of the BFM with heat-kernel techniques provides a manifest covariant and efficient tool⁴. For the connection between the

heat-kernel technique and position space Feynman diagrams in curved spacetime (see e.g., [60, 61]). For an introduction to the background field method, (see [62–64]). For an overview of flat-space Feynman-diagrammatic calculations in momentum space, see e.g., [65], as well as [66] for an introduction to modern on-shell methods. An explicit illustration of the connection between the different techniques is given section 9 in the context of the one-loop divergences for projectable Hořava gravity.

The heat-kernel technique, originally developed in mathematics in the context of asymptotic expansions, partial differential equations, and geometric analysis of the Laplace operator [67–72], has turned out to be a very useful tool in physics also, especially in the context of renormalization in Quantum Field Theory (QFT) on a curved background [33, 60, 73, 74]. Recalling the ultra-condensed DeWitt notation, the (gauge-fixed) fluctuation operator (25) takes the general form $F_{ij}(\bar{\nabla}) = F_{AB}(\bar{\nabla}_{X_A})\delta(X_A, X_B)$. The operator with proper index positions F^A_B , acting on the fluctuation field $\delta\phi^A(X)$, is obtained from F_{AB} by raising the bundle index A with the (ultra-local) configuration space metric C_{AB} ⁵,

$$\mathbf{F}(\nabla) := F^A_B(\nabla) = C^{AC}F_{CB}(\nabla). \quad (26)$$

Inverse powers and the logarithm of the operator (26), which appear in the perturbative expansion (6), are conveniently expressed in terms of the Schwinger integral representation⁶ over “proper time” s ,

$$\frac{1}{F^n} = \int_0^\infty \frac{ds}{(n-1)!} s^{n-1} e^{-s\mathbf{F}}, \quad \ln \mathbf{F} = - \int_0^\infty \frac{ds}{s} e^{-s\mathbf{F}}. \quad (27)$$

⁵If the configuration space of fields $\mathcal{C}$ is viewed as a differentiable manifold, the configuration space metric defines the invariant line element $dS^2 = C_{ij} d\phi^i d\phi^j$. Ultra-locality means that $C_{ij} = C_{AB}\delta(X_A, X_B)$ with C_{AB} involving no derivatives. For $2k$ th-order derivative theories, defined by an action functional (1), the configuration space metric C_{AB} could be defined by the coefficient of the (minimal part of the) highest derivative term in the fluctuation operator $F_{AB} = C_{AB}\Delta^k + \dots$. The inverse is defined via $C^{AC}C_{CB} = \delta_B^A = \mathbf{1}$. The boldface notation is exclusively reserved for matrix-valued operators with proper index positions. Since the content of this section holds for general operators $\mathbf{F}$, no background tensors appear in what follows.

⁶The inverse $\mathbf{F}^{-1}$ of the operator $\mathbf{F}$ is denoted by $1/\mathbf{F}$. It is assumed that $\mathbf{F}$ is positive definite. In the integral relation for the logarithm (27), an (infinite) constant has been neglected. The precise relation can be defined by a regularizing mass damping factor, i.e., by defining $\mathbf{G}(m^2) := \int_0^\infty ds e^{-sm^2} e^{-s\mathbf{F}}$, with the logarithm of $\mathbf{F}$ obtained as a limit, $\ln \mathbf{F} = \lim_{m^2 \rightarrow \infty} [\ln m^2 \mathbf{1} - \int_0^{m^2} d\mu^2 \mathbf{G}(\mu^2)]$.

⁴The heat kernel is in particular very efficient for the extraction of the one-loop divergences. For calculations involving higher loop orders, it is not so well-developed, but see [59].

The *heat kernel* $\mathbf{K}_F(s|X, Y)$ associated with the operator $\mathbf{F}$ formally satisfies the heat equation

$$\begin{aligned} \mathbf{K}_F(s|X, Y) &:= e^{-s\mathbf{F}(\nabla_X)} \delta(X, Y), \\ \left[\frac{\partial}{\partial s} + \mathbf{F}(\nabla) \right] \mathbf{K}_F(s|X, Y) &= 0. \end{aligned} \quad (28)$$

In terms of the heat kernel (28), the one-loop contribution to the effective action (8) takes the form

$$\begin{aligned} \Gamma_1 &= -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} \mathbf{K}_F(s|X, Y) \\ &= -\frac{1}{2} \int_0^\infty \frac{ds}{s} \int d^d x \text{tr} [K_F]^A_B(s|, X, X). \end{aligned} \quad (29)$$

Equation (29) can be viewed as the definition of the functional trace Tr and requires evaluation of the spacetime integral over the internal trace tr of the coincidence limit $y \rightarrow x$ of the matrix-valued two-point kernel $[K_F]^A_B(s|, X, Y)$. Ultraviolet (UV) divergences arise from the lower integration bound in (29), i.e., the $s \rightarrow 0$ limit.

For a *minimal second-order operator* with (positive-definite) Laplacian $\Delta = -g^{\mu\nu} \nabla_\mu \nabla_\nu \mathbf{1}$ and potential $\mathbf{P}$,

$$\mathbf{F}(\nabla) = \Delta + \mathbf{P}, \quad (30)$$

there is an ansatz for the associated heat kernel at *non-coincident points*, introduced in [33]:

$$\mathbf{K}(s|X, Y) = \frac{g^{1/2}(y)}{(4\pi s)^{d/2}} \mathfrak{D}^{1/2}(X, Y) e^{-\frac{\sigma(X, Y)}{2s}} \mathfrak{Q}(s|X, Y). \quad (31)$$

Synge's world function $\sigma(X, Y)$ is a bi-scalar [75], which measures one-half of the geodesic distance squared between the points X and Y , and $\mathfrak{D}(X, Y)$ is the de-densitized Van Vleck determinant, a bi-scalar defined as

$$\mathfrak{D}(X, Y) := g^{-1/2}(X) \det \left(\frac{\partial^2 \sigma(X, Y)}{\partial X^\mu \partial Y^\nu} \right) g^{-1/2}(Y). \quad (32)$$

The bi-tensor $\mathfrak{Q}$ can be obtained in the form of an asymptotic expansion in proper time,

$$\begin{aligned} \mathfrak{Q}(s|X, Y) &:= \sum_{n=0}^{\infty} \mathbf{a}_n(X, Y) s^n, \\ \mathbf{a}_n(X, X) &\propto \underbrace{\nabla_X \dots \nabla_X}_{2p} \underbrace{\mathfrak{R} \dots \mathfrak{R}}_m, \quad n = p + m. \end{aligned} \quad (33)$$

The *Schwinger-DeWitt* (SDW) *coefficients* at coincidence points $\mathbf{a}_n(X, X)$ are local functions of the background fields, and the generalized curvature $\mathfrak{R}$ encompasses three different types of *background curvature*, $\mathfrak{R} = \{R_{\mu\nu\rho\sigma} \mathbf{1}, \mathcal{R}_{\mu\nu}, \mathbf{P}\}$.

For the minimal second-order operators (30), a closed-form algorithm for calculating the one-loop divergences Γ_1^{div} is available. In general, dimensional regularization annihilates all power-law divergences and is sensitive only to logarithmic divergences, which are isolated as poles in dimension

TABLE 1 | Coincidence limits required for the calculation of $\mathbf{a}_2(X, X)$.

$\mathfrak{R}$	$\mathfrak{R}^0$	$\mathfrak{R}^{1/2}$	$\mathfrak{R}$	$\mathfrak{R}^{3/2}$	$\mathfrak{R}^2$
σ	$\nabla^2 \sigma$	$\nabla^3 \sigma$	$\nabla^4 \sigma$	$\nabla^5 \sigma$	$\nabla^6 \sigma$
$\mathfrak{D}$	$\mathfrak{D}$	$\nabla \mathfrak{D}$	$\nabla^2 \mathfrak{D}$	$\nabla^3 \mathfrak{D}$	$\nabla^4 \mathfrak{D}$
$\mathbf{a}_0$	$\mathbf{a}_0$	$\nabla \mathbf{a}_0$	$\nabla^2 \mathbf{a}_0$	$\nabla^3 \mathbf{a}_0$	$\nabla^4 \mathbf{a}_0$
$\mathbf{a}_1$			$\mathbf{a}_1$	$\nabla \mathbf{a}_1$	$\nabla^2 \mathbf{a}_1$
$\mathbf{a}_2$					$\mathbf{a}_2$

$\epsilon^{-1} = 2/(4 - D)$. In $D = 4$, the logarithmically UV-divergent part of the one-loop contributions to the effective action (29) for the minimal second-order operator (30) is determined by the coincidence limit of $\mathbf{a}_2(x, x)$ [33],

$$\Gamma_1^{\text{div}} = -\frac{1}{\epsilon} \frac{1}{32\pi^2} \int d^4 X g^{1/2} \text{tr} \mathbf{a}_2(X, X). \quad (34)$$

The coincidence limits of the SDW coefficients $\mathbf{a}_n(x, x)$ can be calculated iteratively by inserting the ansatz (31) into the heat Equation (28), leading to the recurrence relation (for $n \geq 0$),

$$[(n+1) + \sigma^\mu \nabla_\mu] \mathbf{a}_{n+1} = \mathfrak{D}^{-1/2} \mathbf{F}(\nabla) (\mathfrak{D}^{1/2} \mathbf{a}_n) = 0. \quad (35)$$

In order to obtain $\mathbf{a}_2(X, X)$ in this way, the coincidence limits of σ , $\mathfrak{D}$, $\mathbf{a}_0$, $\mathbf{a}_1$, and their derivatives must be calculated. The successive pattern of this calculation is illustrated in **Table 1**.

The coincidence limits of σ , $\mathfrak{D}$, $\mathbf{a}_0$, and their derivatives can be obtained by successive differentiation of the “defining equations” for σ , $\mathfrak{D}$, and $\mathbf{a}_0$,

$$\sigma^\mu \sigma_\mu = 2\sigma, \quad \mathfrak{D}^{-1} \nabla^\mu (\mathfrak{D} \sigma_\mu) = d, \quad \sigma^\mu \nabla_\mu \mathbf{a}_0 = 0, \quad (36)$$

with the “initial conditions” $\sigma|_{y=x} = 0$, $\mathfrak{D}|_{y=x} = 1$, and $\mathbf{a}_0|_{y=x} = \mathbf{1}$. In this way, the coincidence limit of $\mathbf{a}_2(X, Y)$ is found as [33, 60],

$$\begin{aligned} \mathbf{a}_2(X, X) &= \frac{1}{180} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu} - 6\Delta R) \mathbf{1} \\ &+ \frac{1}{2} \left(\mathbf{P}^2 - \frac{1}{6} R \mathbf{P} \right) + \frac{1}{12} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{1}{6} \Delta \mathbf{P}. \end{aligned} \quad (37)$$

For higher-order and non-minimal operators there is no closed-form expression for the one-loop divergences (34) in terms of a single SDW coefficient as for the minimal second-order operator (30). Nevertheless, in [60] a closed algorithm was developed, which reduces the calculation of the one-loop divergences for higher-order and non-minimal operators to the heat kernel of the second-order minimal operator (31) and a few *universal functional traces*,

$$\mathcal{W}_{\mu_1 \dots \mu_p}^{(p,n)} := \nabla_{\mu_1} \dots \nabla_{\mu_p} \frac{1}{\Delta^n} \Big|_{Y=X}^{\text{div}}. \quad (38)$$

The perturbative algorithm underlying the generalized SDW technique relies on the non-degeneracy of the *principal symbol* $\mathbf{D}$ of the operator $\mathbf{F}$. There are, however, important

physical theories for which the principal symbol of the fluctuation operator is degenerate so that the (generalized) SDW algorithm is not directly applicable. In such cases, more general methods are required; see [76–78] for heat-kernel calculations involving operators with degenerate principal part and [79, 80] for operators with Laplacians constructed from an effective (background field-dependent) metric. In the context of Lifshitz theories, the development of heat-kernel techniques for anisotropic operators has recently been initiated [81–83].

3. PERTURBATIVE QUANTUM GENERAL RELATIVITY

3.1. Classical General Relativity

In the theory of general relativity (GR), the gravitational interaction manifests itself *geometrically* as curvature of spacetime and couples *universally* to all fields, which, combined with the attractive nature of gravity, implies that it cannot be shielded. In Einstein's theory, the *dynamical* character of the spacetime geometry is encoded in the dynamics of the metric field $g_{\mu\nu}(X)$. The action functional of GR is the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{M_{\text{P}}^2}{2} \int d^D X \sqrt{-g} (R - 2\Lambda). \quad (39)$$

The action (39) involves the invariant volume element with determinant $g = \det(g_{\mu\nu})$, the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$, and the cosmological constant Λ ⁷. The dynamics of $g_{\mu\nu}$ is determined by Einstein's field equations, obtained from extremizing the total action $S[g, \Psi] = S_{\text{EH}}[g] + S_{\text{M}}[g, \Psi]$ with respect to $g_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = M_{\text{P}}^{-2} T_{\mu\nu}. \quad (40)$$

The energy momentum tensor $T_{\mu\nu}$ derives from the “matter” action $S_{\text{M}}[\Psi]$, with all non-geometrical “matter” fields collectively denoted by Ψ :

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{M}}[g, \Psi]}{\delta g^{\mu\nu}}. \quad (41)$$

Infinitesimal spacetime distances ds measured by the metric field $g_{\mu\nu}$ are defined by the line element

$$ds^2 = g_{\mu\nu}(X) dX^\mu dX^\nu. \quad (42)$$

⁷I work on a D -dimensional (pseudo)-Riemannian manifold $\mathcal{M}$ with local coordinates X^μ , $\mu = 0, 1, 2, 3$, a metric structure $g_{\mu\nu}$ with inverse $g^{\mu\nu}$ defined by $g^{\mu\rho} g_{\rho\nu} = \delta_\nu^\mu$, and the torsion-free metric-compatible Christoffel connection $\Gamma_{\mu\nu}^\rho = g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})/2$, which defines the covariant derivative ∇_μ . I use the following conventions for the Lorentzian signature $\text{sig}(g) = \text{diag}(-1, 1, 1, \dots, 1)$, the Riemann curvature tensor $R_{\mu\sigma\nu}^\rho = \partial_\sigma \Gamma_{\mu\nu}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\sigma}^\rho - \Gamma_{\mu\sigma}^\lambda \Gamma_{\lambda\nu}^\rho$, and the Ricci tensor $R_{\mu\nu} = R_{\mu\rho\nu}^\rho$. I use natural units in which the speed of light c and Planck's constant $\hbar$ are set to one, $c = \hbar = 1$, and Newton's constant G_{N} can be expressed in terms of the reduced Planck mass, $M_{\text{P}} := 1/\sqrt{8\pi G_{\text{N}}}$.

Denoting the *mass dimension* by $[\cdot]_{\text{M}}$ and assigning coordinates X^μ the dimension of a length, $[X]_{\text{M}} = -1$, implies that

$$\begin{aligned} [\partial_\mu]_{\text{M}} &= 1, & [g_{\mu\nu}]_{\text{M}} &= 0, & [R_{\mu\nu\rho\sigma}]_{\text{M}} &= 2, \\ [G_{\text{N}}]_{\text{M}} &= -(D-2), & [\Lambda]_{\text{M}} &= 2. \end{aligned} \quad (43)$$

The Ricci scalar R is the only curvature invariant involving exactly two spacetime derivatives. Except for the cosmological constant, all other curvature invariants necessarily contain higher derivatives. In $D = 4$, these are the only two classically relevant local curvature operators⁸.

The metric field transforms as a rank-(0,2) tensor under D -dimensional coordinate transformations $X^\mu \rightarrow \tilde{X}^\mu(X)$,

$$g_{\mu\nu}(X) \mapsto \tilde{g}_{\mu\nu}(\tilde{X}) = g_{\alpha\beta}(X) \frac{\partial X^\alpha}{\partial \tilde{X}^\mu} \frac{\partial X^\beta}{\partial \tilde{X}^\nu}. \quad (44)$$

The invariance group of GR consists of the D -dimensional diffeomorphisms $\text{Diff}(\mathcal{M})$. The change of the metric field under an infinitesimal diffeomorphism δ_ξ generated by the vector field ξ^μ is given by the Lie derivative of $g_{\mu\nu}$ along ξ^μ :

$$\delta_\xi g_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2g_{\rho(\nu} \partial_{\mu)} \xi^\rho = 2\nabla_{(\mu} \xi_{\nu)}. \quad (45)$$

Round brackets in (45) denote symmetrization among the enclosed indices with unit weight and $\xi_\mu = g_{\mu\rho} \xi^\rho$. Since the gravitational field equations (40) relate geometry with matter, consistency requires that $S_{\text{M}}[g, \Psi]$ must also be invariant under $\text{Diff}(\mathcal{M})$, which implies the “on-shell” covariant conservation of the energy-momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$.

3.2. Quantum GR

In order to establish a connection with the general formalism of perturbative QFT reviewed in section 2, the generalized field ϕ^i in GR is identified with the metric field, $\phi^i \mapsto g_{\mu\nu}(X)$. Comparison of (1) with the Einstein-Hilbert action (39) implies that the operators $\mathcal{O}_i(g, \partial)$ and the coupling constants c_i should be identified as follows:

$$\begin{aligned} \mathcal{O}_1(g) &\mapsto \sqrt{-g}, & c_1 &\mapsto -M_{\text{P}}^2 \Lambda, \\ \mathcal{O}_2(g, \partial) &\mapsto \sqrt{-g} R, & c_2 &\mapsto \frac{M_{\text{P}}^2}{2}. \end{aligned} \quad (46)$$

The particle spectrum of GR is derived by expanding the action (39) to quadratic order in the linear perturbations

$$h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} \quad (47)$$

around a flat background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ ⁹. Absorbing a factor of $M_{\text{P}}/2$ in the definition of $h_{\mu\nu}$, i.e., $h_{\mu\nu} \mapsto 2h_{\mu\nu}/M_{\text{P}}$, and defining

⁸I call an operator O classically relevant if $[O]_{\text{M}} < D$, classically marginal if $[O]_{\text{M}} = D$, and classically irrelevant if $[O]_{\text{M}} > D$.

⁹The particle spectrum of a QFT is usually derived by expanding the action up to quadratic order in the linear perturbation around the vacuum. In relativistic QFTs, the natural vacuum is Minkowski space, which, even in the presence of gravity, could be justified locally by the equivalence principle. Minkowski space is a maximal symmetric space whose isometries are generated by the $D(D+1)/2$ linearly independent Killing vectors, which correspond to the generators of

$h = \eta^{\mu\nu} h_{\mu\nu}$ and $\partial^2 := \eta^{\mu\nu} \partial_\mu \partial_\nu$, upon integration by parts the result reads

$$S_{\text{EH}}^{(2)}|_{\bar{g}=\eta} = \int d^D X \left[h^{\mu\nu} \partial^2 h_{\mu\nu} - h \partial^2 h - 2 h^{\mu\nu} \partial_\nu \partial_\rho h_\mu{}^\rho + 2 h^{\mu\nu} \partial_\nu \partial_\mu h \right]. \quad (48)$$

After Fourier transformation $\partial_\mu \mapsto i P_\mu$ with four momentum P^μ and square $P^2 = \eta_{\mu\nu} P^\mu P^\nu$, the fluctuation operator (10) in momentum space can be expressed in terms of spin-projection operators as

$$F^{\mu\nu,\rho\sigma}(-P^2) = \left[\Pi^{(2)\mu\nu\rho\sigma} - (D-2)\Pi^{(0,ss)\mu\nu\rho\sigma} \right] (-P^2). \quad (49)$$

The spin-projection operators acting on the symmetric rank-two tensor $h_{\mu\nu}$ read

$$\Pi^{(2)}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2} \left(\Pi^{(T)}_{\mu}{}^{\rho} \Pi^{(T)}_{\nu}{}^{\sigma} + \Pi^{(T)}_{\mu}{}^{\sigma} \Pi^{(T)}_{\nu}{}^{\rho} \right) - \frac{1}{D-1} \Pi^{(T)}_{\mu\nu} \Pi^{(T)\rho\sigma}, \quad (50)$$

$$\Pi^{(1)}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2} \left(\Pi^{(T)}_{\mu}{}^{\rho} \Pi^{(L)}_{\nu}{}^{\sigma} + \Pi^{(T)}_{\mu}{}^{\sigma} \Pi^{(L)}_{\nu}{}^{\rho} + \Pi^{(T)}_{\nu}{}^{\rho} \Pi^{(L)}_{\mu}{}^{\sigma} + \Pi^{(T)}_{\nu}{}^{\sigma} \Pi^{(L)}_{\mu}{}^{\rho} \right), \quad (51)$$

$$\Pi^{(0,ss)}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{D-1} \Pi^{(T)}_{\mu\nu} \Pi^{(T)\rho\sigma}, \quad (52)$$

$$\Pi^{(0,ww)}_{\mu\nu}{}^{\rho\sigma} = \Pi^{(L)}_{\mu\nu} \Pi^{(L)\rho\sigma}, \quad (53)$$

$$\Pi^{(0,sw)}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\sqrt{D-1}} \Pi^{(T)}_{\mu\nu} \Pi^{(L)\rho\sigma}, \quad (54)$$

$$\Pi^{(0,ws)}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\sqrt{D-1}} \Pi^{(L)\rho\sigma} \Pi^{(T)}_{\mu\nu}. \quad (55)$$

Here, $\Pi^{(T)}$ and $\Pi^{(L)}$ are the transversal and longitudinal vector field projectors

$$\Pi^{(T)}_{\mu}{}^{\nu} = \delta_{\mu}^{\nu} - \frac{P_{\mu} P^{\nu}}{P^2}, \quad \Pi^{(L)}_{\mu}{}^{\nu} = \frac{P_{\mu} P^{\nu}}{P^2}. \quad (56)$$

Note that the scalar sector (52–55) is non-diagonal, such that apart from the diagonal projection operators $P^{(0,ss)}$ and $P^{(0,ww)}$ there are the two intertwining operators $\Pi^{(0,sw)}$ and $\Pi^{(0,ws)}$ which connect the two spin-0 representations s and w . The operators satisfy the algebra (orthogonality and idempotency relations)

$$\Pi^{(I,ij)}_{\mu\nu}{}^{\alpha\beta} \Pi^{(J,kl)}_{\alpha\beta}{}^{\rho\sigma} = \delta^{IJ} \delta^{ik} \Pi^{(J,jl)}_{\mu\nu}{}^{\rho\sigma}, \quad (57)$$

infinitesimal transformations of the Poincaré group. In this way, the Minkowski vacuum is connected to the representation theory of the Poincaré group, ultimately giving rise to Wigner's classification [84], in which particles are classified according to their mass and their spin, i.e., the eigenvalues of the Casimir operators of the Poincaré group. A positive cosmological constant $\Lambda > 0$ suggests, however, that the global vacuum is De Sitter space rather than Minkowski space. De Sitter space is also a maximally symmetric space, whose Killing vectors are the generators of the De Sitter group. More generally, this also suggests that for an arbitrary spacetime without any symmetry, the very concept of a particle is not really well-defined.

with $J = 2, 1, 0$ labeling the spin of the representation and $i, j, k, l = s, w$ labeling the different spin-0 operators. In addition, the diagonal operators (50–53) satisfy the completeness relation

$$\Pi^{(2)}_{\mu\nu}{}^{\rho\sigma} + \Pi^{(1)}_{\mu\nu}{}^{\rho\sigma} + \Pi^{(0,ss)}_{\mu\nu}{}^{\rho\sigma} + \Pi^{(0,ww)}_{\mu\nu}{}^{\rho\sigma} = \delta_{\mu\nu}^{\rho\sigma}, \quad (58)$$

with $\delta_{\mu\nu}^{\rho\sigma} = (\delta_{\mu}^{\rho} \delta_{\nu}^{\sigma} + \delta_{\nu}^{\rho} \delta_{\mu}^{\sigma})/2$ denoting the identity in the space of symmetric rank-two tensors. Finally, the traces of the operators (50–53) yield the dimensions of the invariant subspaces, which, according to (58), add up to the $D(D+1)/2$ components of a symmetric rank-two tensor $h_{\mu\nu}$,

$$\begin{aligned} \text{tr} \Pi^{(2)} &= \frac{1}{2} (D+1)(D-2), & \text{tr} \Pi^{(1)} &= D-1, \\ \text{tr} \Pi^{(0,ss)} &= 1, & \text{tr} \Pi^{(0,ww)} &= 1. \end{aligned} \quad (59)$$

Despite the appearance of the spin-0 projector in (49), the spectrum of propagating particles in GR in D dimensions encompasses only the massless spin-2 graviton; the scalar mode can be eliminated by a residual gauge transformation and is not a physical degree of freedom. As explained in (18), the operator (49) is degenerate and a gauge-fixing is required for its inversion. Choosing $O_{\mu\nu} = -\eta_{\mu\nu} \delta(x-y)$ for the operator in (19) and the DeDonder gauge condition on a flat background,

$$\chi^{\mu}[\eta, g] = \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \frac{1}{2} \eta^{\rho\sigma} \eta^{\mu\nu} \right) \partial_{\nu} h_{\rho\sigma}, \quad (60)$$

the flat gauge-fixed fluctuation operator (25) of GR in momentum space reads

$$F_{\text{gf}}^{\mu\nu,\rho\sigma}(-P^2) = \frac{1}{2} [\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}] (-P^2). \quad (61)$$

Inversion of (61) leads to the spin-2 propagator on a flat background¹⁰,

$$\mathcal{P}_{\mu\nu,\rho\sigma}(-P^2) = \frac{1}{2} \left(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{2}{D-2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \frac{1}{(-P^2)}. \quad (62)$$

The propagator $\mathcal{P}_{\mu\nu,\rho\sigma}$ defines the free theory and hence the particle spectrum in perturbation theory. The massless graviton in D dimensions has $D(D-3)/2$ polarization states, obtained by subtracting the $2D$ components of the independent ghost fields in (21) from the $D(D+1)/2$ independent components of the symmetric rank-two tensor $h_{\mu\nu}$.

The interactions in momentum space are defined by the higher n -point functions $\mathcal{V}_{\mu_1 \nu_1 \dots \mu_n \nu_n}^{(n)}(P_1, \dots, P_n)$, which derive from the Fourier transforms of the n th functional derivative of the action

$$\mathcal{V}_{\mu_1 \nu_1 \dots \mu_n \nu_n}^{(n)}(X_1, \dots, X_n) := \frac{\delta^n S_{\text{EH}}[g]}{\delta g_{\mu_1 \nu_1}(X_1) \dots \delta g_{\mu_n \nu_n}(X_n)}, \quad n > 2. \quad (63)$$

¹⁰I reserve the symbol G for the general Green's function in position space defined in (9), and use $\mathcal{P}$ instead for the flat-space Green's function in momentum space.

The essential non-linearity of GR [i.e., the non-polynomial dependence of (39) on $g_{\mu\nu}$] is the origin of the infinite tower of interaction vertices (63) with an increasing number of legs n . The diagrammatic representation of the propagator and the interaction vertices in GR are shown in **Figure 2**¹¹.

The fact that the Einstein-Hilbert action is linear in the scalar curvature implies that GR is a second-order derivative theory, such that (suppressing the index structure) the propagators have a momentum scaling of $\mathcal{P} \propto P^{-2}$, while all n -point vertices in momentum space scale as $\mathcal{V}^{(n)} \propto P^2$. Feynman diagrams with loops, such as in **Figure 1**, correspond to a momentum space integral $\mathcal{I}$ that could diverge in UV. A generic Feynman integral $\mathcal{I}$ in GR with L -loops, I internal propagators, and V vertices has the momentum scaling

$$\mathcal{I} \propto \int (d^D P)^L \frac{1}{(P^2)^I} (P^2)^V. \quad (64)$$

The superficial degree of divergence $D^{\text{div}}(\mathcal{I})$ provides a simple way of estimating the leading divergence of $\mathcal{I}$ by power counting. Scaling each loop momentum by a constant factor b , taking the limit $b \rightarrow \infty$, and counting powers of b defines $D^{\text{div}}(\mathcal{I})$. If $D^{\text{div}}(\mathcal{I}) < 0$, the associated diagram is superficially finite (i.e., finite modulo subdivergences); and if $D^{\text{div}}(\mathcal{I}) \geq 0$, it is divergent. Using the topological relation $I - V = L - 1$, valid on an abstract graph level (i.e., independent of the underlying physical theory), the superficial degree of divergence of quantum GR reads

$$D_{\text{GR}}^{\text{div}} = DL - 2(L - 1). \quad (65)$$

This equality shows that in $D = 4$, the degree of divergence grows with the number of loops L as $D_{\text{GR}}^{\text{div}} = 2(L + 1)$ and signals the perturbatively non-renormalizable character of GR, which in $D = 4$ is directly connected to the negative mass dimension (43) of the gravitational coupling constant $G_N = M_{\text{P}}^{-2}$.

In addition to this simple power-counting argument, the UV divergences of GR and its coupling to matter fields have been calculated in various approximations. For GR with and without a scalar field, the one-loop divergences were first derived in [36]. In subsequent works, the one-loop divergences were extended, including to GR coupled to abelian and non-abelian gauge fields [91, 92], GR coupled to fermions [93], GR with a cosmological constant [94, 95], GR with non-minimal gauges [60], and GR coupled non-minimally to a scalar field [96–98]. At the two-loop order, the calculations of the UV divergences for pure gravity were first performed in [37, 99] and later confirmed in [100]; see also [59].

¹¹This can also be seen as follows: Starting from a spin-2 particle freely propagating in flat spacetime with a linear field equation, locality and diffeomorphism invariance require non-linear self-interactions to be added iteratively in a consistent way such that, when summed, the full non-linear theory of GR is recovered (see [85]). The explicit expressions for the vertices in momentum space are rather lengthy and not very illuminating. The expressions for the three-point and four-point vertices can be found in [35], for example. For these calculations computer-algebra programs, such as FORM or the Mathematica-based xAct bundle (in particular, the core package xTensor and the extension packages xPert and xTras) are indispensable [86–90].

In order to make connections with the general formalism outlined in section 2, I briefly illustrate the calculation of the one-loop divergences for the Euclidean version of the Einstein-Hilbert action (39) in $D = 4$,

$$S_{\text{EH}}[g] = -\frac{M_{\text{P}}^2}{2} \int d^4 X \sqrt{g} (R - 2\Lambda). \quad (66)$$

The gauge-breaking action (19) for the second-order theory (66) is given by

$$S_{\text{gb}}[\bar{g}_{\mu\nu}; h_{\mu\nu}] = -\frac{1}{2} \int d^4 X \chi^\mu g_{\mu\nu} \chi^\nu, \quad (67)$$

where the ultra-local operator $O_{\alpha\beta}$ and De Donder gauge condition χ^α are

$$O_{\alpha\beta} = -\frac{\sqrt{\bar{g}}}{2} \bar{g}_{\mu\nu} \delta^{(4)}(X, Y),$$

$$\chi^\mu[\bar{g}_{\mu\nu}; h_{\mu\nu}] = \left(\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\mu\nu} \right) \bar{\nabla}_\nu h_{\rho\sigma}. \quad (68)$$

Adding (67) to (66) results in a gauge-fixed fluctuation operator (25), which is of the minimal second-order type (30),

$$F^{\mu\nu, \rho\sigma} = \bar{G}^{\mu\nu, \tau\lambda} F_{\tau\lambda}^{\rho\sigma} = \bar{G}^{\mu\nu, \tau\lambda} (\bar{\Delta} \delta_{\tau\lambda}^{\rho\sigma} + \bar{P}_{\tau\lambda}^{\rho\sigma}), \quad (69)$$

where $\bar{\Delta} = -\bar{g}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu$ is the positive-definite background Laplacian and the background values of the DeWitt metric $\bar{G}^{\mu\nu, \rho\sigma}$ and the potential $\bar{P}_{\tau\lambda}^{\rho\sigma}$ are defined as

$$\bar{G}^{\mu\nu, \rho\sigma} := \frac{g^{1/2}}{4} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\rho\sigma}), \quad (70)$$

$$P_{\mu\nu}^{\rho\sigma} := -2R_{(\mu}^{\rho}{}_{\nu)}^{\sigma} - 2\delta_{(\mu}^{(\rho} R_{\nu)}^{\sigma)} + g_{\mu\nu} R^{\rho\sigma} + g^{\rho\sigma} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R + (R - 2\Lambda) \delta_{\mu\nu}^{\rho\sigma}. \quad (71)$$

According to (22), the ghost operator derives from (68) and reads

$$Q_\mu{}^\nu = \delta_\mu^\nu \bar{\Delta} - \bar{R}_\mu{}^\nu. \quad (72)$$

The divergent part of the one-loop approximation (8) reduces to the evaluation of the two functional traces

$$\Gamma_1^{\text{div}} = \frac{1}{2} \text{Tr} \ln(F^{\mu\nu}{}_{\rho\sigma}) \Big|_{\text{div}} - \text{Tr} \ln(Q_\mu{}^\nu) \Big|_{\text{div}}. \quad (73)$$

Terms proportional to $\delta^{(4)}(0)$ that arise from $\text{Tr} \ln(\bar{G}_{\mu\nu\rho\sigma})$ are zero in dimensional regularization. The divergent parts of the functional traces (73) are most efficiently evaluated by the heat-kernel techniques presented in section 2.3. The operators (69) and (72) in (73) are both of the form (30), for which the divergent part is given by (34). The final result for the one-loop divergences (73) reads

$$\Gamma_1^{\text{div}} = \frac{1}{16\pi^2 \varepsilon} \int d^4 X \sqrt{\bar{g}} \left[-\frac{53}{90} \bar{\mathfrak{G}} - \frac{7}{20} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} - \frac{1}{120} \bar{R}^2 + \frac{13}{6} \Lambda \bar{R} - \frac{5}{2} \Lambda^2 \right]. \quad (74)$$

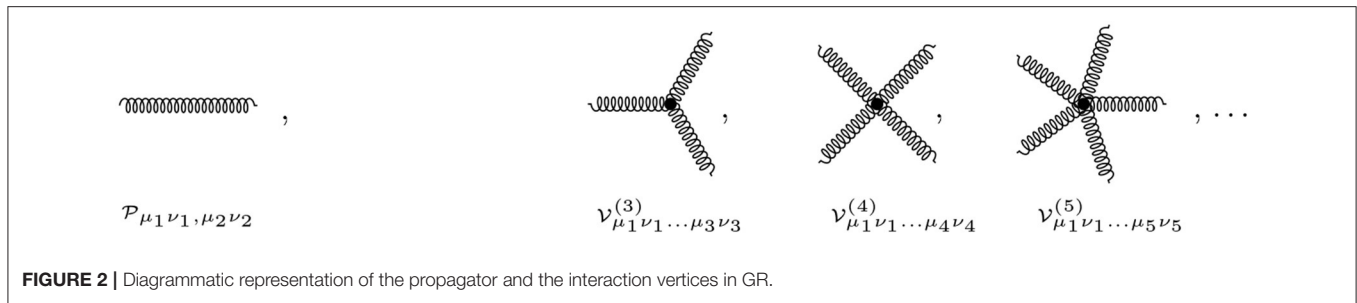


FIGURE 2 | Diagrammatic representation of the propagator and the interaction vertices in GR.

The Euler characteristic $\chi(\mathcal{M})$ is a topological invariant, defined in terms of the quadratic Gauss-Bonnet invariant $\mathfrak{G}$ as

$$\chi(\mathcal{M}) := \frac{1}{32\pi^2} \int_{\mathcal{M}} d^4 X \sqrt{-g} \mathfrak{G},$$

$$\mathfrak{G} := R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2. \quad (75)$$

This allows us to eliminate squares of the Riemann tensor in (74) in favor of squares of the Ricci tensor and squares of the Ricci scalar. For gravity with a cosmological constant in vacuum, the field equations (40) imply $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Therefore, on-shell, quantum Einstein gravity with a cosmological constant at the one-loop order can be expressed in terms of the Euler characteristic (75) and the volume $\mathcal{V}(\mathcal{M}) := \int d^D X \sqrt{\bar{g}}$ as

$$\Gamma_{1,\text{on-shell}}^{\text{div}} = \frac{1}{\varepsilon} \left[-\frac{53}{45} \chi(\mathcal{M}) + \frac{87}{20} \frac{\Lambda^2}{12\pi^2} \mathcal{V}(\mathcal{M}) \right]. \quad (76)$$

As discussed in [95], the result (76) shows that, within the one-loop approximation, pure Einstein gravity in $D = 4$ is on-shell renormalizable, as the divergences in (76) can be absorbed by adding the topological term $\chi(\mathcal{M})$ (which does not affect the field equations) with some coefficient to the action (66) and renormalizing this coefficient as well as the cosmological constant Λ . For the case of a vanishing cosmological constant, the fact that Einstein gravity is on-shell one-loop finite was first found in [36]. However, as soon as matter fields are coupled, the one-loop divergences remain even on-shell [36]. For example, the one-loop divergences of GR with a minimally coupled scalar field φ with quartic self-interaction induce a non-minimal coupling to gravity proportional to $R\varphi^2$, an operator not present in the original action [36, 96–98]. At the two-loop order, even for a vanishing cosmological constant $\Lambda = 0$, a divergent contribution of a single operator among the cubic curvature invariants survives the on-shell reduction [37, 99, 100],

$$\Gamma_{2,\text{on-shell}}^{\text{div}} = \frac{1}{\varepsilon} \frac{1}{(16\pi^2)^2} \frac{209}{1470} \frac{1}{M_{\text{P}}^2} \int d^4 X \sqrt{-g} \bar{C}_{\mu\nu}{}^{\rho\sigma} \bar{C}_{\rho\sigma}{}^{\alpha\beta} \bar{C}_{\alpha\beta}{}^{\mu\nu}, \quad (77)$$

thereby showing explicitly that GR is perturbatively non-renormalizable¹². In (77), the cubic Riemann curvature invariant

is expressed in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$, which on-shell coincides with the Riemann tensor $R_{\mu\nu\rho\sigma}$ in view of the vacuum on-shell identity $R_{\mu\nu} = 0$,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2} (R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma} + R_{\mu\sigma} g_{\nu\rho} + R_{\nu\sigma} g_{\mu\rho}) - \frac{R}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}). \quad (78)$$

In a perturbatively renormalizable QFT, finitely many free parameters (fields, masses, and coupling constants) are sufficient to absorb all UV divergences to all orders in the perturbative expansion. As demonstrated in (65) based on power-counting arguments and in (77) based on explicit calculations, GR is not of that form. New higher-dimensional operators with divergent coefficients are induced at every loop order and have to be renormalized by introducing the corresponding counterterms, each of which introduces a new coupling constant with a finite part that needs to be determined by a measurement. In this way, more and more free parameters are introduced at each order in the perturbative expansion, and the theory ultimately loses its predictive power.

4. EFFECTIVE FIELD THEORY OF GRAVITY

For many physical systems, an effective *coarse-grained* description is sufficient to accurately describe phenomena at low energies by the relevant degrees of freedom [38]. Such an effective description might arise in two complementary ways, often referred to as the *top-down* and *bottom-up* approaches. In the case where a (more) fundamental theory is known at high energy scales, a top-down approach leads to an effective low-energy theory by “integrating out” the heavy degrees of freedom¹³. Denoting the heavy degrees of freedom collectively by Φ , with characteristic mass scale M_Φ , and denoting the light degrees of freedom by ϕ , with characteristic mass scale M_ϕ , in a “top-down” scenario there is a natural mass hierarchy $M_\Phi \gg M_\phi$. Integrating out the Φ -fields from the combined

¹²In a recent calculation of the two-loop divergences with modern on-shell methods, it was found that by using dimensional regularization, evanescence operators (such as the Gauss-Bonnet term) in divergent subdiagrams can alter the coefficient of the pole term [101].

¹³Only when the more fundamental theory is valid up to arbitrarily high energy scales does it qualify as UV-complete theory. Instead of integrating out certain heavy particles, in the Wilsonian approach the effective action is defined at a given energy scale E by integrating out *all* particles with momenta $P^2 > E$.

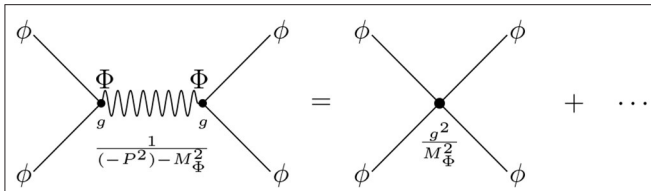


FIGURE 3 | In the diagrammatic representation, to first order in the expansion (80), the Φ -propagator is shrunk to a point, leading to an effective four-point contact interaction between the ϕ -fields.

action $S[\Phi, \phi]$ in the path integral defines the effective action $S_{\text{eff}}[\phi]$ for the ϕ -fields,

$$\int \mathcal{D}[\phi] e^{-S_{\text{eff}}[\phi]} := \int \mathcal{D}[\Phi, \phi] e^{-S[\Phi, \phi]}.$$

In general, the process of integrating out Φ -fields results in a non-local effective action $S_{\text{eff}}[\phi]$. Within an energy expansion $E/M_\Phi \ll 1$, it can be expanded in terms of *local* operators $\mathcal{O}_n(\phi, \partial)$ for the ϕ -fields as

$$S_{\text{eff}}[\phi] = S[\phi] + \sum_n \int d^D X w_n \frac{\mathcal{O}_n(\phi, \partial)}{M_\Phi^{n-D}},$$

$$[\mathcal{O}_n(\phi, \partial)]_M = n, \quad [w_n]_M = 0. \quad (79)$$

The higher-dimensional local operators $\mathcal{O}_n(\phi, \partial)$ parameterize the impact of the heavy degrees of freedom Φ on the effective low-energy theory for the light degrees of freedom ϕ , and their interacting strength is characterized by the dimensionless *Wilson coefficients* w_n . In terms of momentum space Feynman integrals, this expansion is associated with an expansion of the Φ -propagators in inverse powers of the heavy mass scale M_Φ ,

$$\frac{1}{(-P^2) - M_\Phi^2} = -\frac{1}{M_\Phi^2} - \frac{1}{(-M_\Phi)} (-P^2) \frac{1}{(-M_\Phi^2)} + \dots \quad (80)$$

For example, in this way, a $\phi\phi$ - $\phi\phi$ interaction from a trivalent vertex $\propto g\Phi\phi^2$ in $S[\Phi, \phi]$ leads to an effective quartic contact interaction between the ϕ -fields $\propto (g^2/M_\Phi^2)\phi^4$ in $S_{\text{eff}}[\phi]$, as illustrated in **Figure 3**.

Since in the top-down approach calculations can be performed both ways, i.e., in the more fundamental theory as well as in the effective theory, scattering amplitudes can be compared at some scale below (but usually close to) M_Φ in order to fix the Wilson coefficients in terms of the parameters of the more fundamental theory, a procedure called *matching*. Assuming $w_n = \mathcal{O}(1)$, the accuracy of the effective description is limited only by the ratio E/M_Φ , which controls the energy expansion, and completely breaks down for energies $E \approx M_\Phi$, where the propagation of the Φ particles is no longer suppressed.

Importantly, the effective field theory (EFT) description is still applicable, even if no (more) fundamental theory in the UV is known. This is the situation for GR, i.e., the EFT approach to gravity is necessarily a bottom-up one [39, 40]. In this case, the cutoff scale M that limits the range of validity of the effective

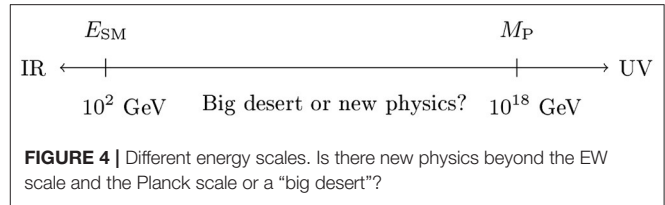


FIGURE 4 | Different energy scales. Is there new physics beyond the EW scale and the Planck scale or a “big desert”?

description is not known a priori. Assuming no new physics at scales in between the electroweak (EW) scale of the standard model (SM) of particle physics and the scale at which gravity becomes comparable to the other interactions (see **Figure 4**), the Planck scale might be the natural cutoff scale, $M = M_P$ ¹⁴.

It could be considered a particular strength of the bottom-up approach that it is agnostic about the gravitational degrees of freedom in the UV: the low-energy limit of the EFT defines the field variables, symmetries, and particle spectrum. In the case of GR, these are the metric field, the diffeomorphisms, and the massless spin-2 graviton. The ignorance of a more fundamental theory in the UV is parameterized by the systematic inclusion of higher-dimensional operators, which are compatible with the symmetries of the defining low-energy theory and suppressed by inverse powers of the cutoff scale. In the case of gravity, diffeomorphism invariance requires that the higher-dimensional purely gravitational operators $\mathcal{O}(g, \partial)$ have the form of curvature invariants proportional to $g^{1/2} \nabla^{2n} R^m / M^{2(n+m)-D}$. For energy scales well below the cutoff $\nabla/M \ll 1$, $R/M^2 \ll 1$, these higher-dimensional operators are strongly suppressed and the expansion can be truncated at a finite order determined by the required accuracy of the EFT. In contrast to a fundamental theory, the higher-dimensional operators in an EFT are viewed merely as correction terms, i.e., they lead to additional interaction vertices but do not modify the propagators of the theory and hence do not affect the particle spectrum, which is defined by the relevant operators at low energy¹⁵. While the higher-dimensional operators in an EFT are included in a controlled way, the precise way in which such an expansion scheme is realized can differ. Depending on the requirements of the underlying physical model, such an expansion could be realized as a derivative expansion, as a vertex expansion, as the aforementioned combined “energy expansion,” or according to a different scheme.

In principle, the presence of the infinite tower of operators $\nabla^{2n} R^m / M^{2(n+m)-D}$ is required in an EFT to absorb all UV divergences by renormalizing the w_i . However, according to the GR power counting (65), the L th loop correction in $D = 4$ induces divergent operators of the form $\nabla^{2n} R^m / M^{2(n+m)-D}$ with $n+m = L+1$. Thus, within a *finite truncation*, the EFT of gravity can be perturbatively renormalized in the standard way, and only

¹⁴This naive estimate might be modified in the presence of matter; see e.g., the discussion in the context of scalar-tensor theories with a strong non-minimal coupling, such as in the model of Higgs inflation [102–107].

¹⁵Note, however, that a summation of operators with a fixed number of external fields but an arbitrary number of derivatives results in non-local form factors that lead to IR modifications of the propagator. For a discussion of these non-local form factors in the context of gravity and the heat kernel (see e.g., [108, 109]).

finitely many renormalized parameters w_i have to be measured, ultimately rendering the EFT predictive¹⁶.

However, ultimately absorbing the UV divergences within a finite truncation provides a consistency condition rather than a prediction. In contrast to the local but unphysical UV divergences, true predictions of the quantum theory are connected with infrared (IR) effects that arise from long-range interactions dominated by massless particles. These contributions are connected to the non-analytic parts in scattering amplitudes. The most prominent example of how such IR effects can be extracted from QFT scattering amplitudes within the EFT of GR concerns the corrections to the Newtonian potential for two point masses M_1 and M_2 , which after Fourier transformation read [111]

$$V(r) = -\frac{G_N M_1 M_2}{r} \left[1 + 3 \frac{G_N (M_1 + M_2)}{rc^2} + \frac{41}{10\pi} \frac{G_N \hbar}{r^2 c^3} + \dots \right]. \quad (81)$$

The second term is a purely classical relativistic correction related to the $\sqrt{P^2}$ part, while the third term is of genuine quantum origin and related to the $P^2 \log(P^2)$ part of the one-loop contribution [111]. Both contributions correspond to those parts of the scattering amplitude that have a non-analytic momentum dependence. They are independent of the higher curvature terms in the EFT expansion and therefore do not depend on a UV completion. While the general structure of the correction terms in (81) follows from dimensional analysis, the coefficients (in particular the signs) have to be calculated and provide a true prediction of quantum gravity.

While the *quantum* gravitational corrections are accompanied by powers of $G_N \hbar$ and are therefore very hard to measure, classical post-Minkowskian (PM) corrections are in powers of G_N . High-order PM corrections have been calculated with classical techniques [112–115]. Since the advent of gravitational wave astronomy, there have been increasing efforts to extract the classical PM corrections within an EFT framework from QFT scattering amplitudes, which, in turn, can be efficiently calculated using modern on-shell techniques (see e.g., [116–122]).

The EFT of GR is a powerful and universal approach that yields universal quantum gravitational predictions from long-range effects of massless particles, but its scope of applicability is limited by construction. Therefore, certain questions cannot be addressed within this framework but require a fundamental quantum theory of gravity.

5. ASYMPTOTIC SAFETY

Although the question about a fundamental theory of gravity cannot be addressed in the framework of the perturbative EFT

approach, the asymptotic safety (AS) program, initiated in [123, 124], might offer a UV-complete theory of quantum gravity. The basic underlying idea is that the renormalization group (RG) flow drives the (dimensionless) *essential* couplings g_n of a theory toward a UV fixed point g_n^* ¹⁷. In this way, the AS scenario prevents the couplings from running into divergences at finite energy scales (Landau poles) and allows the RG flow to be extrapolated to arbitrary energy scales $k \rightarrow \infty$. However, in contrast to the asymptotic freedom scenario corresponding to a free (i.e., non-interacting or “Gaussian”) UV fixed point $g_n^* = 0$, the AS scenario only requires the weaker condition $g_n^* = \text{const.}$, which includes the possibility of an interacting fixed point for $g_n^* \neq 0$ [123]. In particular, the couplings g_n are not required to remain within the perturbative regime $g_n \ll 1$ and consequently allow for a strongly interacting UV fixed point at which (at least some of) the couplings $g_n^* \gg 1$. Clearly such a strongly interacting UV fixed point cannot be found within a perturbative approach. Thus, the AS scenario is an inherently *non-perturbative* approach, which can be addressed with the Wilsonian approach to the RG [125].

The main object is the *averaged effective action* Γ_k , which defines the full quantum theory at a given RG scale k . The sliding scale k interpolates between the bare action $\Gamma_\infty = S$ in the UV, corresponding to $k = \infty$, and the full effective action $\Gamma_0 = \Gamma$ in the IR, corresponding to $k = 0$. Once the propagating degrees of freedom ϕ^i and their symmetries are identified, Γ_k can be expressed in terms of symmetry-compatible operators $\mathcal{O}_n(\phi, \partial)$ with coupling strengths $g_n(k)$ as

$$\Gamma_k = \sum_{i=1}^{\infty} \int d^D X g_i(k) \mathcal{O}_n(\phi, \partial). \quad (82)$$

The space of all coupling constants g_i is called *theory space*. A suitable tool for a non-perturbative analysis is the Wetterich equation [126–128], which describes the exact functional RG flow of the averaged effective action Γ_k ,

$$k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left(\frac{k \partial_k \mathcal{R}_k}{\Gamma_k^{(2)} + \mathcal{R}_k} \right). \quad (83)$$

Here, Tr is the functional trace, $\mathcal{R}_k$ is a scale-dependent regulator and $\Gamma_k^{(2)}$ is the Hessian of the averaged effective action Γ_k . The Wetterich equation (83) has a similar structure to the one-loop approximation (8) but involves the scale-dependent regulator function $\mathcal{R}_k$ defined such that it acts as an effective mass term of the full propagator for quantum fluctuations with momenta $P^2 \leq$

¹⁶An important technical requirement for the consistent renormalization is that the counterterms have the same structure as the operators in the EFT expansion. Since the latter are restricted by symmetry, the process of renormalization is required to preserve this symmetry; see e.g., the discussion in [110]. This property is non-trivial to show and has been proven for GR and Yang-Mills theory in [58]. Recently, the proof was extended to effective and non-relativistic theories by combining the BRST cohomology with the background field method [52].

¹⁷In this section, I denote the coupling constants by g_n to contrast with the c_n in (1) and the ω_n in (79), although when put in the right context they are all the same objects. The RG flow $g_n(k)$ is defined as the solution to the RG system $k \partial_k g_n = \beta_{g_n}$, with the abstract RG scale k and beta functions β_{g_n} . A fixed point g_n^* is defined by the condition $\beta_{g_n}(g_n^*) = 0$ for all n . The couplings $\tilde{g}_n$, which carry a canonical physical dimension $[g_n]_M = \alpha_n$, are made dimensionless by rescaling with the appropriate power of the RG scale $[k]_M = 1$, i.e., $g_n = \tilde{g}_n k^{-\alpha_n}$, such that $[g_n]_M = 0$. Moreover, since only essential couplings enter physical observables, only they are required to take finite values in the UV. In contrast, inessential couplings, which can be changed by a field redefinition, do not enter physical observables and so may diverge in the UV.

k^2 and vanishes for momenta $P^2 \gg k^2$. Together with the factor $\partial_k \mathcal{R}$ in (83), which cuts off fluctuations with momenta $P^2 \geq k^2$, the presence of the regulator ensures that only fluctuations with momenta peaked around $P^2 \approx k^2$ contribute to the trace in (83), thereby realizing the Wilsonian “shell-by-shell” integration¹⁸. Owing to the presence of the regulator, no divergences occur. In general, the Wetterich equation cannot be solved exactly. Instead of a semiclassical expansion in powers of loops, such as in (6), a finite truncation of the (in general infinite) set of operators included in Γ_k is performed:

$$\Gamma_k = \sum_{n=1}^N \int d^D X g_n(k) \mathcal{O}_n(\phi, \partial). \quad (84)$$

According to which criteria such a truncation is chosen practically may depend on the underlying physical problem. In most applications the operators are organized in terms of an energy expansion, i.e., ordered by increasing canonical mass dimension. There are, however, cases where a derivative expansion or a vertex expansion is more appropriate. In the case of gravity, diffeomorphism invariance requires that the $\mathcal{O}(g, \partial)$ be curvature invariants, schematically $\mathcal{O}(g, \partial) = \sqrt{g}(\nabla)^{2p} R^m$. By substituting the ansatz (84) into (83), choosing a regulator $\mathcal{R}_k$, and evaluating the functional trace on the right-hand side of (83), the RG flow of the couplings $g_n(k)$ can be extracted by “projecting” to the operator basis $\mathcal{O}_n(\phi, \partial)$. Contributions of operators that are induced by the flow and lead out of the truncation (84) are neglected¹⁹.

For a successful realization of the AS scenario, the existence of a UV fixed point g_i^* is only a necessary condition, not a sufficient one. In addition, an appropriate fixed point must have a *finite-dimensional* UV critical surface²⁰. The finiteness of the UV critical surface lies at the very heart of the AS scenario, as it implies that only a finite subset of the (in general infinitely many) coupling constants have to be measured, rendering the theory predictive. It is this feature that might qualify the AS scenario in providing a UV-complete quantum theory of gravity²¹. Therefore, in principle, if all UV-relevant couplings were measured (and thus a particular RG trajectory

emanating from the UV fixed point selected), all other UV-irrelevant couplings would be fixed. They therefore constitute predictions that could be falsified by additional measurements of these couplings. In practice, however, calculations are limited to finite truncations, and one must ensure that the properties of the fixed point (and hence any prediction derived from them) remain stable under an enlargement of the truncation. In principle, if a reliable measure of the quality of a given truncation were to exist, one could try to ultimately prove convergence; but as so far no such measure exists, this is hard to realize in practice and one has to rely on systematic step-by-step enlargements of finite truncations. Nevertheless, as for the perturbative approach (fundamental or EFT), a particular strength of the AS approach to quantum gravity is its universality, i.e., gravity and matter fields are treated within one and the same formalism. This not only allows for a unification but also enables one to test the techniques used in the context of quantum gravity in more controlled environments, for which experimental data are also available.

The functional RG flow in the context of gravity [41–43, 129, 130] has been studied in various truncations, starting with the Einstein-Hilbert truncation [131], encompassing higher curvature invariants [132–137] and matter fields [138–143] as well as closed flow equations for $f(R)$ gravity [144, 145], and general scalar-tensor theories [146, 147]. A pattern that emerges from most of these truncations is that an interacting UV fixed point can be found and the dimension of the associated UV critical surface does not grow upon enlarging the truncation beyond the classically marginal operators. Since this program has been pushed to high orders in various truncations, it might provide some confidence that the observed pattern is a generic feature and not an artifact of the truncation.

Despite these interesting results, there are a number of open questions associated with this program (see e.g., [148]). In general, the off-shell flow defined by Γ_k suffers from a number of ambiguities related to the choice of regulator as well as to the gauge dependence and field parametrization dependence of the beta functions. Since different regulator choices, different gauges, and different field parameterizations can even affect qualitative features, such as the existence of a fixed point, a satisfying resolution of these ambiguities seems to be crucial for establishing the reliability of the predictions following from the AS conjecture.

In connection with the gauge and parameter dependence, a unique off-shell extension of the averaged effective action along the lines of the construction proposed in [149] might offer an interesting option. But even without such a construction, the gauge and parametrization dependence should be absent in an on-shell scheme (see e.g., [150]). However, making use of the equations of motion in general leads to degeneracies among different operators in a given truncation and therefore

¹⁸In particular, once a cutoff is introduced, it does not matter whether the underlying theory is perturbatively renormalizable in the strict sense or not. All operators compatible with the symmetries of the theory have to be considered. This is similar to the EFT case, but in contrast to the EFT treatment, the particle content and the symmetries are not necessarily defined by the relevant operators of the low-energy approximation, but rather are defined along with the averaged effective action (82). In general the theory space is infinite, but if the symmetry restriction is so strong that it only allows for a finite number of operators, the theory space could be finite.

¹⁹This is a consistency requirement of the truncation. If no operators that lead out of the truncation are induced, the flow closes and (83) is really an exact equation.

²⁰The UV critical surface can be thought of as a subspace of the tangent space at g^* , consisting of those RG trajectories which are attracted toward the fixed point. In general there can be more than just one fixed point, and the RG flow may also allow for more exotic phenomena, such as limit cycles. It could also happen that some of the fixed points can be discarded on physical grounds.

²¹Compare this with the perturbative quantization of GR, discussed in section 3. The perturbatively non-renormalizable character requires the measurements of an infinite number of couplings, thereby leading to a loss of predictive power.

Compare this also with the EFT approach to GR, discussed in section 4. While only a finite number of couplings have to be measured within a finite truncation, the EFT cannot be extrapolated beyond a certain energy scale and therefore does not qualify as a UV-complete theory.

the individual RG flow of the couplings for these on-shell degenerate operators cannot be disentangled and resolved²². Nevertheless, extracting, for example, physical observables from the S-matrix will anyway involve an on-shell reduction. By definition only *essential* couplings span the theory space. In this sense, the “on-shellness” is already built into the formalism of the AS conjecture from the very beginning. However, especially in the context of gravity, the situation is more complicated. For example, the question of whether Newton’s constant is an essential or inessential coupling is not so clear and leads to conceptional intricacies (see e.g., the discussion in [151]).

In any case, the starting point for the derivation of observables should be the effective action at $k = 0$, which is independent of the regulator and is formally obtained by integrating out all quantum fluctuations, i.e., integrating the functional flow all the way down to the IR. One might be tempted to extract information from the averaged effective action Γ_k at non-zero k by performing an “RG-improvement” based on a heuristic identification of the abstract coarse-graining RG scale k with some characteristic physical scale. However, aside from the fact that such an identification is typically only possible in highly symmetric backgrounds where a single scale is present, such as the radius in the context of spherically symmetric black hole backgrounds, the Hubble parameter in the context of an isotropic and homogeneous cosmological Friedmann-Lemaître-Robertson-Walker background, the value of the scalar field in the Coleman-Weinberg-like radiatively induced symmetry breaking in a classically scale-invariant theory, or the momentum transfer in the context of scattering amplitudes—it does not seem that such a naive identification can be based on a more general solid theoretical ground. However, even when working with the effective action at $k = 0$, another problem arises: the effective action is non-local (and non-analytic) and therefore not appropriately described by the finite number of local operators in a given truncation that do not capture essential IR contributions. In this context, the introduction of form factors in the AS program provides a more promising route. Including form factors in the truncation goes beyond a finite derivative expansion, as it captures the full momentum dependence of propagators and vertices, which can be studied either using a flat-space vertex expansion [152–154] or in a general background by an expansion of the effective action in powers of external fields (curvatures in the context of gravity) [155, 156]. The manifest covariant calculations of these non-local form factors are technically challenging and require heat-kernel-based methods developed in [108, 109, 157–159].

The analysis of form factors in the AS program may also shed light on the status of the particle content, a problem shared by higher-derivative theories of gravity, discussed in section 6. Any truncation based on a finite derivative expansion will in general lead to additional propagating degrees of freedom in the particle spectrum (defined by the quadratic action expanded around a

flat background) and will almost always include higher-derivative ghosts among them. Having access to the pole structure of the propagators, including the full momentum dependence carried by the form factors, may ultimately reveal the status of the ghost degrees of freedom as an artifact of the finite truncation (realized, for example, when the full propagators have only a single pole with positive residue). Technically, this program is closely related to the (ghost-free) non-local approach to quantum gravity (see e.g., [18, 20–22, 25, 26]).

6. HIGHER-DERIVATIVE GRAVITY

Before giving up on finding a fundamental theory of quantum gravity or abandoning the framework of perturbative QFT, another obvious approach to try is to modify the underlying classical theory of gravity and investigate the impact of these modifications on the resulting quantum theory. Adding higher-dimensional curvature invariants to the action might be the most natural generalization of GR. In contrast to the EFT treatment, when treating the modified theory as fundamental, the higher-dimensional operators are no longer considered as perturbations, and so they not only modify the interaction vertices but also the propagators. Ultimately, this leads to new additional propagating degrees of freedom. There are many ways to modify GR. A simple and phenomenologically important extension of GR is $f(R)$ gravity, allowing for an arbitrary function f of the Ricci scalar R ,

$$S_f[g] = \int d^4X \sqrt{g} f(R). \quad (85)$$

In particular, (85) encompasses the Starobinsky model [160], which is highly relevant for inflationary cosmology,

$$f_{\text{Star}} = \frac{M_{\text{P}}^2}{2} \left[R + \frac{1}{6M_0^2} R^2 \right]. \quad (86)$$

In fact, (86) was the first model of inflation and is strongly favored by the latest Planck data [161]. The one-loop divergences for $f(R)$ gravity (85) have recently been calculated on an arbitrary background [76], thereby essentially generalizing previous calculations obtained for spaces of constant curvature [144, 145, 162],

$$\begin{aligned} \Gamma_1^{\text{div}} = & \frac{1}{32\pi^2\epsilon} \int d^4X \sqrt{g} \left[-\frac{71}{60} \mathfrak{G} - \frac{259}{180} R_{\mu\nu} R^{\mu\nu} - \frac{9}{2} \left(\frac{f}{f_1} \right)^2 \right. \\ & - \frac{1}{18} \left(\frac{f_1}{f_2} \right)^2 + \frac{9f}{2f_1} R + \frac{1}{3} \frac{f}{f_2} - \frac{59}{360} R^2 + \frac{21f}{2f_1} (\gamma_\mu{}^{;\mu}) \\ & - \frac{33}{4} R (\gamma_\mu{}^{;\mu}) - \frac{371}{72} R (\gamma_\mu \gamma^\mu) + \frac{27f}{4f_1} (\gamma_\mu \gamma^\mu) \\ & + \frac{20}{9} R_{\mu\nu} \gamma^\mu \gamma^\nu - \frac{137}{24} (\gamma_\mu{}^{;\mu})^2 - \frac{9}{8} (\gamma_\mu \gamma^\mu) (\gamma_\nu{}^{;\nu}) \\ & \left. - \frac{769}{96} (\gamma_\mu \gamma^\mu)^2 \right]. \quad (87) \end{aligned}$$

²²A similar problem occurs when working on special (in general highly symmetric) backgrounds, even if they do not correspond to on-shell configurations.

The derivatives of the function f are defined by $f_n := \partial^n f / \partial R^n$, and the vector Υ_μ is defined as $\Upsilon_\mu := R_{;\mu} f_2 / f_1$. Even for a general function f , the result (87) shows that $f(R)$ gravity is perturbatively non-renormalizable on a general background. Although divergences accompanied by arbitrary functions of R can be absorbed by renormalizing $f(R)$, due to the absence of the derivative structures Υ_μ and the quadratic curvature structure $R_{\mu\mu} R^{\mu\nu}$ in (85), the associated divergences cannot be absorbed²³. The higher derivatives in (85) lead to a fourth-order fluctuation operator and imply the presence of an additional propagating scalar degree of freedom, the scalaron. In the context of the cosmological model (86), the scalaron drives the accelerated expansion of the early universe, and its mass $M_0 \approx 10^{-5} M_{\text{Pl}}$ is fixed by the observed anisotropy spectrum in the cosmic microwave background radiation [161].

What is the required extension of GR that qualifies as a candidate for a perturbatively renormalizable quantum theory of gravity? The power counting performed in (65) for GR can easily be generalized to higher-derivative theories of gravity. Diffeomorphism invariance requires that all higher curvature invariants have a schematic structure $\sqrt{-g} \nabla^{2n} R^m$ (suppressing indices) with a total number of derivatives $p = 2(n + m)$. The natural candidate higher-derivative gravity (HDG) theory is the one which includes all classically relevant and marginal operators, i.e., in the $D = 4$ case all operators with $p \leq 4$. Aside from the relevant operators (46) that are already present in the Einstein-Hilbert action (39), the marginal operators with $p = 4$ have either $m = 2$ and $n = 0$ or $m = 1$ and $n = 1$. For the latter case, there is only one scalar invariant $\mathcal{O}_3(g, \partial) = \sqrt{-g} \nabla_\mu \nabla^\mu R$, which is a total derivative. For the former case there are three possible scalar invariants that are quadratic in the curvature,

$$\begin{aligned}\mathcal{O}_4(g, \partial) &= \sqrt{-g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \\ \mathcal{O}_5(g, \partial) &= \sqrt{-g} R_{\mu\nu} R^{\mu\nu}, \\ \mathcal{O}_6(g, \partial) &= \sqrt{-g} R^2.\end{aligned}\quad (88)$$

The three curvature invariants in (88) can be more conveniently parameterized in a different basis of quadratic curvature invariants involving the Gauss-Bonnet term, the Weyl tensor, and the Ricci scalar, as the latter two are more directly related to the particle content:

$$\begin{aligned}S_{\text{QDG}}[g] &= S_{\text{EH}}[g] + \int d^4 X \sqrt{-g} \\ &\times [c_1 \mathfrak{G} + c_2 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + c_3 R^2].\end{aligned}\quad (89)$$

The power counting in the UV is dominated by the marginal quadratic curvature operators, and the momentum scaling of the propagator is $\mathcal{P} \propto P^{-4}$, while that of the vertices is $\mathcal{V}^{(n)} \propto P^4$.

²³Even on-shell, there remain divergences associated with operators involving derivatives of the Ricci scalar, which are not total derivatives and cannot be absorbed into the function $f(R)$ [76]. On a constant-curvature background $g_{\mu\nu}^0$, for which $R_{\mu\nu\rho\sigma}^0 = R_0(g_{\mu\rho}^0 g_{\nu\sigma}^0 - g_{\mu\sigma}^0 g_{\nu\rho}^0)$, $\Upsilon_\mu = 0$, $\int d^4 X \sqrt{g^0} = 384\pi^2 / R_0^2$, and the equations of motion reduce to the algebraic equation $2f - R_0 f_1 = 0$, the one-loop divergences $\Gamma_1^{\text{div}}|_0^{\text{on-shell}} = (1/\epsilon)[\frac{97}{20} + 4f/R_0^2 f_2 - 8f^2/3(R_0 f_2)^2]$ can be absorbed by a renormalization of $f(R_0)$.

Consequently, the superficial degree of divergence in quadratic gravity (QDG) in $D = 4$ is

$$D_{\text{QDG}}^{\text{div}} = 4L - 4(L - 1) = 4. \quad (90)$$

Hence, in $D = 4$, QDG is power-counting renormalizable, suggesting that QDG is indeed the required extension of GR. Going beyond this simple power-counting argument requires more advanced methods; a strict proof that the QDG (89) is a perturbatively renormalizable quantum theory of gravity was given in [45].

However, even if the perturbative renormalizability of QDG has been established, it remains to show that QDG is UV-complete, i.e., that the theory can be extended to an arbitrary energy scale. To answer this question requires studying the RG flow determined by the divergence structure of the theory. In particular, for a UV-complete theory the absence of Landau poles, where couplings diverge at finite energies, must be assured. The one-loop divergences of QDG were first calculated in [163] and later corrected in [164]. The authors of [164] considered the Euclidean version of (89) with a different parametrization and basis for the quadratic curvature invariants,

$$\begin{aligned}S_{\text{QDG}}[g] &= \int d^4 X \sqrt{g} \left[\frac{2}{k^4} \lambda - \frac{1}{k^2} R + \frac{1}{v^2} \mathfrak{G} \right. \\ &\quad \left. + \frac{1}{f^2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) - \frac{\omega}{3f^2} R^2 \right],\end{aligned}\quad (91)$$

with $1/k^2 = M_{\text{Pl}}^2/2$ and the dimensionless cosmological constant $\lambda = 2\Lambda/M_{\text{Pl}}^2$. The beta functions can directly be read off from the one-loop divergences and determine the running of the coupling constants with the logarithmic parameter $t := 1/(4\pi^2) \ln(\mu/\mu_0)$. Here μ is the sliding scale and μ_0 an arbitrary renormalization point. Within the standard framework with the “ordinary” definition of the effective action as in (5), it was found in [164] that the *essential* couplings $1/v^2(t)$, $1/f^2(t)$, and $\omega/f^2(t)$ are asymptotically free, provided that $1/v^2 > 0$, $1/f^2 > 0$, and $\omega/f^2 < 0$, while λ grows in the UV limit $t \rightarrow \infty$. Note, however, that in [165] it was found that $\omega/f^2 > 0$ is required in the Lorentzian regime to avoid a tachyonic instability of the scalaron. Fixing the correct sign, the running is no longer asymptotically free.

Newton’s constant, or k^2 in terms of the parametrization in (91), is an inessential coupling and does not run. In order to access the running of all couplings separately, including the running of k^2 , an off-shell extension is required, which renders the effective action gauge-independent and parametrization-invariant²⁴. Such an off-shell extension was proposed in [149] by a geometrically defined (field-covariant) “unique” effective action. At the one-loop level, the difference between the “ordinary” definition of the effective action and the “unique” effective action is a correction term proportional to the equations of motion. The “unique” off-shell one-loop beta functions for (91) were calculated in [164] and the running of $1/k^2(t)$ was extracted,

²⁴See also [166–170] for a discussion of the quantum parametrization dependence of the effective action in cosmology.

with the result that $\lim_{t \rightarrow \infty} 1/k^2(t) = 0$ and $\lim_{t \rightarrow \infty} \Lambda(t) = 0$. Thus, the UV limit $t \rightarrow \infty$ found in this way corresponds to the induced gravity scenario $M_{\text{P}}^2 \rightarrow 0$ (i.e., $G_{\text{N}} \rightarrow \infty$) with vanishing (dimensional) cosmological constant $\Lambda \rightarrow 0^{25}$.

While the above quoted results support the status of QDG in $D = 4$ as a perturbative renormalizable theory of quantum gravity, the reason QDG is not usually regarded as a consistent theory of quantum gravity relates to its problem with the additional propagating spin-2 ghost degrees of freedom. In analogy to (49), the momentum space fluctuation operator of QDG defined in the parametrization (89) for arbitrary D on a flat background can be expressed in terms of the projectors (50) and (53) and reads [171]

$$F^{\mu\nu,\rho\sigma}(-P^2) = \frac{(-P^2)}{2} \left[1 + 8c_2 \frac{D-3}{D-2} \frac{(-P^2)}{M_{\text{P}}^2} \right] P^{(2)\mu\nu\rho\sigma} - (D-2) \frac{(-P^2)}{2} \left[1 - 8c_3 \frac{D-1}{D-2} \frac{(-P^2)}{M_{\text{P}}^2} \right] P^{(0,ss)\mu\nu\rho\sigma}. \quad (92)$$

Clearly, this reduces to (49) for $c_2 = c_3 = 0$. Moreover, because of the topological nature of the GB term $\mathfrak{G}$, c_1 does not enter (92). Just as in GR, the diffeomorphism invariance of QDG renders the fluctuation operator (92) degenerate, and a gauge-fixing is required to obtain the propagators. Nevertheless, the tree-level particle spectrum of QDG can already be analyzed on the basis of the pole structure in (92). Defining the two effective masses for $D > 3$,

$$M_2^2 := -\frac{1}{8c_2} \frac{D-2}{D-3} M_{\text{P}}^2, \quad M_0^2 := \frac{1}{8c_3} \frac{D-2}{D-1} M_{\text{P}}^2, \quad (93)$$

the pole structure of the propagators in the spin-2 and spin-0 sectors becomes more transparent [45]:

$$P_{(2)} \propto \frac{-M_2^2}{(-P^2)[(-P^2) - M_2^2]} = \frac{1}{(-P^2)} - \frac{1}{(-P^2) - M_2^2}, \quad (94)$$

$$P_{(0)} \propto \frac{M_0^2}{(-P^2)[(-P^2) - M_0^2]} = -\frac{1}{(-P^2)} + \frac{1}{(-P^2) - M_0^2}. \quad (95)$$

The partial fraction in the second equality reveals that, compared to GR, in QDG there are two additional propagating particles with masses M_2 and M_0 . The first term in (94) corresponds to a massless spin-2 particle and, just as in GR, combines with the first term in (95) to become the massless graviton. The second term of (94) indicates the presence of a propagating massive spin-2 particle originating from the $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ term in (89),

while the second term in (95) indicates the presence of a massive spin-0 particle originating from the R^2 term in (89). Excluding tachyons requires $M_2^2 > 0$ ($c_2 < 0$) and $M_0^2 > 0$ ($c_3 > 0$). The massive spin-0 particle, which can be identified with the scalaron in model (86), is “healthy” (neither a ghost nor a tachyon), while the overall minus sign in the second term of (94) shows that the massive spin-2 particle is a higher-derivative ghost. The presence of ghosts corresponds to states of negative norm, leading to a violation of unitarity [45] (see also [172–175]).

Within an effective low-energy treatment $P^2/M_2^2 \ll 1$, the propagation of the massive spin-2 ghost is strongly suppressed. Whether such an EFT, which still includes the scalaron as a propagating degree of freedom (since the R^2 would not be treated as a perturbation compared to the R term), can be realized depends strongly on the characteristic mass scales M_2 and M_0 , i.e., the values of c_2 and c_3 , respectively. It requires that M_2^2 be large enough that the effective description is valid up to energy scales at which the additional propagating scalaron has interesting phenomenology, such as in the inflationary model (86), but at the same time $M_0^2 \ll 1$ must be sufficiently small that the scalaron can be considered a propagating degree of freedom; see e.g., [176] for discussion of such a scenario in the context of the scalaron-Higgs model. Solar system-based experimental constraints on both c_2 and c_3 are extremely weak. However, while c_2 is practically unconstrained, a large $c_3 = M_{\text{P}}^2/(12M_0^2) \approx 10^9$ is required in (86) if the scalaron is supposed to drive inflation. But even if the problem with the spin-2 ghost can effectively be neglected at sufficiently “low” energies, without a mechanism that prevents the occurrence of the higher-derivative ghost at arbitrarily high energy scales, QDG cannot be considered a fundamental theory.

Recently, the negative conclusion about the ghost-related loss of unitarity in QDG at the fundamental level has been questioned. The questions are related to early proposals about different quantization prescriptions, which modify the pole structure of the propagators in higher-derivative theories [177, 178]. In [47, 179] a new quantization prescription was proposed which turns higher-derivative ghosts into “fakeons” at the expense of a loss of micro-causality. Another resolution of the unitarity problem was suggested in [48, 180]. A key point in this proposal is that the coupling of light matter particles to gravity renders the heavy spin-2 ghost unstable, such that the ghost is not part of the asymptotic particle spectrum. Extending the conclusion that unstable particles must be excluded from the sum of the unitarity relation [181] to the case of unstable ghost particles (which are nevertheless identified as such by the free-particle spectrum), it is concluded in [48] that there is no violation of unitarity in QDG. Nevertheless, in [48, 180] it is also found that the ghosts “propagate backwards in time,” leading to a violation of micro-causality. While this effect can in principle be tested experimentally, it becomes unobservably small for sufficiently heavy ghost masses, such as in QDG if $M_2 \approx M_{\text{P}}$.

Summarizing, the proposals [47, 179] and [48, 180] about the correct treatment of higher-derivative ghost particles both led to the conclusion that the unitarity violation can be avoided at the expense of violating micro-causality, but it seems that a conclusive agreement on this controversially debated issue has

²⁵Since Newton’s constant $G_{\text{N}}(t) \sim 1/k^2(t)$ exceeds the perturbative regime, a perturbative treatment does not seem reliable in the asymptotic limit $t \rightarrow \infty$. However, because Newton’s coupling is an inessential coupling in the ordinary perturbative approach (even if it runs in the covariant Vilkovisky off-shell extension), it should never enter an on-shell observable in an isolated way, but only via a dimensionless combination with other couplings [including $\Lambda(t)$] whose beta function is gauge-independent. Thus, independently of whether G_{N} itself grows beyond perturbative control in the limit $t \rightarrow \infty$, the question should then rather be whether the RG running of this dimensionless combination stays under perturbative control.

not yet been reached. For related work on higher-derivative ghosts (see also [182–194]). For a discussion of the ghost problem in the context of the non-perturbative AS program for quantum gravity, see e.g., [135, 195–199]. For the non-local approach to a ghost-free quantum theory of gravity (see [17–26]).

7. HOŘAVA GRAVITY

The picture emerging from the previously described approaches to providing a consistent fundamental local quantum theory of gravity suggests that the basic principles of relativistic invariance, renormalizability, and unitarity are incompatible in the context of the perturbative quantization of the gravitational interaction: quantum GR is a relativistic and unitary but perturbatively non-renormalizable QFT, while quantum QDG is a relativistic and perturbatively renormalizable but non-unitary QFT. Therefore, in [49, 50], Petr Hořava suggested exploring the consequences of abandoning relativistic invariance while trying to preserve unitarity and perturbative renormalizability.

One of the key motivations for Hořava's proposal comes from the discussion of QDG. While the higher derivatives help to improve the UV behavior of the theory, the higher *time* derivatives are responsible for the occurrence of the additional higher-derivative ghost degrees of freedom and the associated problems with unitarity. The desire to keep the UV-improving effect of the higher derivatives, but at the same time avoid the ghost problem, leads to the idea of allowing *higher spatial derivatives* but restricting to *second-order time derivatives*. Obviously, such a proposal is not compatible with relativistic invariance. It is clear that “sacrosanct” principles, such as relativistic invariance should not be recklessly sacrificed—not only because this changes the fundamental structure of spacetime but also since there are very strong experimental constraints on Lorentz-violating effects.

With this proviso, I first review how the above idea can be formalized by the notion of an anisotropic Lifshitz scaling between space and time and how it can be incorporated into a consistent mathematical framework by formulating the resulting anisotropic theory of gravity in terms of the geometric Arnowitt-Deser-Misner (ADM) variables, giving rise to the Lifshitz theory of gravity, Hořava gravity (HG). Within the ADM formulation, the main difference between GR and HG is the weaker invariance group underlying HG, the *foliation-preserving diffeomorphisms* $\text{Diff}_{\mathcal{F}}$, which form a subgroup of the full diffeomorphisms.

Important consequences of the anisotropic scaling and the less restrictive invariance group in HG are the modified dispersion relations and the presence of an additional propagating gravitational scalar degree of freedom. After a brief discussion of their phenomenological consequences in $D = 2+1$ and $D = 3+1$ dimensions, I review the quantum properties of HG. I first discuss the gauge and propagator structure of the theory and then review the essential steps in the proof of perturbative renormalizability of the projectable version of HG.

Finally, I discuss the UV properties of quantum HG based on the RG flow of the projectable theory in $D = 2 + 1$ dimensions,

which requires explicit calculation of the one-loop divergences within a Lifshitz theory of gravity [53]. I close with a brief summary and an outlook on future perspectives of quantum HG. For earlier reviews of HG with a different focus, especially on the phenomenological constraints and the cosmological applications (see [200–203]).

7.1. Anisotropic Scaling and Modified Propagators

As briefly outlined before, the basic idea of Hořava gravity is to allow higher spatial derivatives but restrict to second-order time derivatives. Obviously, such a proposal implies that relativistic invariance will be lost at the fundamental level. How precisely Lorentz invariance is broken in a manner compatible with this proposal can be made concrete by introducing the anisotropic Lifshitz scaling between time and space [49, 50, 204],

$$t \rightarrow b^{-z} t, \quad x^i \rightarrow b^{-1} x^i. \quad (96)$$

Here, b is a constant scaling parameter and z a dynamical scaling exponent. In analogy to the mass dimension $[\cdot]_{\text{M}}$ introduced in section 3.1, the *anisotropic scaling dimension* is denoted by $[\cdot]_{\text{S}}$. According to the anisotropic scaling law (96), the scaling dimensions of time and space are $[t]_{\text{S}} = -z$ and $[x]_{\text{S}} = -1$. This implies the scaling relations

$$[\partial_t]_{\text{S}} = z, \quad [\partial_i]_{\text{S}} = 1, \quad [\omega]_{\text{S}} = z, \quad [k_i]_{\text{S}} = 1, \quad (97)$$

where ω and k_i are the frequency and spatial momentum, Fourier conjugates to ∂_t and ∂_i . The dynamical scaling exponent z can be thought of as measuring the degree of anisotropy between space and time, with $z = 1$ restoring relativistic invariance. In view of (97), the (Euclidean) anisotropic propagator takes the form

$$\mathcal{P} \propto \frac{1}{\omega^2 + k^2 + \dots + G(k^2)^z} \simeq \begin{cases} \frac{1}{\omega^2 + k^2} = \frac{1}{p^2} & \text{in the IR,} \\ \frac{1}{\omega^2 + G(k^2)^z} & \text{in the UV,} \end{cases} \quad (98)$$

with some coupling constant G such that $[G]_{\text{M}} = -2(z-1)$ and $[G]_{\text{S}} = 0$. This propagator illustrates the basic idea that Lorentz invariance is completely broken by the anisotropic scaling exponent z for $G(k^2)^z \gg k^2$ in the UV limit and is effectively restored in a natural way for $k^2 \gg G(k^2)^z$ in the IR limit [50]²⁶.

7.2. Geometrical Formulation in Terms of ADM Variables

The anisotropic Lifshitz theory of gravity can be consistently formulated within a geometrical framework when described in terms of ADM variables. Following the presentation in [205], I briefly review the ADM formulation in the context

²⁶In general, relevant deformations also lead to different coupling constants in front of different powers of k^2 in the propagator (98), which, as discussed in the context of HG in section 7.4, might prevent a direct restoration of Lorentz invariance in the IR.

of GR, and highlight the differences in HG when the full diffeomorphism invariance $\text{Diff}(\mathcal{M})$ is reduced to the foliation-preserving diffeomorphism $\text{Diff}_{\mathcal{F}}(\mathcal{M})$.

7.2.1. ADM Variables and GR

A point $X \in \mathcal{M}$ in the D -dimensional ambient spacetime $\mathcal{M}$ can be described by local coordinates X^μ . For a globally hyperbolic ambient space, $\mathcal{M}$ can be foliated by a one-parameter family of d -dimensional spatial hypersurfaces Σ_t of constant time t , where $d = D - 1$. The hypersurfaces Σ_t can be thought of as level surfaces of a time field t . The gradient of t defines a natural unit covector field

$$\begin{aligned} n_\mu &:= -\frac{\nabla_\mu t}{\sqrt{-g^{\mu\nu} \nabla_\mu t \nabla_\nu t}}, \\ n^\mu &= g^{\mu\nu} n_\nu, \quad n^\mu n_\mu = -1. \end{aligned} \quad (99)$$

By construction, at each point, the normal vector field $n^\mu(\mathbf{x}, t)$ is orthogonal to Σ_t and therefore allows an orthogonal decomposition of tensor fields with respect to n^μ . In particular, the ambient metric decomposes as

$$g_{\mu\nu} = \gamma_{\mu\nu} - n_\mu n_\nu. \quad (100)$$

Here, $\gamma_{\mu\nu}$ is the tangential part of $g_{\mu\nu}$, i.e., $\gamma_{\mu\nu} n^\mu = 0$. The hypersurfaces Σ_t can be viewed as embeddings of an intrinsically d -dimensional manifold $\tilde{\Sigma}_t$ into the ambient space $\mathcal{M}$. A point $x \in \tilde{\Sigma}_t$ can be described by the local coordinates x^i , $i = 1, \dots, d$. The D -dimensional coordinates $X^\mu = X^\mu(t, \mathbf{x})$ can be parameterized in terms of the time field t and the spatial coordinates x^i . The change of X^μ with respect to t and x^i is given by the coordinate one-form

$$dX^\mu = t^\mu dt + e^\mu_i dx^i. \quad (101)$$

The time vector field t^μ and the soldering form e^μ_i appearing in (101) are defined as

$$t^\mu := \frac{\partial X^\mu(t, \mathbf{x})}{\partial t}, \quad e^\mu_i := \frac{\partial X^\mu(t, \mathbf{x})}{\partial x^i}. \quad (102)$$

As illustrated in **Figure 5**, the lapse function $N(t, \mathbf{x})$ and the shift vector $N^\mu(t, \mathbf{x})$ are defined as the coefficients of the orthogonal decomposition of $t^\mu := N n^\mu + N^\mu$ in the directions normal and tangential to Σ_t , respectively.

The soldering form e^μ_i transforms like a D -dimensional tangential vector with respect to the μ index, i.e., $e^\mu_i n_\mu = 0$, and a d -dimensional vector with respect to the i index. It defines the pullback of tangential tensors in $\mathcal{M}$ to tensors in $\tilde{\Sigma}_t$:

$$e^\mu_i e^\nu_j = \delta^\mu_{\nu}, \quad e^\mu_i e^\mu_j = \delta^i_j. \quad (103)$$

The pullbacks of $\gamma_{\mu\nu}$ and N^μ define the spatial metric γ_{ij} and the spatial shift-vector N^i ,

$$\gamma_{ij} := e^\mu_i e^\nu_j \gamma_{\mu\nu}, \quad N^i := e^\mu_i N^\mu. \quad (104)$$

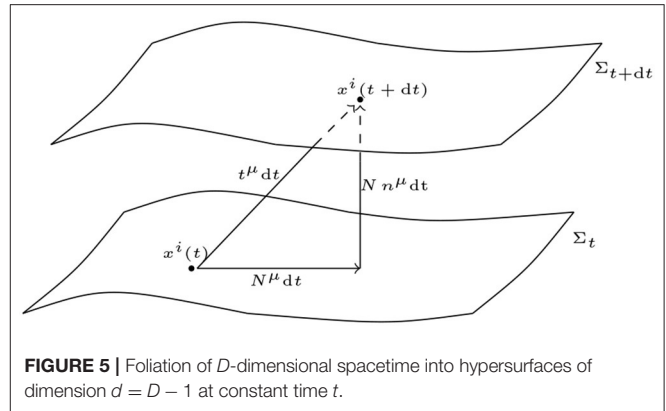


FIGURE 5 | Foliation of D -dimensional spacetime into hypersurfaces of dimension $d = D - 1$ at constant time t .

In terms of dt and dx^i , the ambient space coordinate one-form is expressed as

$$dX^\mu = N n^\mu dt + e^\mu_i (N^i dt + dx^i). \quad (105)$$

Inserting this into (42), the ambient space line element takes the familiar ADM form [206]

$$ds^2 = -N^2 dt^2 + \gamma_{ij} (N^i dt + dx^i) (N^j dt + dx^j). \quad (106)$$

On $\tilde{\Sigma}_t$, the commutator of the (torsion-free and metric-compatible $\nabla_k \gamma_{ij} = 0$) spatial covariant derivative ∇_i defines the d -dimensional spatial curvature tensor by its action on a spatial vector field v^k ,

$$[\nabla_i, \nabla_j] v^k = R^k_{lij}(\gamma) v^l. \quad (107)$$

The relation between the scalar curvature of the D -dimensional ambient space $R(g)$ and the scalar curvature $R(\gamma)$ of the d -dimensional embedded space is given by the Gauss-Codazzi relation (see e.g., [207])

$$\begin{aligned} R(g) &= R(\gamma) - (K^2 - K_{ij} K^{ij}) \\ &\quad - 2 (\nabla_i + a_i) a^i + 2 (D_t + K) K. \end{aligned} \quad (108)$$

Here, $K := \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature K_{ij} , defined via the covariant time derivative D_t as

$$\begin{aligned} K_{ij} &:= \frac{1}{2} D_t \gamma_{ij} = \frac{1}{2N} (\partial_t \gamma_{ij} - \nabla_i N_j - \nabla_j N_i), \\ D_t &:= \frac{1}{N} (\partial_t - \mathcal{L}_N), \end{aligned} \quad (109)$$

where $\mathcal{L}_N$ is the Lie derivative along the spatial shift vector N^i . The acceleration vector a^i in (108) is defined as

$$a_i := \partial_i \ln N. \quad (110)$$

Note that the D -dimensional diffeomorphisms $\text{Diff}(\mathcal{M})$ completely fix the structure and the numerical coefficients of the individual terms in (108). In terms of the ADM variables

(106), the volume element of $\mathcal{M}$ reads $\sqrt{-g} = N\sqrt{\gamma}$, and, modulo surface terms, the Einstein-Hilbert action (39) takes the ADM form

$$S_{\text{EH}} = \frac{M_{\text{P}}^2}{2} \int dt d^3x N \sqrt{\gamma} [K_{ij}K^{ij} - K^2 + R(\gamma)]. \quad (111)$$

It is natural to think of the first two terms in (111), which involve the square of the “velocities” $\partial_t \gamma_{ij}$, as the “kinetic term” for γ_{ij} , and to view $R(\gamma)$ as the “potential,” which involves only spatial derivatives $\partial_k \gamma_{ij}$. In particular, the invariance of the action (39) under $\text{Diff}(\mathcal{M})$ implies that only the very specific combination of ADM operators in (111) is $\text{Diff}(\mathcal{M})$ -invariant. This illustrates how strongly the underlying $\text{Diff}(\mathcal{M})$ invariance in GR restricts the possible operators allowed in the Einstein-Hilbert action when expressed in terms of ADM variables.

7.2.2. Symmetry in GR and HG

In GR, the ADM variables derive from the decomposition of the D -dimensional ambient space metric $g_{\mu\nu}$. Consequently, in this case, the symmetry group acting on the ADM variables consists of the full D -dimensional spacetime diffeomorphisms $\text{Diff}(\mathcal{M})$, or general coordinate transformations,

$$x^i \mapsto \tilde{x}^i(t, x), \quad t \mapsto \tilde{t}(t, x). \quad (112)$$

In general, operators $\mathcal{O}(g_{\mu\nu}, \partial_\nu)$, invariant under $\text{Diff}(\mathcal{M})$, are constructed by scalar contractions of covariant derivatives ∇_μ and curvature tensors $R_{\mu\nu\rho\sigma}$. While the action of $\text{Diff}(\mathcal{M})$ on the D -dimensional ambient metric $g_{\mu\nu}$ is realized linearly as in (45), in view of (106), the action of $\text{Diff}(\mathcal{M})$ on the ADM variables N , N^i , and γ_{ij} is non-linearly realized. Thus, only very particular combinations of $\text{Diff}(\mathcal{M})$ -invariant operators $\mathcal{O}(N, N^i, \gamma_{ij}, \partial_i, \partial_t)$ constructed by scalar contractions of the time and space derivatives ∂_t and ∂_i of the ADM variables N , N^i , and γ_{ij} are allowed.

In contrast to the general coordinate transformations (112), the coordinate transformations that preserve the foliation include the d -dimensional time-dependent spatial diffeomorphisms and the reparameterizations of time,

$$x^i \mapsto \tilde{x}^i(t, x), \quad t \mapsto \tilde{t}(t). \quad (113)$$

Under (113), the ADM fields N , N^i , and γ_{ij} transform as

$$\begin{aligned} N &\mapsto \tilde{N} = N \frac{dt}{d\tilde{t}}, \\ N^i &\mapsto \tilde{N}^i = \left(N^j \frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{x}^i}{\partial t} \right) \frac{dt}{d\tilde{t}}, \\ \gamma_{ij} &\mapsto \tilde{\gamma}_{ij} = \gamma_{kl} \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j}. \end{aligned} \quad (114)$$

Combining the action of an infinitesimal diffeomorphism (45) on the ambient metric $g_{\mu\nu}$, its decomposition in ADM variables (100), and the decomposition of the generator of infinitesimal diffeomorphisms $\varepsilon^\mu = (\varepsilon, \varepsilon^i)$ with $\varepsilon^i(t, x) = \varepsilon^\mu e_\mu^i$ and

$\varepsilon(t, x) = t_\mu \varepsilon^\mu$ the action of an infinitesimal $\text{Diff}(\mathcal{M})$ on the ADM fields γ_{ij} , N^i , and N is derived as

$$\delta_\varepsilon N = \partial_t (\varepsilon N) + \mathcal{L}_\varepsilon N - NN^i \partial_i \varepsilon, \quad (115)$$

$$\delta_\varepsilon N^i = \partial_t (\varepsilon N^i) + \partial_t \varepsilon^i + (\mathcal{L}_\varepsilon N)^i - (N^i N^j + N^2 \gamma^{ij}) \partial_j \varepsilon, \quad (116)$$

$$\delta_\varepsilon \gamma_{ij} = \varepsilon \partial_t \gamma_{ij} + (\mathcal{L}_\varepsilon \gamma)_{ij} + 2N_{(i} \partial_{j)} \varepsilon. \quad (117)$$

Here $\mathcal{L}_\varepsilon$ denotes the Lie derivative along ε^i . The transformation law for the shift vector with covariant index position $N_i = \gamma_{ij} N^j$ can be obtained by combining the transformation laws (116) and (117), giving

$$\delta_\varepsilon N_i = \partial_t (\varepsilon N_i) + (\mathcal{L}_\varepsilon N)_i + \gamma_{ij} \partial_t \varepsilon^j + (N_j N^j - N^2) \partial_i \varepsilon. \quad (118)$$

In contrast to the linear transformation (45) of the ambient metric $g_{\mu\nu}$, the transformations (115–117) of the ADM variables under an infinitesimal $\text{Diff}(\mathcal{M})$ is not linear. The transformations of the ADM variables under $\text{Diff}_{\mathcal{F}}(\mathcal{M})$, for which the time component ε of the generator $\varepsilon^\mu = (\varepsilon, \varepsilon^i)$ is a function of time only, $\varepsilon(t, x) = \varepsilon(t)$, are derived from (115–117) by neglecting terms involving $\partial_i \varepsilon$, and the action of an infinitesimal $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ on the ADM variables is given by

$$\delta_\varepsilon N = \partial_t (\varepsilon N) + \mathcal{L}_\varepsilon N, \quad (119)$$

$$\delta_\varepsilon N^i = \partial_t (\varepsilon N^i) + \partial_t \varepsilon^i + (\mathcal{L}_\varepsilon N)^i, \quad (120)$$

$$\delta_\varepsilon \gamma_{ij} = \varepsilon \partial_t \gamma_{ij} + (\mathcal{L}_\varepsilon \gamma)_{ij}. \quad (121)$$

Likewise, the transformation (118) reduces to

$$\delta_\varepsilon N_i = \partial_t (\varepsilon N_i) + (\mathcal{L}_\varepsilon N)_i + \gamma_{ij} \partial_t \varepsilon^j. \quad (122)$$

Hence, the $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ form a subgroup of the $\text{Diff}(\mathcal{M})$, and the absence of terms proportional to $\partial_i \varepsilon$ has the effect that the transformations (119–122) act linearly on the ADM variables [50, 208].

Mathematically, the $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ are diffeomorphisms that respect the preferred codimension-one foliation $\mathcal{F}$ of $(d+1)$ -dimensional spacetime $\mathcal{M}$ into spatial d -dimensional leaves [50]. On such a foliation, two classes of functions can be defined: functions that depend on all coordinates (t, x^i) and functions that are constant on each spatial leaf, i.e., which depend only on time t . The latter are called “projectable.” From a canonical perspective with a fundamental dynamical field γ_{ij} , the shift vector N^i can be viewed as the gauge field associated with the time-dependent spatial diffeomorphisms with infinitesimal generator $\varepsilon^i(t, x)$, and the lapse function N can be viewed as the gauge field of the reparameterizations of time with infinitesimal generator $\varepsilon(t)$. It therefore seems natural to restrict $N(x, t)$ to be a function of time only, although the versions $N(t, x)$ and $N(t)$ are both compatible with the $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry, essentially leading to two variants of HG.

i) Projectable HG:

The lapse function depends only on time, i.e., $N(t)$, and is not considered a dynamical field. By choosing a global time slicing corresponding to the gauge in which $N(t) = 1$, the foliation-preserving diffeomorphisms reduce to the time-dependent spatial diffeomorphisms.

ii) Non-projectable HG:

The lapse function depends on space and time, and $N(t, \mathbf{x})$ is a propagating degree of freedom, i.e., an integration variable in the path integral. Compared to the projectable theory, the main technical challenge is the enlarged set of $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ -invariants that involve the acceleration vector (110).

Since the two possibilities lead to two different theories with different particle content and different phenomenology, they have to be investigated separately. In particular, the quantization of the non-projectable theory is complicated due to the presence of the fluctuating lapse function, which leads to non-regular propagators [51]. In this paper I mainly focus on the projectable theory but will highlight at several places important differences from the non-projectable theory.

7.3. Projectable HG in $D = 2 + 1$ and $D = 3 + 1$ Dimensions

The action functional of projectable HG in $D = d + 1$ dimensions can be formulated in terms of the ADM variables. The natural assignment of the anisotropic scaling dimensions to the ADM variables follows from (105) and (106):

$$[\gamma_{ij}]_S = 0, \quad [N^i]_S = z - 1, \quad [N]_S = 0. \quad (123)$$

Compared with the stringent constraints on the ADM operators in GR following from the invariance under $\text{Diff}(\mathcal{M})$, the less restrictive invariance under $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ allows for a richer structure and hence more ADM invariants. Nevertheless, there are a number of conditions which limit the possible $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ -invariants in the projectable HG:

1. Formulated in a manifest $\text{Diff}_{\mathcal{F}}$ -invariant way, the shift vector can only arise in combination with a time derivative of the metric γ_{ij} in the form of the covariant time derivative (109). Thus, the invariants in projectable HG can only be constructed by scalar contractions of covariant time derivatives of the metric field $D_t \gamma_{ij}$ (or, equivalently, extrinsic curvatures K_{ij}), covariant space derivatives ∇_i , and spatial curvature tensors R_{ijkl} .
2. Invariance under time-reversal and parity allows only invariants with an even number of time and space derivatives. Writing $S_{\text{HG}} = \int dt d^d x \mathcal{L}_{\text{HG}}$ and $\mathcal{L}_{\text{HG}} = \sum_n c_n \mathcal{O}_{(n)}(D_t, \nabla_i, \gamma_{ij})$ implies that the operators have the general schematic structure (suppressing the summation index n)

$$\mathcal{O}(D_t, \nabla_i, \gamma_{ij}) = \sqrt{\gamma} (D_t \gamma_{ij})^{2k} (\nabla_i)^{2n} (R_{ijkl})^m. \quad (124)$$

3. For HG to be power-counting renormalizable, the action can only include relevant and marginal operators with respect to the anisotropic scaling [50]. Combining the scaling $[S_{\text{HG}}]_S = 0$ with $[\int dt d^d x]_S = -(d + z)$ implies that $[\mathcal{L}_{\text{HG}}]_S = d + z$. Relevant and marginal operators have scaling $[\mathcal{O}_{(j)}(D_t, \nabla_i, \gamma_{ij})]_S \leq d + z$. Combining this with the structure (124) yields the constraint

$$2(kz + n + m) \leq d + z. \quad (125)$$

4. The original motivation of HG as a means of solving the problems with unitarity caused by higher-derivative ghosts is to restrict the invariants in the action to include time derivatives of the metric only up to second-order. In view of the structure (124), this leaves the two possibilities of $k = 1$ and $k = 0$. For the kinetic term with $k = 1$ and $n = m = 0$ to scale marginally under (96), equality in (125) has to be satisfied and implies the *critical scaling* condition

$$z = d. \quad (126)$$

The operators with $k = 0$ correspond to the potential $\mathcal{V}^d$ and, for the critical scaling (126), are restricted by the condition $2(n + m) \leq 2d$.

The action of projectable HG in $D = d + 1$ dimensions (in the gauge $N = 1$), including all relevant and marginal terms with respect to the critical anisotropic scaling, reads

$$S_{\text{HG}} = \frac{1}{2G} \int dt d^d x \sqrt{\gamma} (K_{ij} K^{ij} - \lambda K^2 - \mathcal{V}^{(d)}). \quad (127)$$

As a consequence of (126), the structure of the kinetic term is universal, i.e., independent of d :

$$\sqrt{\gamma} (K_{ij} K^{ij} - \lambda K^2) = \frac{1}{4} (D_t \gamma_{ij}) \mathcal{G}^{ij,kl} (D_t \gamma_{kl}). \quad (128)$$

Here, $\mathcal{G}^{ij,kl}$ is the one-parameter λ -family of “generalized DeWitt metrics”

$$\mathcal{G}^{ij,kl} := \frac{\sqrt{\gamma}}{2} (\gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} - 2\lambda \gamma^{ij} \gamma^{kl}). \quad (129)$$

There are two special values of λ . The first is the “relativistic” value $\lambda = 1$, which leads to an enhanced symmetry [50]. The second is the “conformal” value $\lambda_c = 1/d$, where $\mathcal{G}^{ij,kl}$ is degenerate, which also leads to an enhanced symmetry, namely local anisotropic Weyl invariance [49]. For non-singular values $\lambda \neq \lambda_c$, the inverse is given by

$$\mathcal{G}_{ij,kl} = \frac{1}{\sqrt{\gamma}} \left(\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - \frac{2\lambda}{d\lambda - 1} \gamma_{ij} \gamma_{kl} \right). \quad (130)$$

For $\lambda < \lambda_c$ (129) is positive definite, and for $\lambda > \lambda_c$ it is indefinite. In the context of GR, this property was found in [209] to be directly related to the attractive or repulsive nature of gravity.

Note the difference between (127) and the Einstein-Hilbert action (111) in ADM variables, where the $\text{Diff}(\mathcal{M})$ invariance completely fixed the structure of the action, i.e., the relative coefficient between the two terms $K_{ij} K^{ij}$ and K^2 in the kinetic terms as well as the coefficient of the potential R . In HG, $K_{ij} K^{ij}$, K^2 , and the terms in $\mathcal{V}^d$ are separately invariant under $\text{Diff}_{\mathcal{F}}(\mathcal{M})$. In particular, λ is a free parameter of the theory.

The potential $\mathcal{V}^{(d)}$ of projectable HG is defined in terms of d -dimensional curvature invariants and, according to (3), includes all relevant and marginal operators with respect to the critical

anisotropic scaling. In contrast to the kinetic term, the potential is not universal and the number and complexity of invariants in the potential grows with increasing d . Restricting to $d = 2$ and $d = 3$, up to total derivatives the possible curvature invariants read [210]

$$\mathcal{V}^{(d=2)} = 2\Lambda + \mu R^2, \quad (131)$$

$$\mathcal{V}^{(d=3)} = 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij}R^{ij} + \nu_1 R^3 + \nu_2 R R_{ij}R^{ij} + \nu_3 R^i_{\ j}R^j_{\ k}R^k_{\ i} + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk}. \quad (132)$$

Note that in $d = 2$ and $d = 3$ all invariants involving the Riemann tensors are absent. In addition, in $d = 2$, the linear Einstein-Hilbert term $\sqrt{\gamma}R$ is a total derivative. In general, the Riemann tensor in d dimensions has $d^2(d^2 - 1)/12$ independent components. Hence, in $d = 2$, there is only one independent component associated with the Ricci scalar,

$$R_{ijkl}^{(d=2)} = \frac{R}{2} (\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \quad (133)$$

Likewise, in $d = 3$, there are only six independent components of the Riemann curvature tensors that are associated with the six components of the Ricci tensor R_{ij} . This can also be seen from the fact that in $d = 3$, the Weyl tensor $C_{ijkl} \equiv 0$ vanishes identically, which allows all curvature tensors R_{ijkl} to be expressed in terms of R_{ij} and R via

$$R_{ijkl}^{(d=3)} = R_{ik}\gamma_{jl} + R_{il}\gamma_{jk} + R_{jk}\gamma_{il} + R_{jl}\gamma_{ik} - \frac{R}{2} (\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \quad (134)$$

The mass dimensions of the coupling constants follow from $[S_{\text{HG}}]_{\text{M}} = 0$, $[\gamma_{ij}]_{\text{M}} = 0$, and $[\partial_i]_{\text{M}} = [\partial_t]_{\text{M}} = [N^i]_{\text{M}} = 1$:

$$\begin{aligned} [G]_{\text{M}} &= 1 - d, & [\Lambda]_{\text{M}} &= 2, & [\lambda]_{\text{M}} &= [\eta]_{\text{M}} = 0, \\ [\mu]_{\text{M}} &= [\mu_1]_{\text{M}} = [\mu_2]_{\text{M}} = -2, \\ [\nu_1]_{\text{M}} &= [\nu_2]_{\text{M}} = [\nu_3]_{\text{M}} = [\nu_4]_{\text{M}} = [\nu_5]_{\text{M}} = -4. \end{aligned} \quad (135)$$

A new set of dimensionless couplings $[\tilde{G}]_{\text{M}} = [\tilde{\Lambda}]_{\text{M}} = [\tilde{\mu}_i]_{\text{M}} = [\tilde{\nu}_i]_{\text{M}} = 0$ is trivially defined by expressing the couplings in units of a common, a priori unspecified, mass scale M_* :

$$\begin{aligned} \tilde{G} &:= \frac{G}{M_*^{1-d}}, & \tilde{\Lambda} &:= \frac{\Lambda}{M_*^2}, \\ \tilde{\mu}_i &:= M_*^2 \mu_i, & \tilde{\nu}_i &:= M_*^4 \nu_i. \end{aligned} \quad (136)$$

The parametrization (136) is useful when discussing phenomenological bounds on HG.

7.4. Particle Spectrum, Dispersion Relations, and Phenomenological Constraints

The particle spectrum of projectable HG in $d = 2$ and $d = 3$ is derived along the same lines as for GR by expanding the action

around flat space $\bar{\gamma}_{ij} = \delta_{ij}$, $\bar{N}^i = 0$ to quadratic order in the linear perturbations²⁷

$$h_{ij} := \gamma_{ij} - \bar{\gamma}_{ij}, \quad n^i := N^i - \bar{N}^i. \quad (137)$$

Substituting the irreducible decomposition of the perturbations,

$$\begin{aligned} n^i &= n^i_{\text{T}} + \partial^i B, \\ h_{ij} &= h_{ij}^{\text{TT}} + 2\partial_{(i}v_{j)}^{\text{T}} + \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right) \Psi + \frac{\partial_i \partial_j}{\partial^2} E \end{aligned} \quad (138)$$

with the three scalars Ψ , E , and B , the differentially constrained transversal vector fields $\partial_i v^i_{\text{T}} = 0$ and $\partial_i n^i_{\text{T}} = 0$, and the transversal traceless tensor field $h_{ij}^{\text{TT}} \delta^{ij} = \partial^i h_{ij}^{\text{TT}} = 0$ into the quadratic action, “integrating out” the non-dynamical modes v^i_{T} and E , fixing the gauges $B = 0$ and $n^i_{\text{T}} = 0$, yields after Fourier transformation to momentum space the dispersion relations for the physical propagating degrees of freedom h_{ij}^{TT} and Ψ . As discussed in the previous section, in $D = 2 + 1$ there are no transversal traceless (TT) modes h_{ij}^{TT} . However, in contrast to GR, which has no local degrees of freedom in $D = 2 + 1$ dimensions, in HG there is an additional propagating scalar degree of freedom, which is a consequence of the reduced $\text{Diff}_{\mathcal{F}}(\mathcal{M})$ invariance of HG; cf. the discussion in section 3.2. The additional scalar mode persists even for low energies such that there is no smooth limit of HG to GR.

In $d = 2$, the additional gravitational scalar has the following non-relativistic dispersion relation expressed in terms of the dimensionless couplings (136):

$$\omega_{\text{S}}^2 = 4\tilde{\mu} \frac{1 - \lambda}{1 - 2\lambda} \frac{k^4}{M_*^2}. \quad (139)$$

Clearly, the dispersion relation for the additional scalar does not reduce to the linear relativistic form at low energies $k^2/M_*^2 \ll 1$, which again is a consequence of the absence of the relevant linear curvature invariant in the potential (131).

In $d = 3$, aside from the additional scalar mode, the spectrum encompasses a propagating TT mode. Both modes have non-relativistic dispersion relations,

$$\begin{aligned} \omega_{\text{TT}}^2 &= k^2 \left[\eta + \tilde{\mu}_2 \frac{k^2}{M_*^2} + \tilde{\nu}_5 \frac{k^4}{M_*^4} \right], \\ \omega_{\text{S}}^2 &= \frac{1 - \lambda}{1 - 3\lambda} k^2 \left[-\eta + (8\tilde{\mu}_1 + 3\tilde{\mu}_2) \frac{k^2}{M_*^2} + (8\tilde{\nu}_4 + 3\tilde{\nu}_5) \frac{k^4}{M_*^4} \right]. \end{aligned} \quad (140)$$

Before discussing experimental constraints on HG, I briefly review several theoretical restrictions:

1. Despite the critical scaling (126), which guarantees that the non-relativistic dispersion relations depend only quadratically on the frequency ω , it is essential to make sure that no unitarity-violating propagating ghost degrees of freedom enter

²⁷This implies $\Lambda = 0$. For a discussion of the cosmological constant in HG, see e.g. [211].

in HG. Demanding the absence of ghosts leads to the condition $G > 0$, which ensures the positivity of the TT kinetic term, and the requirement that λ must lie in the gapped interval $\lambda < 1/d$ or $\lambda > 1$, bounded by the points of enhanced symmetry, to ensure the positivity of the scalar kinetic term.

2. In contrast to the situation in $D = 2 + 1$, thanks to the presence of the relevant operator $\propto R$ in (132), for low energies $k^2/M_*^2 \ll 1$ both dispersion relations (140) in $D = 3 + 1$ reduce to the linear relativistic relations

$$\begin{aligned}\omega_{\text{TT}}^2 &= \eta k^2 + \mathcal{O}(k^2/M_*^2), \\ \omega_S^2 &= -\eta \frac{1-\lambda}{1-3\lambda} k^2 + \mathcal{O}(k^2/M_*^2).\end{aligned}\quad (141)$$

However, because of the requirement that $(1-3\lambda)/(1-\lambda) > 0$, there is no value of $\eta \neq 0$ at which both of the relations in (141) are simultaneously positive, and for $\eta = 0$ the linear relativistic dispersion relation is lost, just as in $D = 2 + 1$. For $\eta > 0$, this leads to a tachyonic instability of the scalar mode at low energies $k^2/M_*^2 \ll 1$. An obvious attempt to circumvent this problem is to keep $\eta > 0$ and tune λ very close to 1, in order to suppress the IR instability of the scalar mode. Unfortunately, this leads to strong coupling for the scalar mode at low energies [212–215], invalidating the perturbative treatment that underlies the power-counting renormalizability [216]; see, however [200, 217–220]. In summary, without a mechanism by which this IR problem can be avoided, the projectable theory seems to be excluded on phenomenological grounds.

3. The IR instability problem can be remedied in the non-projectable version of HG in which the potential (132) involves invariants including the acceleration vector (110), thanks to the propagating lapse function. To illustrate the difference from the projectable case, I present the potential and the dispersion relation for the non-projectable theory in $d = 2$. In the non-projectable case, the action (127) acquires a modified volume element $dt d^d x \sqrt{\gamma} \mapsto dt d^d x N \sqrt{\gamma}$, and the potential (131) for the non-projectable theory in $d = 2$ dimensions is enlarged by additional invariants,

$$\begin{aligned}\mathcal{V}_{\text{np}}^{(2)} &= 2\Lambda - \eta R - \alpha a_i a^i + \mu R + \rho_1 \Delta R + \rho_2 R a_i a^i \\ &\quad + \rho_3 (a_i a^i)^2 + \rho_4 a_i a^i \nabla_j a^j + \rho_5 (\nabla_i a^i)^2 + \rho_6 \nabla_i a_j \nabla^i a^j.\end{aligned}\quad (142)$$

Defining the perturbation of the lapse function $\phi := N - 1$ (with the choice $\bar{N} = 1$ for the background value of the lapse function), expansion of the action around the flat background ($\Lambda = 0$) up to quadratic order in the linear perturbations leads to the dispersion relation for the single scalar propagating degree of freedom [51],

$$\begin{aligned}\omega_S^2 &= \left(\frac{1-\lambda}{1-2\lambda} \right) (\eta^2 k^2 + (4\alpha\mu + 2\eta\rho_1)k^4 \\ &\quad + [\rho_1^2 - 4\mu(\rho_5 + \rho_6)]k^6)^{-\alpha - (\rho_5 + \rho_6)k^2}.\end{aligned}\quad (143)$$

In particular, among the additional invariants in (142), there is a relevant operator proportional to $\alpha N \sqrt{\gamma} a^i a_i$ that leads to the required modifications of the low-energy limit. The

freedom in tuning the additional coupling constant α can be used to avoid the IR instability. In [221] it was found that for $0 < \alpha < 2$ the instability can be avoided in non-projectable HG. However, as already anticipated in [50] and supported by different arguments in [51, 214, 222], the presence of the propagating lapse function N in the non-projectable version leads to essential complications with the quantization, which I briefly comment on in section 8.

Aside from these theoretical restrictions, there are phenomenological constraints stemming from experimental bounds on Lorentz violation (LV) (see e.g., [223–228]). In the context of HG, these can be divided into two regimes:

1. LV in the IR:

Despite the suppression of higher-order terms in the dispersion relations (140) for low energies $k^2/M_*^2 \ll 1$, HG does not smoothly connect to GR in the IR, but rather to a modified theory of gravity with an additional propagating gravitational scalar degree of freedom. Deviations from GR can be quantified by a variety of experiments and mainly lead to restrictions on the couplings of the relevant operators in the IR. Experimental constraints come from deviations of the observed helium abundance during Big Bang nucleosynthesis [221, 229, 230] from post-Newtonian parameters [214, 231, 232], binary pulsars [233], and black holes [234–236]. The most stringent constraint, however, comes from the recent detection of gravitational waves from the binary neutron star merger event GW170817 [237]. The inferred speed of propagation of the TT mode strongly constrains the parameter, $|\eta - 1| \lesssim 10^{-15}$, but the propagation speed of the scalar mode remains largely unconstrained (cf., [238]).

2. LV in the UV:

LV effects in the gravitational sector at high energies are not as strongly restricted as in the matter sector provided by the SM particles. In particular, the scale M_* might naturally be identified with the LV scale in the gravitational sector. Observations sensitive to the higher-order corrections in the dispersion relations (140) provide a lower bound on M_* . However, LV effects in the SM are constrained much more tightly, and a mechanism is needed that would prevent LV effects percolating from the gravitational sector to the matter sector [227]. While several such mechanisms have been suggested (see e.g., [239–246]), it remains an open question as to whether they can ultimately be realized in HG [247, 248]. In the case of there being a universal LV scale (i.e., when the LV scale in the matter sector can be identified with the LV scale M_* in the gravitational sector), the observation of synchrotron radiation from the crab nebula would provide a lower bound on M_* around the grand unification scale $M_* > 10^{16}$ GeV [130].

Summarizing, the “healthy extension” of the non-projectable model is still phenomenologically viable [221, 238], but stronger constraints on the IR parameter, as well as on M_* , have the potential to rule out the theory. Moreover, regarding the quantum theory, these properties will rely on the IR limit of the RG flow for the couplings of the relevant operators, as briefly discussed in section 9 for projectable HG in $d = 2 + 1$ dimensions.

8. QUANTUM HOŘAVA GRAVITY

So far, all considerations in HG have been purely classical. However, the main motivations for proposing a Lifshitz theory of gravity are its unitarity and perturbative renormalizability, which was originally conjectured based on power-counting arguments [50]. While this conjecture has stimulated a vast amount of research devoted to specific applications of HG in various scenarios, the question of whether HG is indeed perturbatively renormalizable beyond power counting remained open for a long time. It was ultimately answered in the affirmative for the projectable version of HG in [51]. Furthermore, for HG to qualify as a UV-complete theory, its RG structure must also be investigated, which in turn requires explicit loop calculations. In this section, I discuss both these aspects. In order to establish a connection with the general formalism in section 2, in the remaining sections I use Euclidean signature by the Wick rotation $t \mapsto it$ and $N^i \mapsto -iN^i$, which effectively leads to a sign flip of the potential in (127).

8.1. Non-local Gauge-Fixing and Propagators

Since HG is a gauge theory with invariance group $\text{Diff}_{\mathcal{F}}(\mathcal{M})$, its fluctuation operator (10) is degenerate and its perturbative quantization requires a gauge-fixing. In contrast to relativistic theories, in Lifshitz theories the situation is more complicated because of the anisotropic scaling between space and time: a standard local gauge-fixing causes the propagators of the theory to behave in an irregular way, ultimately leading to spurious non-local divergences [51]. Even if, on general grounds, it might be expected that these non-local divergences ultimately cancel order by order in the perturbative expansion, their presence would greatly complicate the general analysis of renormalizability as well as the intermediate calculations. Therefore, a new type of non-local gauge-fixing was proposed in [51], which leads to regular propagators.

In the background field method, the geometric fields γ_{ij} and N^i are decomposed according to (137). As in the general case for relativistic theories (19), the gauge-breaking action in HG is quadratic in the gauge condition χ^i ,

$$S_{\text{gb}} = \frac{\sigma}{2G} \int dt d^d x^i \sqrt{\gamma} \chi^i O_{ij} \chi^j, \quad (144)$$

where σ is a gauge parameter. Guidance for finding a suitable gauge condition χ^i can be obtained by looking at the spatial part of the relativistic gauges of type (68), which expressed in terms of ADM variables (106), with the background covariant derivatives $\bar{D}_t$ and $\bar{\nabla}_i$ and the gauge parameter c_1 , have the general structure

$$\chi^i[\bar{\gamma}, \bar{N}; h, n] = \bar{D}_t n^i + (\bar{\gamma}^{ij} \bar{\gamma}^{k\ell} - c_1 \bar{\gamma}^{ik} \bar{\gamma}^{j\ell}) \bar{\nabla}_k h_{j\ell}. \quad (145)$$

A characteristic feature of these “quasi-relativistic gauge conditions” is that they artificially render the shift perturbation n^i propagating, owing to the time derivative $\bar{D}_t n^i$. However, the gauge condition in the form (145) is not adequate, as it does not scale homogeneously under (96), which can be seen by

comparing $[\bar{D}_t n^i]_S = 2d - 1$ with $[\bar{\gamma}^{ij} \bar{\gamma}^{k\ell} \bar{\nabla}_k h_{j\ell}]_S = 1$. A possible solution is to omit the term $\bar{D}_t n^i$ from (145), but this would lead precisely to the aforementioned irregular propagators [51]. Therefore, keeping the $\bar{D}_t n^i$ term, the only option is to increase the scaling dimension of the remaining terms by decorating them with additional spatial derivatives:

$$\chi^i[\bar{\gamma}, \bar{N}; h, n] = \bar{D}_t n^i + B^{ij} \bar{\gamma}^{k\ell} (\bar{\nabla}_k h_{j\ell} - c_1 \bar{\nabla}_j h_{k\ell}). \quad (146)$$

Here, $B^{ij}(\bar{\gamma}; \bar{\nabla})$ is a differential operator of order $2(d - 1)$, which apart from $\bar{\nabla}_i$ involves only the background metric $\bar{\gamma}_{ij}$. Without introducing any new dimensional parameter, S_{gb} should have a marginal anisotropic scaling $[S_{\text{gb}}]_S = 0$, which in view of the critical scaling (126) and $[dt d^d x^i]_S = 2d$ implies $[\chi^i O_{ij} \chi^j]_S = -2d$. Therefore, while (146) with $[B^{ij}]_S = 2(d - 1)$ ensures a homogeneous scaling $[\chi^i]_S = 2d - 1$, it requires a scaling of $[O_{ij}]_S = -2(d - 1)$. Consequently, if the operator $O_{ij}(\bar{\gamma}; \bar{\nabla})$ includes only powers of γ_{ij} and ∇_i , it must be of the non-local form²⁸

$$O_{ij} = (-1)^{d-1} (\bar{\Delta}^{(d-1)} \bar{\gamma}^{ij} + \xi \bar{\nabla}^i \bar{\Delta}^{(d-2)} \bar{\nabla}^j)^{-1}, \quad \xi \neq -1. \quad (147)$$

For the particularly useful choice of $B^{ij} = (O^{-1})^{ij}/2\sigma$ and $c_1 = \lambda$, the metric and shift fluctuations in the quadratic action of projectable HG decouple, leading to the two-parameter family of (ξ, σ) gauge conditions [51],

$$\chi^i[\bar{\gamma}, \bar{N}; h, n] = \bar{D}_t n^i + \frac{1}{2\sigma} (O^{-1})^{ij} \bar{\gamma}^{k\ell} (\bar{\nabla}_k h_{j\ell} - \lambda \bar{\nabla}_j h_{k\ell}). \quad (148)$$

The gauge-fixing given by (147) and (148) leads to the aforementioned regular propagators, discussed in more detail in the next subsection. Unfortunately, the same gauge-fixing does not seem to work in the non-projectable theory; it leads to irregular terms in the propagators involving the lapse function, which is absent in the projectable theory [51].

8.2. Regular Propagators, Superficial Degree of Divergence, and Renormalizability

In the context of Lifshitz theories with anisotropic scaling (96), an important concept is the notion of a *regular* propagator, which also plays a central role in the proof of perturbative renormalizability of HG. A propagator for two generalized fields

²⁸The order of the covariant derivatives in (147) is a matter of choice, as different orders differ only in curvature terms that do not affect the principal part of the fluctuation operator. When lower-derivative parts are included in the operator (147), there may be “preferred choices” that simplify the lower-derivative parts of the fluctuation operator. In (147), a symmetric ordering has been chosen. Another natural symmetric choice is $O_{ij} = -(\bar{\nabla}_{i_1} \bar{\nabla}_{i_2} \dots \bar{\nabla}_{i_{(d-2)/2}} (\bar{\Delta} \gamma^{kl} + \xi \bar{\nabla}^k \bar{\nabla}^l) \bar{\nabla}^{i_{(d-2)/2+1}} \dots \bar{\nabla}^{i_d})^{-1}$.

ϕ_1 and ϕ_2 with anisotropic scaling $[\phi_1]_S = s_1$ and $[\phi_2]_S = s_2$ is of the regular form

$$\langle \phi_1, \phi_2 \rangle = \sum \frac{P(\omega, k)}{D(\omega, k)},$$

$$D = \prod_{m=1}^M \left[A_m \omega^2 + B_m k^{2d} + \dots \right], \quad (149)$$

if and only if $P(\omega, k)$ is a polynomial in ω and k^i with leading anisotropic scaling $[P]_S \leq s_1 + s_2 + 2d(M-1)$ and $A_m > 0$ and $B_m > 0$ are strictly positive constants. The ellipsis represents terms with subleading scaling dimensions, which generically originate from relevant operators in the action. The scaling properties ensure that the propagator has the right fall-off properties at small distances and time intervals, i.e., it scales as $[\langle \phi_1, \phi_2 \rangle]_S \leq s_1 + s_2 - 2d$ in the UV limit for high frequencies and momenta in momentum space.

With the choice (148), the propagators of projectable HG in $D = 2 + 1$ and $D = 3 + 1$ dimensions are derived on a flat background $\bar{\gamma}_{ij} = \delta_{ij}$ and $\bar{N}^i = 0$. Upon inserting the decomposition (138) for the fluctuations h_{ij} and n_i into the gauge-fixed quadratic action, the gauge-fixed fluctuation operator (25) has a block-diagonal form in the scalar, vector, and tensor sectors and can be inverted algebraically in momentum space. The propagators for the original h_{ij} and n_i fields are recovered by using (138) again. In $D = 2 + 1$ the propagators read [51],

$$\langle h_{ij}, h_{kl} \rangle = 2G \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \frac{2\lambda}{1-2\lambda} \delta_{ij}\delta_{kl} \right] \mathcal{P}_S(\omega, k), \quad (150)$$

$$\langle n_i, n_j \rangle = 4\mu G k^2 \left[\frac{2(1-\lambda)}{(1-2\lambda)} \delta_{ij} - \frac{k_i k_j}{k^2} \right] \mathcal{P}_S(\omega, k). \quad (151)$$

The tensor combination in (150) is just the inverse DeWitt metric (130) in $d = 2$ flat space. In order to arrive at the final forms (150) and (151), the gauge parameters (ξ, σ) have to be chosen such that there is a single pole

$$\mathcal{P}_S(\omega, k) = \left[\omega^2 + 4\mu \frac{1-\lambda}{1-2\lambda} k^4 \right]^{-1},$$

$$\sigma = \frac{1-2\lambda}{8\mu(1-\lambda)}, \quad \xi = -\frac{1-2\lambda}{2(1-\lambda)}. \quad (152)$$

Clearly, the propagators (150) and (151) are both of the regular form (149)²⁹.

²⁹The ghost field propagator $\langle c_i^*, c^j \rangle = G \delta_i^j \mathcal{P}_S(\omega, k)$, which is derived from the gauge-fixing (148) according to the general rule (22), also has the regular form [51].

In $D = 3 + 1$ dimensions the analogous procedure yields the propagators for the h_{ij} and n_i fields [51],

$$\langle h_{ij}, h_{kl} \rangle = 2G \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \mathcal{P}_{TT} - 2G \delta_{ij}\delta_{kl} \left[\mathcal{P}_{TT} - \frac{1-\lambda}{1-3\lambda} \mathcal{P}_S \right]$$

$$+ 2G \left(\delta_{ij} \frac{k_k k_l}{k^2} + \delta_{kl} \frac{k_i k_j}{k^2} \right) [\mathcal{P}_{TT} - \mathcal{P}_S]$$

$$+ 2G \frac{k_i k_j k_k k_l}{k^4} \left[\frac{7\lambda-5}{1-\lambda} \mathcal{P}_{TT} + \frac{1-3\lambda}{1-\lambda} \mathcal{P}_S \right], \quad (153)$$

$$\langle n_i, n_j \rangle = G \frac{v_5}{1-\lambda} k^4 \left[2(1-\lambda) \delta_{ij} - (1-2\lambda) \frac{k_i k_j}{k^2} \right] \mathcal{P}_S. \quad (154)$$

Again, in order to arrive at the final forms (153) and (154), the gauge parameters (ξ, σ) have been chosen in such a way that there are only the two physical poles³⁰

$$\mathcal{P}_{TT} = [\omega^2 + v_5 k^6]^{-1},$$

$$\mathcal{P}_S = \left[\omega^2 + \frac{(1-\lambda)(8v_4 + 3v_5)}{1-3\lambda} k^6 \right]^{-1},$$

$$\sigma = \frac{1}{2v_5}, \quad \xi = -\frac{1-2\lambda}{2(1-\lambda)}. \quad (155)$$

The additional second pole in $D = 3 + 1$ is due to the TT mode, which is absent in $D = 2 + 1$ dimensions. Again, the propagators (153) and (154) are of the regular form (149).

The superficial degree of divergence in HG is obtained along the same lines as in (65), but with the anisotropic scaling of loop frequencies and momenta (97). Provided the propagators are of the regular form, it reads [51]

$$D_{\text{HG}}^{\text{div}} = 2d - dT - X - (d-1)l_N, \quad (156)$$

where T and X are the numbers of time derivatives and spatial derivatives, respectively, acting on external legs and l_n is the number of external n -legs. Due to the $\text{Diff}_{\mathcal{F}}$ invariance of the counterterms it is sufficient to focus on diagrams with $l_n = 0$ ³¹. From (156) it follows that $D_{\text{HG}}^{\text{div}} < 0$ with more than two time derivatives or d space derivatives on external h_{ij} -legs. If $D_{\text{HG}}^{\text{div}} < 0$ were indeed to imply the absence of divergences, only local operators with at most two time derivatives or d spatial derivatives acting on h_{ij} would have to be renormalized and HG would be perturbatively renormalizable. However, there are two complications that prevent us from immediately drawing this conclusion. The first is the problem of (overlapping)

³⁰The propagators (153) and (154) with the poles (155) are derived by taking into account only those operators in the potential (132) that have a marginal anisotropic scaling. If the relevant operators in (132) were taken into account, they would lead to relevant deformations in the propagators, i.e., additional terms with lower k -dependence. Positive definiteness of O_{ij} requires $\xi > -1$, which is not satisfied for $\lambda > 1$ in the gauge (155). This, however, does not seem to lead to difficulties in the perturbative approach, at least as far as gauge-independent on-shell quantities are concerned, such as the beta functions of the essential couplings in $(2+1)$ -dimensional HG discussed in section 9.

³¹This statement also relies on the $\text{Diff}_{\mathcal{F}}$ -invariant structure of the counterterms proven in [52], as factors of the shift vector in $\text{Diff}_{\mathcal{F}}$ -invariant operators can only occur in the form of the covariant time derivative (109).

subdivergences, which is also present in non-relativistic theories, i.e., a diagram might diverge despite $D_{\text{HG}}^{\text{div}} < 0$. However, in [249] it was shown that the combinatorics of the recursive order-by-order subtraction of the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) scheme [250–252] works essentially the same as in relativistic theories.

The second problem is similar but inherently related to the non-relativistic nature of the theory. It can be illustrated by considering a generic L -loop Feynman integral that is free of subdivergences and has $D_{\text{HG}}^{\text{div}}(\mathcal{I}) < 0$:

$$\mathcal{I} = \int d\omega_{(L)} d^d k_{(L)} f(\omega_{(L)}, k_{(L)}), \quad (157)$$

$$f(\omega_{(L)}, k_{(L)}) = \int \prod_{\ell=1}^{L-1} d\omega_{(\ell)} d^d k_{(\ell)} \tilde{f}(\{\omega_{(\ell)}\}, \{k_{(\ell)}\}; \omega_{(L)}, l_{(L)}). \quad (158)$$

The absence of subdivergences implies that the integrations over the $L - 1$ loop integrals converge and result in a function $f(\omega_{(L)}, k_{(L)})$ which, upon suppressing the dependence on the external momenta, depends only on the L th loop frequency $\omega_{(L)}$ and spatial momentum $k_{(L)}$. The anisotropic scalings $[\omega]_S = d$ and $[k]_S = 1$ imply that $[f]_S = D^{\text{div}}(\mathcal{I}) - 2d$. However, in contrast to relativistic theories, in which $f(\omega_{(L)}, k_{(L)})$ can depend only on the relativistic combination $p_{(L)}^2 = \omega_{(L)}^2 - k_{(L)}^2$, in Lifshitz theories the anisotropic scaling is less restrictive and $f(\omega_{(L)}, k_{(L)})$ can take different forms, such as

$$f(\omega_{(L)}, k_{(L)}) = \begin{cases} \omega_{(L)}^{-1+n} k_{(L)}^{D^{\text{div}}(\mathcal{I})-d(1+n)}, \\ \omega_{(L)}^{-1-n} k_{(L)}^{D^{\text{div}}(\mathcal{I})-d(1-n)}. \end{cases} \quad (159)$$

The problem is that, despite the fact that $D^{\text{div}}(\mathcal{I}) < 0$, the total integral $\mathcal{I}$ may diverge as the individual integrals over the frequency [as in the first case of (159)] or the spatial momentum [as in the second case of (159)] diverge. In [51] it was shown that this problem is absent if the propagators are of the regular form (156), in which case $D_{\text{HG}}^{\text{div}}(\mathcal{I}) < 0$ really implies convergence of $\mathcal{I}$. As shown earlier, all propagators in projectable HG can be brought into the regular form (149) by the non-local gauge-fixing (147) and (148). Combined with the $\text{Diff}\mathcal{F}(\mathcal{M})$ invariance of the counterterms shown in [52], this completes the proof of perturbative renormalizability of projectable HG [51]. Unfortunately, the proof does not extend to the non-projectable theory, as not all propagators can be brought into the regular form (149) for the gauge-fixing (147) and (148), because of the propagating lapse function. This, of course, does not imply that the non-projectable theory is perturbatively non-renormalizable; it simply means that other methods are needed to investigate the renormalization structure of the non-projectable theory.

8.3. Auxiliary Field, Local Formulation, and Path Integral

The Euclidean path integral (2) for projectable HG has the form

$$Z_{\text{HG}} = (\text{Det } O_{ij})^{1/2} \int \mathcal{D}[N^i, \gamma_{ij}, c^i, c_i^*] e^{-S_{\text{tot}}[N^i, \gamma_{ij}, c^i, c_i^*]} \quad (160)$$

with the total action

$$S_{\text{tot}} = S_{\text{HG}} + S_{\text{gb}} + S_{\text{gh}}, \quad (161)$$

including the HG action (127), the gauge-breaking action (144), and the ghost action S_{gh} , which derives from the gauge condition (148) according to the general definition (21) with the ghost operator (22).

Because of the gauge condition (148) with the non-local operator O_{ij} defined in (147), the gauge-breaking action S_{gb} introduces a non-locality in S_{tot} . However, this non-locality only persists in the shift-shift sector of S_{gb} ,

$$S_{\text{gb}} = \frac{\sigma}{2G} \int dt d^d x \sqrt{\gamma} (\bar{D}_t n^i O_{ij} \bar{D}_t n^j + \text{local terms}). \quad (162)$$

The non-local part can be rendered local by “integrating in” the auxiliary field π_i via the Gaussian functional integral:

$$\begin{aligned} & (\text{Det } O_{ij})^{1/2} \exp \left[- \int dt d^d x \frac{\sigma \sqrt{\gamma}}{2G} \bar{D}_t n^i O_{ij} \bar{D}_t n^j \right] \\ &= \int \mathcal{D}[\pi_i] \exp \left[- \int dt d^d x \frac{\sqrt{\gamma}}{G} \left(\frac{1}{2\sigma} \pi_i (O^{-1})^{ij} \pi_j - i \pi_i \bar{D}_t n^i \right) \right]. \end{aligned} \quad (163)$$

The Hubbard-Stratonovich-type transformation (163) reveals the role of π_i as momentum canonically conjugated to n^i . The field π_i also shares similarities with the Nakanishi-Lautrup field used in the BRST formalism to ensure the off-shell nilpotency of the Slavnov operator (see e.g., [52]).

The field π_i has mass dimensionality $[\pi_i]_M = 1$ and scaling dimensionality $[\pi_i]_S = 1$ (for arbitrary d). In [51] it was verified that the $\langle \pi_i, \pi_j \rangle$ and $\langle \pi_i, n_j \rangle$ propagators are also of the regular form (149) and that the presence of the π_i field does not affect the regularity of the h_{ij} and n_i propagators. Therefore, within the perturbative quantization, the procedure (163) is well-defined such that the apparent non-locality in the shift sector, induced by the gauge-fixing, does not lead to any problems. Moreover, (163) has the effect of absorbing the functional determinant $(\text{Det } O_{ij})^{1/2}$ in (160), such that the partition function takes the simple form

$$Z_{\text{HG}} = \int \mathcal{D}[N^i, \pi_i, \gamma_{ij}, c^i, c_i^*] e^{-S_{\text{tot}}[N^i, \pi_i, \gamma_{ij}, c^i, c_i^*]}, \quad (164)$$

with a local action functional $S_{\text{tot}}[N^i, \pi_i, \gamma_{ij}, c^i, c_i^*]$ that includes the auxiliary π_i field.

9. EXPLICIT CALCULATIONS AND RENORMALIZATION GROUP FLOW

The proof that projectable HG is perturbatively renormalizable beyond power counting [51, 52] is an important step toward a unitary quantum theory of gravity. However, for this theory to qualify as a fundamental theory, it must be extendable to arbitrarily high energy scales. In other words, perturbative renormalizability does not yet ensure the UV completeness, as the RG flow could drive one or more coupling constants into a

Landau pole, leading to divergent interaction strengths at finite energy scales. Another aspect of the RG flow in HG is connected to the IR and the question of whether relativistic invariance can effectively be restored dynamically as an emergent symmetry at low energies. The RG analysis and the logarithmic running of the coupling constants requires calculation of the beta functions determined by the UV divergences of the theory.

Various quantum aspects of Lifshitz theories, in particular in the context of HG, have been considered in [49–51, 81, 83, 208, 253–273]. Here, I focus on the calculation of the beta functions in HG in $D = 2 + 1$ dimensions. Previous work in this context includes the contributions of Lifshitz scalars to the gravitational beta functions [82, 274], the one-loop beta functions for conformally reduced projectable HG in $D = 2 + 1$ dimensions [275], and the renormalization of the cosmological constant in $D = 2 + 1$ projectable HG [276]. In this section I report on the full RG flow of all couplings in projectable HG in $D = 2 + 1$ dimensions, which was derived in [53]. The analogous calculation in $D = 3 + 1$ dimensions is technically much more challenging and has not yet been completed. However, recent partial results provide an important first step in this direction [277].

The Euclidean action for projectable HG in $D = 2 + 1$ dimensions reads³²

$$S_{\text{HG}}^{(d=2)} = \frac{1}{2G} \int dt d^2x \sqrt{\gamma} (K_{ij}K^{ij} - \lambda K^2 + \mu R^2). \quad (165)$$

The background covariant gauge condition (148) and the non-local operator (147) in $D = 2 + 1$ dimensions take the forms

$$\begin{aligned} \chi^i &= \bar{D}_t n^i + \frac{1}{2\sigma} (O^{-1})^{ij} \bar{\gamma}^{k\ell} (\bar{\nabla}_k h_{j\ell} - \lambda \bar{\nabla}_j h_{k\ell}), \\ O_{ij} &= -(\bar{\Delta} \bar{\gamma}^{ij} + \xi \bar{\nabla}^i \bar{\nabla}^j)^{-1}, \quad \xi \neq -1. \end{aligned} \quad (166)$$

In the background field method the “quantum fields” h_{ij} , n^i , π_i , c_i^* , and c^i are integrated out in the path integral, which, within the one-loop approximation, means performing the functional Gaussian integration (8). Therefore, only the part of the total action $S_{\text{tot}} = S_{\text{HG}} + S_{\text{gf}} + S_{\text{gh}}$ that is quadratic in the perturbations $S_{\text{tot}}^{(2)}$ is required. In view of (21) and (22), this means that only the “affine” parts of the gauge transformations on h_{ij} and n_i are required to derive the quadratic part of S_{gh} . In the projectable version of HG in the gauge $N = 1$, the $\text{Diff}_{\mathcal{F}}$ reduce to the time-dependent spatial diffeomorphisms [corresponding to $\varepsilon = 0$ in (120) and (121)], and the required gauge transformations in terms of the background covariant time derivative $\bar{D}_t$ and the background covariant spatial derivative $\bar{\nabla}_i$ are given by

$$\delta_\varepsilon h_{ij} = 2\bar{\nabla}_{(i}\varepsilon_{j)}, \quad \delta_\varepsilon n^i = \bar{D}_t \varepsilon^i. \quad (167)$$

³²Note the flipped sign of the μR^2 term compared to (127).

The vector-ghost operator Q_j^i is derived from (166) according to the general formula (22), and its quadratic part reads

$$\begin{aligned} Q_j^i &= \delta_j^i \bar{D}_t^2 + \frac{1}{4\sigma} \left\{ -2\delta_j^i \bar{\Delta}^2 + 2[1 + 2\xi - 2\lambda(1 + \xi)] \bar{\nabla}^i \bar{\Delta} \bar{\nabla}_j \right. \\ &\quad + \delta_j^i \bar{R} \bar{\Delta} - (1 - 2\lambda + 2\xi) \bar{R} \bar{\nabla}^i \bar{\nabla}_j - 2\xi \bar{R}_{;j} \bar{\nabla}^i \\ &\quad \left. - 2\xi \bar{R}^i \bar{\nabla}_j - 2\delta_j^i \bar{R}_{;k} \bar{\nabla}^k - \delta_j^i \bar{R}_{;k}^k - 2\xi \bar{R}^i{}_{;j} \right\}. \end{aligned} \quad (168)$$

A virtue of the manifest background covariant treatment in the background field method is that, owing to the background $\text{Diff}_{\mathcal{F}}$ invariance, the shift vector $\bar{N}^i$ appears only in combination with the time derivative $\partial_t \gamma_{ij}$ in the form of the extrinsic curvature or, equivalently, in the form of the covariant time derivative of the metric $D_t \gamma_{ij} = 2K_{ij}$. When performing variations of the total action $S_{\text{tot}} = S_{\text{HG}} + S_{\text{gf}} + S_{\text{gh}}$, factors of the shift perturbations n^i arise only from the variation of the covariant time derivative, as can be seen from the operator relation

$$[\delta, D_t] = -\mathcal{L}_{\delta N}. \quad (169)$$

Moreover, a canonical ordering among mixed covariant time derivatives and covariant space derivatives could be chosen in such a way that the covariant time derivatives act first. This requires repeated use of the basic commutator³³

$$[D_t, \nabla_m] T_{i_1 \dots i_s}^{j_1 \dots j_r} = \sum_{j_\ell} \kappa_{mn}^{j_\ell} T_{i_1 \dots i_s}^{j_1 \dots j_r} - \sum_{i_\ell} \kappa_{mi_\ell}^n T_{i_1 \dots n \dots i_s}^{j_1 \dots j_r}, \quad (170)$$

with the “anisotropic commutator curvature” tensor defined in terms of derivatives of the extrinsic curvature,

$$\kappa_{ij}^k := \nabla_i K_j^k + \nabla_j K_i^k - \nabla^k K_{ij}. \quad (171)$$

Upon introducing the auxiliary field π_i according to (163), making use of (169), integrating by parts, sorting derivatives with (170), reducing curvature tensors by the dimension-dependent identity (133), and arranging the fluctuations of the fields h_{ij} , n^i , and π_i in a multiplet $\phi^A = (h_{ij}, n^i, \pi_i)^T$, the gauge-fixed fluctuation operator acquires a block matrix structure and can be represented in the form

$$\begin{aligned} F_{AB}(\bar{D}_t, \bar{\nabla}) &= C_{AB} \bar{D}_t^2 + D_{AB}^{ijkl} \bar{\nabla}_i \bar{\nabla}_j \bar{\nabla}_k \bar{\nabla}_l + T_{AB} \bar{D}_t + W_{AB}^{ij} \bar{\nabla}_i \bar{\nabla}_j \\ &\quad + \Gamma_{AB}^i \bar{\nabla}_i + P_{AB}. \end{aligned} \quad (172)$$

The principal part of (172) is split into a temporal part C_{AB} and a spatial part D_{AB}^{ijkl} for which the derivatives have been made explicit. For brevity, I do not give explicit expressions for the matrices C_{AB} , D_{AB}^{ijkl} , T_{AB} , W_{AB}^{ij} , Γ_{AB}^i , and P_{AB} , which are functions of the background fields. The one-loop renormalization requires

³³The relations (169) and (170) hold for any d , but only in the projectable version of HG. In the non-projectable theory the operator version of (169) reads $[\delta, D_t] = -N^{-1}(\delta N D_t + \mathcal{L}_{\delta N})$. Likewise, (170) yields an additional term $a_m D_t T_{i_1 \dots i_s}^{j_1 \dots j_r}$ on the right-hand side, and the covariant spatial derivatives in the definition (171) must be shifted by the acceleration vector $\nabla_i \mapsto \nabla_i + a_i$.

calculation of the divergent part of the functional traces for the operators (172) and (168),

$$\Gamma_1^{\text{div}} = \frac{1}{2} \text{Tr} \ln F_{AB} |^{\text{div}} - \text{Tr} \ln Q_i^j |^{\text{div}}. \quad (173)$$

In contrast to the relativistic case, standard heat-kernel techniques are not available for the anisotropic case; in particular, there is no closed algorithm based on an SDW representation (31) for the off-diagonal kernel of (172). In addition to the anisotropic character of these operators, they suffer from further complications. First, the matrices in the principal parts C_{AB} and D_{AB} are degenerate, as n^i and π^i enter F_{AB} only with lower derivatives and the h - h block of D_{AB}^{ijkl} is a non-minimal fourth-order operator³⁴.

Nevertheless, initial attempts to deal with anisotropic operators using the heat-kernel technique were suggested in [81, 274]. A general algorithm for anisotropic operators, based on the resolvent method, was proposed in [83]. In the most general case, however, this algorithm requires the evaluation of a large number of products of nested multi-commutators as well as non-trivial parameter integrals, which is technically challenging.

Therefore, an alternative way of calculating the one-loop divergences might be more suitable, especially since the number of invariants in HG in $D = 2 + 1$ dimensions is reasonably small and the one-loop calculation using Feynman-diagrammatic techniques is still manageable, particularly when combined with the background field method. After integrating out the “quantum fields” h_{ij} and n^i as well as π_i , c_i^* , and c^i in the path integral, the effective action becomes a functional of the mean fields, which at the one-loop level can be identified with the background fields. In particular, the divergent part of the effective action is a sum of local operators of the background fields $\bar{\gamma}_{ij}$ and $\bar{N}^i$ together with their time and space derivatives, which, owing to the renormalizability of projectable HG, are of the same form as the manifestly $\text{Diff}_{\mathcal{F}(\mathcal{M})}$ -invariant operators already present in the bare action (165). This allows us to extract the one-loop renormalization of G , λ , and μ in a simpler way by expanding the general background field $\bar{\gamma}_{ij}$ around a flat background in which $\bar{N}^i = 0$:

$$\bar{\gamma}_{ij} = \delta_{ij} + H_{ij}. \quad (174)$$

Evaluating the bare action (165) on the background (174) and expanding up to quadratic order in H_{ij} yields

$$S_{\text{HG}}[\bar{\gamma}_{ij}, \bar{N}^i] = \frac{1}{2G} \int dt d^2x \left\{ \frac{1}{4} (\dot{H}_{ij} \dot{H}^{ij} - \lambda \dot{H} \dot{H}) + \mu [\partial^2 H \partial^2 H - \partial_k \partial_l H^{kl} (2\partial^2 H - \partial_j \partial_i H^{ij})] + \mathcal{O}(H^3) \right\}, \quad (175)$$

³⁴The degeneracy is a consequence of the anisotropic scaling: in contrast to $[h_{ij}]_S = 0$, the fields n^i and π^i carry non-zero scaling dimension $[\pi_i]_S = [n^i]_S$, such that the overall homogeneous scaling $[F_{AB}]_S = 4$ only allows for lower derivatives of n^i and π^i .

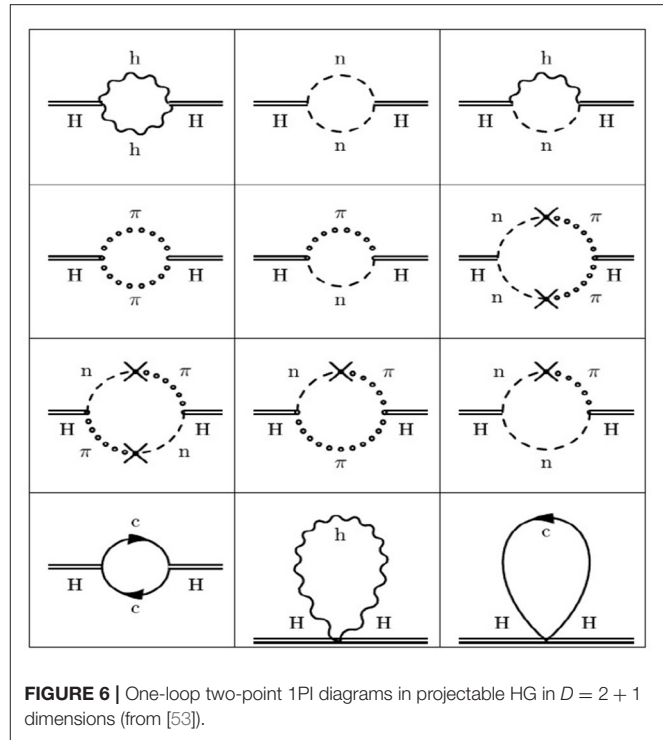


FIGURE 6 | One-loop two-point 1PI diagrams in projectable HG in $D = 2 + 1$ dimensions (from [53]).

with $\partial^2 := \delta^{\mu\nu} \partial_\mu \partial_\nu$. The divergent part of the effective action can be expanded in the same way:

$$\Gamma^{\text{div}}[\bar{\gamma}_{ij}, \bar{N}^i] = \int dt d^2x \left\{ c_1^{\text{div}} \dot{H}_{ij} \dot{H}^{ij} + c_2^{\text{div}} \dot{H} \dot{H} + c_3^{\text{div}} [\partial^2 H \partial^2 H - \mu \partial_k \partial_l H^{kl} (2\partial^2 H - \partial_j \partial_i H^{ij})] + \mathcal{O}(H^3) \right\}. \quad (176)$$

In order to obtain the renormalizations of the couplings G , λ , and μ , it is sufficient to calculate the divergent coefficients c_1^{div} , c_2^{div} , and c_3^{div} of the operators quadratic in H_{ij} . The renormalization of G is extracted from c_1^{div} , the renormalization of λ/G from c_2^{div} , and the renormalization of μ/G by any of the three operators in (176). Disentangling this system enables extraction of the individual renormalizations of G , λ , and μ . Diagrammatically, the background fields H_{ij} appear only at external legs, while the quantum fields h_{ij} and n^i , as well as π_i , c_i^* , and c^i , propagate in the loops. Hence, according to (175) and (176), the one-loop renormalization of G , λ , and μ requires one to calculate the divergent part of the 1PI diagrams with two external H_{ij} -legs, shown in **Figure 6**.

For the regular gauge (166) with gauge parameters (ξ, σ) and pole $\mathcal{P}_S$ as in (152), the propagators of the quantum fields h_{ij} and n^i are the same as in (150) and (151), while those including the π_i , c_i^* , and c^i fields read [51],

$$\begin{aligned} \langle \pi_i, n^j \rangle &= G \omega \delta_i^j \mathcal{P}_S(\omega, k), \\ \langle \pi_i, \pi_j \rangle &= \frac{G k^2}{2} \left[\delta_{ij} + (1 - 2\lambda) \frac{k_i k_j}{k^2} \right] \mathcal{P}_S(\omega, k), \\ \langle c_i^*, c^j \rangle &= G \delta_i^j \mathcal{P}_S(\omega, k). \end{aligned} \quad (177)$$

The required three-point and four-point vertices in the gauge $\bar{N}^i = 0$ are obtained by expanding the background fields in $\mathcal{L}^{(2)} = \delta\phi^A F_{AB} \delta\phi^B$, with F_{AB} given in (172), according to (174) up to second order in H_{ij} . The explicit results for the vertices are rather lengthy and therefore not presented here. Within dimensional regularization, the divergent part of the one-loop diagrams in **Figure 6** can be extracted by expanding the propagators in the corresponding integrals around vanishing external frequency and momenta, resulting in a sum of vacuum diagrams from which the logarithmically divergent contributions can easily be extracted by power counting³⁵. The one-loop beta functions β_G , β_λ , and β_μ , which determine the RG running of the couplings G , λ , and μ , are obtained directly from the logarithmic one-loop divergences, i.e., from the corresponding coefficient of the pole $1/\varepsilon$ in dimension.

Finally, in order to discuss the physical implications of the RG flow, it is important to extract the gauge-independent physical information from the RG system. In general, the off-shell effective action is parametrization- and gauge-dependent. On the one hand, a change of the gauge-fixing induces a change $\Gamma^{\text{div}} \mapsto \Gamma^{\text{div}} + \mathcal{E} \delta \Gamma^{\text{div}}$, which is proportional to the equations of motion $\delta \Gamma^{\text{div}} = S_i X^i$ with an arbitrary constant $\mathcal{E}$ [34, 279, 280]. On the other hand, this change could be compensated for by the change $\delta \Gamma^{\text{div}} = (\partial \Gamma^{\text{div}} / \partial G) \delta G + (\partial \Gamma^{\text{div}} / \partial \lambda) \delta \lambda + (\partial \Gamma^{\text{div}} / \partial \mu) \delta \mu$, which is induced by a change in the couplings. The combinations of couplings for which the corresponding beta function is gauge-independent are called essential; all other couplings are called inessential and do not enter physical observables. The problem is therefore to tell apart and disentangle the essential from the inessential couplings. In order to find X^i explicitly, one could exploit power counting, as $S_i X^i$ must be a local functional with the same scaling as Γ^{div} ; that is, in the context of $D = 2 + 1$ projectable HG, $S_i X^i$ can only involve marginal operators with respect to the anisotropic scaling. Since the scaling and the index structure of the S_i are known, this corresponds to a strong constraint on the possible structure of the X^i . In [53], it was found that the unique combination $X^i S_i$ that vanishes on-shell is

$$\delta \Gamma^{\text{div}} = \mathcal{E} \int dt d^2x [K_{ij} K^{ij} - \lambda K^2 - \mu R^2]. \quad (178)$$

The variation of Γ^{div} with respect to the couplings reads

$$\delta \Gamma^{\text{div}} = \frac{1}{2G} \int dt d^2x \sqrt{\gamma} \left[-\frac{\delta G}{G} K_{ij} K^{ij} - \lambda \left(\frac{\delta \lambda}{\lambda} - \frac{\delta G}{G} \right) K^2 + \mu \left(\frac{\delta \mu}{\mu} - \frac{\delta G}{G} \right) R^2 \right]. \quad (179)$$

Equating (178) and (179) yields the desired transformations of the couplings [53],

$$\delta G = -2G^2 \mathcal{E}, \quad \delta \lambda = 0, \quad \delta \mu = -4G\mu \mathcal{E}. \quad (180)$$

³⁵See e.g., [278] for an application of this method with a particular focus on the combinatorial aspects in the context of relativistic higher-derivative theories.

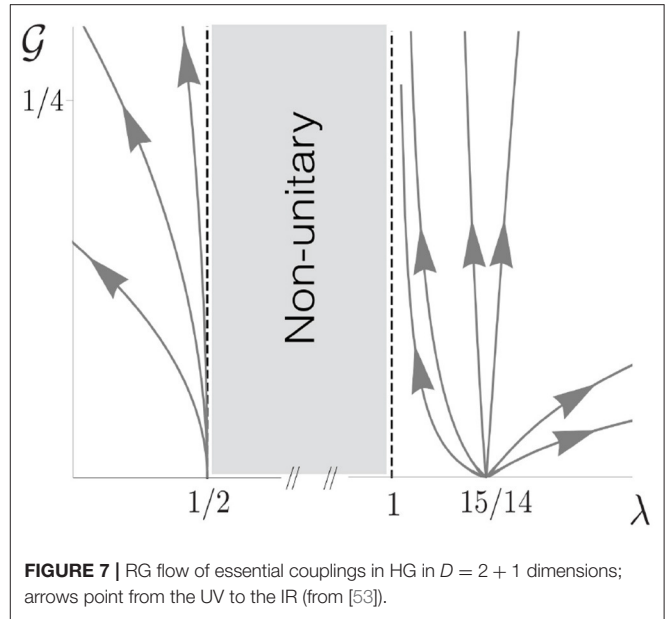


FIGURE 7 | RG flow of essential couplings in HG in $D = 2 + 1$ dimensions; arrows point from the UV to the IR (from [53]).

Thus, only λ and the combination $\mathcal{G} = G/\sqrt{\mu}$ are essential couplings ($\delta\lambda = \delta G = 0$), with beta functions [53]

$$\begin{aligned} \beta_\lambda &= \frac{15 - 14\lambda}{64\pi} \sqrt{\frac{1 - 2\lambda}{1 - \lambda}} \mathcal{G}, \\ \beta_{\mathcal{G}} &= -\frac{(16 - 33\lambda + 18\lambda^2)}{64\pi (1 - \lambda)^2} \sqrt{\frac{1 - \lambda}{1 - 2\lambda}} \mathcal{G}^2. \end{aligned} \quad (181)$$

The RG flow driven by the beta functions (181) is shown in **Figure 7**. There are two UV fixed points at

$$(\lambda_1^*, \mathcal{G}_1^*) = (1/2, 0), \quad (\lambda_2^*, \mathcal{G}_2^*) = (15/14, 0). \quad (182)$$

The first fixed point $(\lambda_1^*, \mathcal{G}_1^*)$ lies exactly on the lower boundary of the non-unitary interval $1/2 < \lambda < 1$ for which the gravitational scalar degree of freedom behaves like a ghost; cf. the discussion in section 7.4. For fixed $\mathcal{G}$, the beta function $\beta_{\mathcal{G}}$ develops a divergence in the limit $\lambda \rightarrow 1/2$. At the same time, however, the limit $\lambda \rightarrow 1/2$ is accompanied by $\mathcal{G} \rightarrow 0$, implying that the relevant expansion parameter in this limit is $\tilde{\mathcal{G}} = \mathcal{G}(1 - 2\lambda)^{-1/2}$. The beta function $\beta_{\tilde{\mathcal{G}}}$ vanishes for $\lambda \rightarrow 1/2$, which means that there is a one-parameter family of UV fixed points parameterized by the asymptotic value of $\tilde{\mathcal{G}}$. Summarizing, the status of this fixed point remains inconclusive and higher loop corrections or contributions from matter loops are required to resolve the situation and to decide whether the fixed point is merely an artifact of the approximation or has physical significance.

In contrast, the second fixed point $(\lambda_2^*, \mathcal{G}_2^*)$ is regular, lies in the unitary region $\lambda > 1$, and is asymptotically free [53]. Although projectable HG in $D = 2 + 1$ dimensions only has the status of a toy model without propagating spin-2 particles, it provides the first unitary, perturbatively renormalizable, and UV-complete quantum theory of gravitational propagating degrees of freedom. In previous calculations of the one-loop divergences

in $D = 2 + 1$ projectable HG, the dynamical content of the metric field was restricted to the conformal mode [275]. In this conformally reduced model, only the fixed point at $(1/2, 0)$ has been found. This shows that the formation of the regular fixed point at $(15/14, 0)$ requires the full theory [53].

Another interesting feature of the RG flow is that there are RG trajectories which emanate from the regular UV fixed point and asymptotically approach the “relativistic value” $\lambda \rightarrow 1$ in the IR. In addition to the problems with the IR $\lambda \rightarrow 1$ limit discussed in section 7.4, the “gravitational coupling” $\mathcal{G}$ becomes strongly coupled along these trajectories, necessitating a non-perturbative analysis in this regime. Nevertheless, the observed flow toward $\lambda = 1$ suggests that the possibility of a dynamical mechanism for an emergent restoration of relativistic symmetry at low energies should be investigated in more detail. First, the phenomenon that a theory which is asymptotically free in the UV develops a strong coupling in the IR is well-known. Second, the strong coupling of $\mathcal{G}$ in the IR could just be an artifact of the absence of relevant curvature operators in $D = 2 + 1$. In $D = 3 + 1$ dimensions relevant deformations might be expected to naturally cut off the strong coupling of $\mathcal{G}$.

All these interesting and encouraging results justify the hope that the RG flow of the more realistic and physically relevant theory in $D = 3 + 1$ dimensions exhibits similar features. Although there are no conceptual problems associated with the analogous calculation in $D = 3 + 1$ dimensions, in view of the increased number and complexity of the independent curvature invariants, it is technically much more challenging. A first step toward the RG flow of projectable HG in $D = 3 + 1$ dimensions has been taken in [54], where the one-loop beta functions of G and λ were derived with Feynman-diagrammatic methods in a similar way to (175) and (176), by exploiting the gauge invariance of counterterms, which allows one to restrict to a flat metric background and focus only on diagrams with background shift fields at the external legs. However, the gauge-invariant beta functions for the essential coupling constants and the fixed point structure of the theory can only be derived by having access to the renormalizations of all couplings, including those in the potential sector. Thus, the complete calculation of the one-loop divergences in $D = 3 + 1$ projectable HG is an important task.

10. CONCLUSIONS AND OUTLOOK

In this article I have reviewed various attempts to quantize gravity within the framework of perturbative quantum field theory, with a particular focus on Hořava gravity. I have highlighted the merits and difficulties that come with each of the approaches. The different approaches to quantum gravity discussed in this work might best be characterized by the property that each does *not* share with the other approaches, as shown in Table 2.

The status of HG with critical anisotropic scaling can be roughly summarized by dividing the discussion into “projectable” vs. “non-projectable” and “phenomenology of the classical theory” vs. “properties of the quantum theory.”

From a phenomenological point of view, projectable HG does not seem to qualify as a viable theory, mainly because it suffers

TABLE 2 | Approaches to quantum gravity characterized by properties that they do not have.

Approach	Property
General relativity	Not renormalizable
Effective field theory	Not fundamental
Asymptotic safety	Not perturbative
Quadratic gravity	Not unitary or not satisfying micro-causality
Hořava gravity	Not relativistic

from an IR instability of the additional scalar gravitational mode [212–215]. Although other proposals with a more optimistic conclusion for this problem have been made [200, 217, 218], they are based on non-perturbative effects which are outside the scope of the weak coupling regime where perturbation theory is applicable.

In contrast, the non-projectable model does not suffer from an IR instability because additional relevant operators that include powers of the acceleration vector (spatial derivatives of the lapse function) can remedy the IR instability [221]. Even if the low-energy sector of the non-projectable model is strongly constrained by observational data and a mechanism to avoid percolation of LV effects from the gravitational sector to the matter sector seems necessary to avoid conflicts with bounds on LV in the matter sector [227], the non-projectable model is still phenomenologically viable [238].

From a theoretical point of view, regarding the status of HG as a consistent quantum theory of gravity, the situation is somewhat opposite to that of the phenomenological assessment. The projectable theory has been proven to be perturbatively renormalizable (for any dimension $D = d + 1$) in the strict sense [51, 52]. Moreover, the model in $D = 2 + 1$ dimensions has been shown to be asymptotically free, and its RG flow features interesting RG trajectories which emanate from the UV fixed point and asymptotically approach the relativistic value $\lambda = 1$ in the IR [53]. Even if the model in $D = 2 + 1$ dimensions must be considered a toy model without propagating TT modes, it is a unitary, perturbatively renormalizable, and UV-complete quantum theory of non-trivial propagating degrees of freedom and captures essential features of HG, which are expected to carry over to the physically relevant $D = 3 + 1$ case. The situation in $D = 3 + 1$ dimensions has not yet been conclusively clarified and requires calculation of the one-loop beta functions. A first step in this direction has been taken in [54], but in order to extract the gauge-independent physical information about the running of the essential couplings, the renormalization of all couplings is needed. While there are no new conceptual difficulties, the analogous calculation is technically much more complex than in the $D = 2 + 1$ case and requires more efficient methods, such as newly developed heat-kernel techniques for anisotropic operators [81, 83, 274]. In any case, the calculation of the one-loop divergences of projectable HG in $D = 3 + 1$ dimensions is certainly a very important endeavor that will provide new insights into the structure of the theory.

The situation with the quantization of the non-projectable model is less clear. Unfortunately, the proof of perturbative

renormalizability for the projectable theory [51, 52] does not extend to the non-projectable theory, mainly because it relies on the regular form of all propagators and no gauge-fixing could be found in the non-projectable model that would render all propagators regular. In particular, there seems to be no gauge-fixing that could remove all irregular contributions to the propagator involving the lapse function—interpreted in [51] as a reflection of the instantaneous interaction induced by the lapse function [214]. Therefore new ideas seem to be necessary for dealing with the perturbative quantization of the non-projectable theory.

In summary, HG is an interesting proposal, but, closing with the words of Bryce DeWitt, the theory does not yet seem to have been “pushed to its logical conclusion” [281]. Further important calculations in $D = 3 + 1$ dimensions are required and may decide the fate of Hořava’s proposal for a unitary, perturbatively renormalizable, and UV-complete quantum theory of gravity.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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Critical Reflections on Asymptotically Safe Gravity

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Asymptotic safety is a theoretical proposal for the ultraviolet completion of quantum field theories, in particular for quantum gravity. Significant progress on this program has led to a first characterization of the Reuter fixed point. Further advancement in our understanding of the nature of quantum spacetime requires addressing a number of open questions and challenges. Here, we aim at providing a critical reflection on the state of the art in the asymptotic safety program, specifying and elaborating on open questions of both technical and conceptual nature. We also point out systematic pathways, in various stages of practical implementation, toward answering them. Finally, we also take the opportunity to clarify some common misunderstandings regarding the program.

Keywords: quantum gravitation, asymptotic safety, renormalization group, running couplings, observables, effective field theory, unitarity

1. INTRODUCTION AND CONCLUSIONS

Asymptotic Safety [1–3] is a candidate for a quantum theory of the gravitational interactions. It does not require physics beyond the framework of relativistic Quantum Field Theory (QFT) nor does it require fields beyond the metric to describe the quantum geometry of spacetime. Moreover, the inclusion of matter degrees of freedom, like the standard model or its extensions, is conceptually straightforward. Thus, ultimately, Asymptotic Safety may develop into a quantum theory comprising all fundamental fields and their interactions.

The core idea of Asymptotic Safety was formulated by Weinberg [4, 5] in the late seventies. It builds on the insight of Wilson [6], linking the renormalizability and predictive power of a quantum field theory to fixed points of its Renormalization Group (RG) flow: a theory whose ultraviolet (UV) behavior is controlled by an RG fixed point does not suffer from unphysical UV divergences in physical processes like scattering events. The prototypical example for such a behavior is Quantum Chromodynamics (QCD) where the UV completion is provided by the free theory. In technical terms QCD is asymptotically free with the UV completion provided by a Gaussian fixed point¹. It was then stressed in [5] that a valid UV completion could also be obtained from fixed points corresponding to actions with non-vanishing interactions, so-called non-Gaussian fixed points. In order to contrast this situation to asymptotic freedom, this non-trivial generalization

¹The terminology Gaussian fixed point reflects that the action associated with the fixed point does not contain interactions and is thus quadratic in the fields.

has been termed “asymptotic safety.” Remarkably, the space of diffeomorphism invariant actions constructed from a four-dimensional (Euclidean) spacetime metric indeed seems to contain a non-Gaussian fixed point suitable for Asymptotic Safety, the so-called Reuter fixed point [7, 8].

As in other approaches to quantum gravity, substantial progress has brought the program to a point where a fair-minded assessment of its achievements and shortcomings will be useful. Therefore, the purpose of this article is to provide a critical review of the current status of the field, of the key open questions and challenges, and to point out directions for future research. By necessity, the discussion also covers questions of a more technical nature which is reflected in the character of some of the sections. This also entails that the article does not serve as an introduction to the asymptotic safety program, for which we refer the reader to the textbooks [2, 3] and reviews [9–15]. A list of key references related to the open questions is provided within each section, pointing the reader toward the broader discussion in the literature.

The rest of the paper is organized as follows. In section 2 we start with a concise introduction to asymptotic safety, also giving examples of non-Gaussian fixed points providing a UV completion in non-gravitational settings. The subsequent sections critically review open questions along the following lines:

1. Issues related to the use of the functional RG (FRG) (“uncontrollable approximations,” use of the background field method) are discussed in section 3.
2. Because of these theoretical uncertainties, it is important to cross-check the results with different methods. This is discussed in section 4.
3. The difficulty of computing observables, and comparing with observations, is discussed in section 6.
4. Closely related to this is the, partly semantic, issue of the physical meaning of running couplings (can Λ and G run? If so, what are the physical implications of this running?) and other aspects where the literature on asymptotic safety deviates from standard particle physics procedures (power vs. log running, use of dimensional regularization). These points are discussed in section 5.
5. In section 7, we discuss whether and in what way asymptotic safety could be matched to effective field theory (EFT) at low energy. Here we also discuss the limitations of the procedure of “RG improvement.”
6. In section 8 we address the relation between scale symmetry and conformal symmetry and the FRG. (How can one have scale invariance in the presence of G ?) We also critically review the argument that the entropy of black holes is incompatible with gravity being described by Asymptotic Safety (“Gravity cannot be Wilsonian” or “Gravity cannot be a conformal field theory”).
7. The unsolved issue of unitarity is discussed in section 9 (in particular: do higher derivatives imply ghosts?).
8. Finally, we stress the need of calculations in Lorentzian signature in section 10.

The goal of this paper is three-fold:

- (i) Reinforcing progress in the research field by clearly spelling out key open questions,
- (ii) Strengthening a critical and constructive dialogue on asymptotically safe gravity within a larger community,
- (iii) Contributing to a broad and critical assessment of the current status and future prospects of research avenues in quantum gravity.

2. ASYMPTOTIC SAFETY

2.1. The Main Idea

...where we recall the notion of quantum scale invariance and the predictive power of RG fixed points.

Asymptotic Safety [2, 3] builds on Wilson’s modern view of renormalization, which links the renormalizability and predictive power of a quantum field theory to fixed points of its RG flow². It is equivalent to the notions of “quantum scale invariance in the UV” and also to “non-perturbative renormalizability,” resulting in a theory that is fully specified by only a finite number of free parameters.

In practice, asymptotic safety is studied in the following way. One has a functional of the fields, that could be either a Wilsonian action S_Λ depending on a UV cutoff Λ or a generating functional Γ_k for the one-particle irreducible (1PI) correlation functions depending on an IR cutoff k . We shall focus on the latter for definiteness, but at this stage the discussion is more general. For the present purposes, let us assume that this functional can be expanded in a suitable basis of operators $\{\mathcal{O}_i\}$, integrals of monomials in the field and its derivatives

$$\Gamma_k = \sum_i \bar{u}_i(k) \mathcal{O}_i. \quad (1)$$

The beta functions of the, generally dimensionful, couplings $\bar{u}_i(k)$ are given by the derivatives of $\bar{u}_i(k)$ with respect to $t = \log k$. Then, one converts the dimensionful couplings³ $\bar{u}_i(k)$ into dimensionless ones by a suitable rescaling with the coarse-graining scale k ,

$$u_i \equiv \bar{u}_i k^{-d_i}, \quad (2)$$

where d_i is the canonical mass dimension of $\bar{u}_i(k)$. In this way one obtains a coupled set of autonomous differential equations

$$k \partial_k u_i(k) = \beta_{u_i}(\{u_j\}). \quad (3)$$

²Generically, a fixed point will be neither UV nor IR, since it typically has both IR attractive (irrelevant) and IR repulsive (relevant) directions. Depending on the choice of RG trajectory, the fixed point can therefore induce a UV or an IR scaling regime. Given two fixed points connected by an RG trajectory, the direction of the flow between them is fixed and the designation of UV and IR fixed point becomes unambiguous.

³The notation u_i for dimensionful quantities and $\bar{u}_i$ for dimensionless quantities can also sometimes be found in the literature.

The solutions of this system are the RG trajectories and each trajectory corresponds to a single physical theory. In general, it may happen that physical observables diverge along a trajectory as $k \rightarrow \infty$ (e.g., at a Landau pole). One simple way to avoid this is to require that the trajectory describing the physical world emanates from a fixed point as k is lowered from the UV to the IR. At a fixed point $\{u_{j*}\}$ all beta functions vanish simultaneously, $\beta_{u_i}(\{u_{j*}\}) = 0, \forall i$ and, as we shall discuss in more detail in section 8, scale invariance is realized⁴. Such RG trajectories are said to be either asymptotically free or asymptotically safe theories. This should be contrasted to the case where physical observables blow up at a finite value of k which indicates that one deals with an effective field theory.

The predictive power of asymptotic safety originates from the properties of the fixed point. Linearizing the beta functions (Equation 3) about the fixed point, and diagonalizing the stability matrix $\mathbf{B}_{ij} \equiv \partial_{u_j} \beta_{u_i}|_{u=u_*}$, one can determine which directions are attractive and which ones are repulsive. Eigenvalues with positive (negative) real parts correspond to eigenvectors along which the flow (from UV to IR) is dragged toward (repelled by) the fixed point. One typically works with the scaling exponents⁵ $\theta_I = -\text{eig } \mathbf{B}$. Every irrelevant (IR attractive/UV repulsive/ $\theta_I < 0$) direction fixes one parameter in the initial conditions⁶ for Γ_k , whereas relevant (IR repulsive/UV attractive/ $\theta_I > 0$) directions correspond to free parameters. Marginal directions ($\theta_I = 0$) typically only occur at Gaussian fixed points. Thus, the number of independent free parameters of an asymptotically safe theory is equal to the number of relevant directions of the fixed point that it originates from in the UV. At a free (Gaussian) fixed point, the relevant directions correspond to couplings with positive mass dimension. In a local theory, there is only a finite number of such parameters. In principle, an interacting fixed point could have even fewer relevant directions, and hence greater predictive power. If one could integrate the RG flow to the IR, one could test if the low-energy relations implied by these properties of the UV fixed point are verified or not, cf. section 6 for further discussion.

2.2. Non-gravitational Examples

... where we provide a list of non-gravitational, asymptotically safe theories together with the corresponding mechanism for asymptotic safety and we discuss how several techniques are used to study these examples.

Whereas the existence of UV-complete quantum field theories based on the mechanism of asymptotic safety has been anticipated already in the early days of the RG [16, 17], concrete examples have been identified only much later, as a parametric control beyond perturbation theory is typically required. A paradigmatic class of examples is given by fermionic models in

$d = 3$ dimensional spacetime including, for instance, the Gross-Neveu model: though interactions of the type $\sim (\bar{\psi} \mathbf{m} \psi)^2$ (with $\mathbf{m}$ carrying some internal spin and/or flavor structure) belong to the class of perturbatively non-renormalizable models, there is by now convincing evidence that a large class of such models are in fact asymptotically safe in $2 < d < 4$ dimensional spacetime. Initially, the existence of the underlying non-Gaussian fixed points has been demonstrated by means of $1/N$ expansions [18, 19]; indeed, non-perturbative renormalizability has been proved for specific models to all orders in the $1/N$ expansion [20] with explicit results for higher orders being worked out, e.g., in [21–24]. Further quantitative evidence subsequently came from $2 + \epsilon$ or $4 - \epsilon$ expansions [25–30]; the FRG for the first time facilitated analytic computations directly in $d = 3$ [31–38]. For the asymptotic safety program, these models are instructive for several reasons:

- (i) The fermionic non-Gaussian fixed point is typically connected to a quantum phase transition. The latter is characterized by universal critical exponents which can also be studied using simulational methods [39–49] or the conformal bootstrap [22, 50]. In this way, the variety of available approaches have led to a confirmation of asymptotic safety of these models to a substantial degree of quantitative precision, summarized, e.g., in [51].
- (ii) While analytical as well as path integral Monte Carlo computations are typically performed in Euclidean spacetime, these models are relevant for layered condensed-matter “Dirac materials” [52, 53], corresponding to a $d = 2 + 1$ dimensional spacetime with Lorentzian signature. The quantitative agreement also with Quantum Monte Carlo methods (based on a Hamiltonian formulation) [44–46], demonstrates that asymptotic safety of these systems is visible in Euclidean as well as Lorentzian formulations.
- (iii) As a generic mechanism of asymptotic safety in these models, an irrelevant (i.e., perturbatively non-renormalizable) operator such as the fermionic interaction $\sim (\bar{\psi} \mathbf{m} \psi)^2$ becomes relevant as a consequence of strong fluctuations. Correspondingly, the anomalous dimension of this and subsequent operators is shifted by an amount of $\mathcal{O}(1)$; see, e.g., [33, 54] for a determination of an infinite set of scaling dimensions for large N . As a consequence, strongly power-counting irrelevant operators remain irrelevant and do not introduce an unlimited set of new physical parameters. The same pattern is also observed in many studies of asymptotically safe gravity [55–60].
- (iv) The comparative simplicity of these models has enabled a first study of the momentum dependence of 4-point correlation functions at the non-Gaussian fixed point [61]. For instance, the Gross-Neveu model ($\mathbf{m} = \mathbb{1}$) in $d = 2 + 1$ at the non-Gaussian fixed point can be analyzed in terms of an s -channel non-trivially and non-analytically on the dimensionless s variable at the UV fixed point. In fact, the s channel dependence can be shown to dominate over possible t and u channel dependences in a quantifiable manner at large N , resulting in a simpler form factor-like structure of the 1PI 4-vertex at the UV fixed point.

⁴In most cases this also implies conformal invariance.

⁵Note that the opposite sign convention, where the θ are defined without the additional negative sign, is also sometimes used in the literature.

⁶More precisely, the “memory” of the initial condition for an irrelevant direction is washed out by the RG flow and plays no role for the physics at $k = 0$.

This illustrates that scattering properties in the scaling regime can develop non-trivial features beyond the scaling suggested by naive power-counting.

Further examples for asymptotic safety include Yang-Mills theory in $d = 4 + \epsilon$ [62–65], and non-linear sigma models in $d = 2 + \epsilon$ [66–71]; for the latter, there is clear evidence for asymptotic safety even in $d = 3$ from lattice simulations [72]. The limit of large number of fermions N_f in gauge theories has recently seen a resurgence of interest, e.g., [73–76], with early work in [77, 78], see also [79].

Another recently discovered set of asymptotically safe models is given by gauged Yukawa models in the Veneziano limit of a suitably arranged large number of vector fermions N_f adjusted to the number of colors N_c of the gauge group [80–86] in $d = 3 + 1$ dimensional spacetime. Contrary to the lower-dimensional fermionic models, these gauged Yukawa models are power-counting renormalizable to all orders in perturbation theory. Because of the large number of fermions, fermionic screening dominates the running of the gauge coupling, such that asymptotic freedom is lost. The RG flow at high energies nevertheless remains bounded, as it is controlled by a UV fixed point appearing in all RG marginal couplings. Whereas perturbative renormalizability of these models supports the use of perturbative RG beta functions in the first place, the existence of non-Gaussian UV fixed points is parametrically controlled by a suitably small Veneziano parameter, e.g., $\epsilon = \frac{N_f}{N_c} - \frac{11}{2}$ as in [80]. Despite this technical vicinity to perturbative computations, the behavior of the theory near the fixed point is very different from the perturbative behavior near the Gaussian fixed point. For instance, the perturbatively marginal operators turn into (ir-)relevant operators with anomalous dimensions reaching up to $\mathcal{O}(1)$ for $\epsilon \lesssim \mathcal{O}(0.1)$. The couplings therefore scale with a power of the RG scale rather than logarithmically. Also, higher-order operators—though remaining RG irrelevant—generically acquire non-trivial fixed-point values and can thus exert an influence on scattering properties at highest energies.

3. FUNCTIONAL RENORMALIZATION GROUP

In section 2, we have discussed the asymptotic-safety mechanism without referring to any specific calculation method. Now we introduce the Functional Renormalization Group (FRG), which has been the main tool enabling progress in Asymptotic Safety in the last 20 years. It has been successfully applied to a large number of other theories and physical phenomena, in particular non-perturbative ones. Applications range from the phase structure of condensed matter systems, to confinement and chiral symmetry breaking in QCD, to the electroweak phase transition in the early universe and beyond Standard Model physics. In cases, where results from other non-perturbative methods (lattice simulations, Dyson-Schwinger equations, Resurgence etc.) exist, the FRG results compare well to those obtained by other methods. It is also worth

emphasizing that, while the combination of conceptual and technical challenges in quantum gravity is certainly unique, many of the technical challenges and physical effects encountered here have counterparts in other theories, most notably in non-Abelian gauge theories, where they can also be tested against other non-perturbative methods.

3.1. Brief Introduction to the FRG

...where we briefly introduce the FRG as a tool to calculate the effective action.

Currently, the primary tool to investigate Asymptotic Safety is the Functional Renormalization Group (FRG) equation for the effective average action Γ_k introduced in [87–89] (Wetterich equation), and in [7] for gravity. Γ_k depends on the content of the theory at hand, in quantum gravity it contains the metric degrees of freedom, Faddeev-Popov ghosts and possibly also matter fields. In the FRG approach the scale k is an infrared cutoff scale below which quantum fluctuations are suppressed. Thus, Γ_k encodes the physics of quantum fluctuations above the cutoff scale. For $k \rightarrow 0$, all quantum fluctuations have been taken into account and $\Gamma_{k=0}$ is the full quantum effective action,

$$\Gamma = \lim_{k \rightarrow 0} \Gamma_k \quad (4)$$

whose minimum is the vacuum state of the QFT. The flow equation for Γ_k encodes the response of the effective average action Γ_k to the process of integrating out quantum fluctuations within a momentum shell,

$$k \partial_k \Gamma_k[\Phi; \bar{\Phi}] = \frac{1}{2} \text{Tr} \left[\frac{1}{\Gamma_k^{(2)}[\Phi; \bar{\Phi}] + R_k} k \partial_k R_k \right]. \quad (5)$$

The term $(\Gamma_k^{(2)} + R_k)^{-1}$ on the right hand side of Equation (5) is the propagator in the regularized theory. Here, we have introduced $\Gamma_k^{(2)} = \Gamma^{(\Phi\Phi)}$, the second derivative of Γ_k w.r.t. the fields Φ . In Equation (5), we have also introduced a generic background $\bar{\Phi}$ which typically is chosen as the solution to the quantum equations of motion. Then, the fluctuation field Φ encodes the fluctuations about this background, and the 1PI correlation functions of the fluctuation fields $\langle \Phi_{i_1} \cdots \Phi_{i_n} \rangle_{1\text{PI}}$ (proper vertices) in a given background $\bar{\Phi}$ are given by

$$\Gamma_k^{(\Phi_{i_1} \cdots \Phi_{i_n})}[\bar{\Phi}] \equiv \frac{\delta}{\delta \Phi_{i_1}} \cdots \frac{\delta}{\delta \Phi_{i_n}} \Gamma_k[\Phi; \bar{\Phi}] \Big|_{\Phi=0}. \quad (6)$$

The term R_k is a cutoff scale k - and momentum-dependent infrared regulator which suppresses fluctuations with momenta $p^2 \lesssim k^2$, decays rapidly for momenta $p^2 \gtrsim k^2$, and vanishes at $k^2 = 0$. The second property renders the flow Equation (5) finite due to the decay of $k \partial_k R_k$ for large momenta. The regulator R_k is independent of the fluctuation field, but may carry a dependence on the background field. In a quantum field theory in flat space typically p^2 is the plain momentum squared, while in gravity and gauge theories p^2 may be associated

with a background-covariant Laplacian. Finally, Tr comprises a sum over all fluctuation fields and an integral over (covariant) loop momenta. The corresponding loop integration is peaked about momenta $p^2 \approx k^2$, leading to the momentum-shell integration. In summary, the flow Equation (5) transforms the task of performing the path integral into the task of solving a functional differential equation.

Conceptually, the Wetterich equation implements the idea of the Wilsonian Renormalization Group: lowering k corresponds to integrating out quantum fluctuations shell by shell in momentum space. For $k \rightarrow \infty$, the theory approaches the bare or renormalized ultraviolet action, depending on the underlying renormalization procedure, for a detailed analysis see, e.g., [90–94]. The fact that Equation (5) does not require specifying a bare action a priori makes it a powerful tool to scan for (interacting) RG fixed points and study their properties. The bare action can then be reconstructed from the RG fixed point along the lines of [91, 93]. Essentially, the Wetterich equation can be viewed as a tool to systematically test which choice of bare action gives rise to a well-defined and predictive path integral for quantum gravity.

Notably, if one approximates $\Gamma_k^{(2)}$ by the k -independent second functional derivative of a given bare action $S^{(2)}$, one obtains

$$\Gamma_k \approx S + \frac{1}{2} \text{Tr} \log (S^{(2)} + R_k), \quad (7)$$

which reduces to the standard one-loop effective action for $k = 0$. Accordingly, approximations to the FRG always contain one-loop results in a natural way.

3.2. FRG Approach to Quantum Gravity

...where we review the Functional Renormalization Group approach to quantum gravity, with a particular focus on background-field techniques.

In the gravitational context, the construction of Equation (5) makes use of the background field method, decomposing the physical metric $g_{\mu\nu}$ into a fixed, but arbitrary background metric $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$, see [9] for technical details⁷. The typical example is the linear split,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (8)$$

In the literature, the fluctuation field $h_{\mu\nu}$ is commonly multiplied with the square root of the Newton constant which makes it a standard dimension-one tensor field in four spacetime dimensions. The linear split (Equation 8) is the common choice not only in quantum gravity but also in applications of the background field method to gauge theories or non-linear sigma models. In gravity it comes at the price that the fluctuation field $h_{\mu\nu}$ is not a metric field, indeed it has no geometrical meaning.

⁷Notably, the asymptotic-safety mechanism is not tied to the spacetime metric carrying the gravitational degrees of freedom. While explored in far less detail, the vielbein and the Palatini formalisms may also lead to a theory which is asymptotically safe [95–101].

While this is not necessary, alternative parameterizations have been used. These have the general form

$$g_{\mu\nu} = f(h, \bar{g})_{\mu}{}^{\kappa} \bar{g}_{\kappa\nu}. \quad (9)$$

Of these alternative cases, the exponential split with $f(h, \bar{g}) = \exp[\bar{g}^{-1}h]$ has been explored, e.g., in [59, 102–105]. Further, the geometrical split in the Vilkovisky-deWitt approach with a diffeomorphism invariant flow has been studied in [90, 106–109], for applications to non-linear sigma models see [110, 111].

Different parameterizations (Equation 9) only constitute the same quantization if they (i) cover the same configuration space and (ii) the Jacobian that arises in the path integral is taken into account (see [104] for a related discussion). Condition (i) does not hold, e.g., for linear parameterization and exponential parameterization, see, e.g., [102, 104], while the linear split and the geometrical one with the Vilkovisky connection at least agree locally. However, it is well-known from two-dimensional gauge theories, that quantizations on the algebra and on the group can differ, see, e.g., [112]. Moreover, studies of the parameterization dependence of results in truncations, e.g., [113–116], so far do not account for (ii).

The presence of the background allows to discriminate “high-” and “low-momentum” modes by, e.g., comparing their eigenvalues with respect to the background Laplacian to the coarse-graining scale k . Moreover, it also necessarily enters gauge-fixing terms for the fluctuation field. As a consequence, the effective action Γ_k inherits two arguments, the set of fluctuation fields Φ and the corresponding background fields $\bar{\Phi}$ for all cutoff scales k . We emphasize that this also holds true for vanishing cutoff scale, $k = 0$, due to the gauge fixing.

Conceptually, the Wetterich equation lives on the so-called theory space, the space containing all action functionals constructable from the field content of the theory and compatible with its symmetry requirements. The FRG then defines a vector field generating the RG flow on this space. We proceed by discussing two systematic expansion schemes commonly used in quantum gravity (as well as other systems): the vertex expansion and the (covariant) derivative expansion.

The proper vertices of the effective average action (Equation 6) can be used as coordinates in theory space as the set of (1PI) correlation functions $\{\Gamma_k^{(\Phi_{i_1} \dots \Phi_{i_n})}[\bar{\Phi}]\}$ defines a given action and hence a theory. The vertex expansion is the expansion in the order of the fluctuation correlation functions and hence in powers of the fluctuation field,

$$\Gamma_k[\Phi; \bar{\Phi}] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^n [d^d x_i \Phi_{j_i}(x_i)] \times \Gamma_k^{(\Phi_{j_1} \dots \Phi_{j_n})}[\bar{\Phi}](x_1, \dots, x_n), \quad (10)$$

where $\Gamma_k^{(\Phi_{j_1} \dots \Phi_{j_n})}[\bar{\Phi}]$, for $n > 2$, are the proper vertices (Equation 6), that carry the measure factors $\prod_i \sqrt{\bar{g}(x_i)}$.

The derivative expansion is best explained in the case of the diffeomorphism invariant background effective action $\bar{\Gamma}_k[\bar{\Phi}] = \Gamma_k[\Phi = 0; \bar{\Phi}]$. This object can be expanded in diffeomorphism

invariant operators such as powers of the curvature scalar and other invariants. Then, in the derivative expansion the sum in Equation (1) contains all diffeomorphism invariant terms with less than a certain number of derivatives. The leading order of this expansion is

$$\bar{\Gamma}_k[g_{\mu\nu}] \simeq \frac{1}{16\pi G_k} \int d^d x \sqrt{g} [2\Lambda_k - R]. \quad (11)$$

At the next order one has to add four-derivative terms including R^2 , $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and so on. In this light, it should be understood that the Einstein-Hilbert action just provides the leading terms in the derivative expansion of $\bar{\Gamma}_k[g_{\mu\nu}]$ and does not constitute the bare action underlying Asymptotic Safety. It has to be supplemented by gauge-fixing and ghost terms, and, if the approximation is extended, additional terms $\hat{\Gamma}_k[h; \bar{g}]$ depending on two arguments separately. “Bimetric” studies distinguishing $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ for the Einstein-Hilbert truncation can be found in [117–119].

3.3. Results for Asymptotically Safe Gravity

...where we give a brief overview of the results obtained with the truncated FRG and provide a sketch of the full flow from the UV fixed point down to the IR.

Most work has been done in the background-field approximation, that is $\Gamma_k[\Phi; \bar{\Phi}] = \bar{\Gamma}_k[\Phi + \bar{\Phi}] + \text{gauge fixing} + \text{ghosts}$. If one evaluates the FRG in a one-loop approximation, including terms quadratic in curvature, the known universal beta functions of the four-derivative couplings are reproduced, but additionally the cosmological and Newton constant have a non-trivial fixed point [120–122]. Going beyond one loop, the following classes of operators have been studied in pure gravity: The Einstein-Hilbert truncation has been explored extensively [7, 8, 113, 123–131]. Einstein-Hilbert action plus R^2 [132, 133]; Einstein-Hilbert action plus R^2 and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ [56, 134–137]; Einstein-Hilbert action plus R^2 , $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ [138]; Einstein-Hilbert action plus the Goroff-Sagnotti counterterm $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu}$ [139]; polynomial functions of the scalar curvature (polynomial “ $f(R)$ truncation”) up to orders $N = 6$ [140, 141], $N = 8$ [55], $N = 35$ [57, 58], and lately also $N = 71$ [142], or effective actions of the form $f_1(R_{\mu\nu}R^{\mu\nu}) + f_2(R_{\mu\nu}R^{\mu\nu})R$, where f_1 and f_2 are polynomials [143], effective actions of the form $f_1(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + f_2(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})R$ where f_1 and f_2 are polynomials or finally effective actions consisting of a single trace of n Ricci tensors $(R^\mu{}_\nu R^\nu{}_\rho \dots R^\alpha{}_\mu)$ with n up to 35 [144]. The case of an “infinite number” of couplings has been addressed in the $f(R)$ truncation by solving [109, 145–159] a non-linear differential equation for f [58, 109, 116, 140–143, 145–161]. Global solutions for such “infinite” truncations can also be found for gravity coupled to a scalar field, see, e.g., [162]. For a more general overview of the situation in gravity-matter systems we refer to the review [14]. Notably, a fixed point suitable for Asymptotic Safety has been identified in all these works.

As is clear from this list, the terms included do not reflect the systematics of a derivative expansion. It has also to be

said that in many of these calculations the beta functions that one obtains are only unknown linear combinations of the beta functions that would be obtained if all curvature invariants of the same order were included. This is because the calculations are done on spheres, e.g., [55, 57, 58, 116, 132, 133, 140–143, 146–148, 150–153, 155, 157, 160, 163], a hyperbolic background [161] or sometimes on Einstein backgrounds, e.g., [56, 134], and this does not permit to differentiate between functions of Ricci tensor and of the Ricci scalar, for example.

In terms of the vertex expansion, most work has been built on an expansion around flat space while keeping part of the full momentum dependence of propagators and vertices. For the vertices typically the symmetric point configuration is considered. For results in pure gravity and gravity-matter systems see [164–174]. These works have revealed the existence of a non-trivial fixed point in the two-, three-, and four-point functions compatible with the findings in the background approximations. Analogous calculations with compatible results have also been done for the two- and three-point functions on a spherical background [156, 159]. The results in [156, 159] for background curvature and background momentum-dependent two- and three-point function of the fluctuation field have then been used to compute the full $f(R)$ -potential in pure gravity and in the gravity-scalar system beyond the background approximation.

Just like in the derivative expansion in asymptotically safe gravity, it has also not been possible to fully and systematically implement the vertex expansion beyond the lowest order: In particular, the three- and four-point functions have only been calculated for a special kinematical configuration and the symmetric background does not allow to fully disentangle different operators.

To connect the UV fixed point to physics at $k = 0$, complete trajectories must be constructed. Currently, this part of the program is less advanced than the characterization of the fixed point itself; UV-IR flows have been computed, e.g., in [108, 164, 166, 169, 175]. It is expected that complete solutions are most likely characterized by several regimes [125, 133, 176, 177], see also, e.g., [178, 179] for matter-gravity systems:

- The first part of the flow from the Reuter fixed point in the UV down to some scale M_1 is in a linear regime close to the fixed point. At these extreme UV scales, the system could *a priori* either be in a strongly interacting non-perturbative regime or be characterized by weak interactions. There are some tentative hints for the latter (see section 3.4), but a conclusive statement regarding the nature of the fixed point cannot yet be made.
- Close to the Planck scale, the flow has potentially already left the linear regime around the fixed point. In simple approximations, $M_1 = M_{\text{Pl}}$, i.e., the transition scale at which fixed-point scaling stops, actually comes out equal to the Planck scale. The regime around the Planck scale could again be characterized by either non-perturbative or near-perturbative physics—irrespective of the nature of the fixed point. Once one leaves the fixed-point regime, non-localities

of order $1/k$, or dynamically generated scales are expected to play a role⁸.

- Below the Planck scale, the description of the purely gravitational sector is expected to become much simpler. Once near the Gaussian fixed point, the flow is dominated by the canonical scaling terms. For instance, the dimensionful Newton constant becomes scale independent. One expects that corrections obtained within the effective-field theory approach to quantum gravity are recovered in this regime. Of course there remains the issue of the cosmological constant. In particular, it is still being debated whether De Sitter space is stable under radiative corrections, including the ones coming from graviton fluctuations. A proposal that the instability in the graviton propagator drive the cosmological constant to zero has been put forward in [181], see also [104] for a different point of view on the interpretation of this property of the graviton propagator, and [123] for an earlier discussion of the effect of IR-fluctuations on the cosmological constant. A general effective field theory approach to study this problem has yet to come.

In many cases these works on asymptotic safety based on the FRG can be compared to, or substantiated by, other approximation methods or techniques. We defer a discussion of such relations to sections 4 and 7.

3.4. The Convergence Question

...where we discuss the convergence (or lack thereof) of systematic expansion schemes in the FRG.

In practical applications, one has to work in truncations of the theory space. These can also be infinite dimensional, if a closed form for the flow of an appropriate functional can be found. In the gravitational case, closed flow equations for $f(R)$ truncations constitute an example [58, 109, 116, 140–143, 145–161]. Further examples are the scalar potential and a non-minimal functional in scalar-tensor theories, see, e.g., [104, 105, 182–184].

A reasonable expansion scheme should capture the relevant physics already at low orders of the expansion. For a fixed point, this includes the relevant operators. At the free fixed point one simply expands according to canonical power counting. At a truly non-perturbative fixed point, the relevant operators are not known. Therefore, simple truncations that correctly model non-perturbative physics can be difficult to devise. It is in such setups

⁸It is important to realize that non-local operators, i.e., operators with inverse powers of derivatives, proliferate under the flow and are canonically increasingly relevant. They are therefore likely to destroy the predictive nature of the fixed point, if included in theory space explicitly. On the other hand, the flow never generates an operator with inverse powers of derivatives within a quasi-local theory space, i.e., the requirement of quasilocality can be imposed consistently on the theory space. Of course it is well-known that the full effective action contains physically important non-localities. These arise in the limit $k \rightarrow 0$ and are expected to be captured through resummation of quasi-local operators, see, e.g., [180] for an example. This intricacy potentially makes this regime of the flow difficult to describe in a quantitatively robust way. It is generally expected that an expansion of the effective action in terms of vertex functions or form factors is best suited to this regime, as it can automatically capture non-localities of order $1/k$, and also encode the presence of dynamically generated scales.

that the concerted use of several techniques can be most useful; the IR regime of QCD constitutes an excellent example. Finally, at an interacting, but near-perturbative fixed point, canonical power counting constitutes a viable guiding principle to set up truncations. Here, near-perturbative refers to the fact that the spectrum of critical exponents exhibits deviations of $\mathcal{O}(1)$ from the canonical spectrum of scaling dimensions, but not significantly larger, in other words, the anomalous contribution to the scaling of operators is $\eta_{\mathcal{O}} \lesssim \mathcal{O}(1)$.

The strategy that has (implicitly or explicitly) been followed for the choice of truncations for the Reuter fixed point has been based on the assumption of near-perturbativity. This motivates a choice of truncation based on canonical power counting. The self-consistency of this assumption has to be checked by the results within explicit truncations. Indeed, [57, 58, 142, 143] find a near-canonical scaling spectrum in the $f(R)$ truncation. Moreover, [172–174] find close agreement of various “avatars” of the Newton coupling, something that is not expected in a truly non-perturbative regime.

As a self-consistent truncation scheme appears to be available for quantum gravity, the apparent convergence of fixed-point results is a key goal. It is fair to say that the status of results is rather encouraging with regard to this question, see [91, 118, 119, 124, 125, 132, 175, 185–200]. This has given rise to the general expectation that the Reuter fixed point indeed exists in full theory space, and provides a universality class for quantum gravity. Nevertheless, it should be pointed out that due to the technically very challenging nature of these calculations, the inclusion of a complete set of curvature-cube operators remains an outstanding task. In the vertex expansion, higher order derivative terms are captured by momentum-dependent correlation functions, which exhibit robust evidence for the Reuter fixed point [156, 164, 166–170, 172–174, 199–201].

3.5. Do Backgrounds Matter?

...where we highlight the technical challenges one faces when attempting to reconcile the use of a local coarse-graining procedure with the background independence expected of a non-perturbative quantum gravity approach.

When setting up the Wetterich equation for gravity [7] the background field formalism plays an essential role. The background metric $\bar{g}_{\mu\nu}$ serves the double purpose of i) introducing a gauge fixing which is invariant under background-transformations, and ii) introducing a regulator, as required to implement a local notion of coarse graining. At the same time, the decomposition of the physical metric into a fixed, but arbitrary background and fluctuations introduces a new symmetry, so-called split-symmetry transformations: the linear split (Equation 8) is invariant under

$$\bar{g}_{\mu\nu} \mapsto \bar{g}_{\mu\nu} + \epsilon_{\mu\nu}, \quad h_{\mu\nu} \mapsto h_{\mu\nu} - \epsilon_{\mu\nu}. \quad (12)$$

While actions of the form Equation (11) are invariant under these transformations, the gauge-fixing and the regulator terms

$$\Delta S_k = \int d^4x \sqrt{\bar{g}} h_{\mu\nu} [R_k(-\bar{D}^2)]^{\mu\nu\kappa\lambda} h_{\kappa\lambda}, \quad (13)$$

with $\bar{D}_\mu$ denoting the covariant derivative constructed from $\bar{g}_{\mu\nu}$ violate this symmetry. Thus $\Gamma_k[h_{\mu\nu}; \bar{g}_{\mu\nu}] \equiv \Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}]$ genuinely depends on two metric-type arguments.

Nevertheless, the gravitational effective average action [7] provides a background-independent approach to quantum gravity. The background metric $\bar{g}_{\mu\nu}$ is not an “absolute element” of the theory but rather a second, freely variable metric-type argument which is determined from its own equations of motion. At the most conservative level this feature follows from standard properties of the background field method satisfied by the effective action Γ and their extension to the effective average action Γ_k [107, 108, 155, 156, 202–206]. Alternatively, it has been proposed to achieve background independence not by quantization *in the absence* of a background, but rather by quantization *on all background simultaneously* [119]. We now review these arguments.

The fact that Γ_k and the resulting effective action Γ depend on two arguments allows to derive a background as well as a quantum equation of motion from Γ

$$\left. \frac{\delta \Gamma[h; \bar{g}]}{\delta \bar{g}_{\mu\nu}} \right|_{\bar{g}=\bar{g}^{\text{geom}}, h=0} = 0, \quad \left. \frac{\delta \Gamma[h; \bar{g}]}{\delta h_{\mu\nu}} \right|_{\bar{g}=\bar{g}^{\text{geom}}, h=0} = 0. \quad (14)$$

The Ward identity following from the transformation (Equation 12) then relates these two equations implying that a solution of one is also a solution of the other. This allows to fix $\bar{g}$ in a dynamical way. In particular, it shows that at $k = 0$ the background metric does not have the status of an absolute element. At finite values of k , the Ward identity satisfied by Γ_k receives additional contributions from the regulator (Equation 13) which introduce a genuine dependence on the background field. From these arguments, it is then clear that “background independence” is restored at $k = 0$ only.

We highlight that background independence is encoded in modified Ward identities which are completely general and work (in principle) for any correlator. The most important issue is to determine a physically motivated background, around which one can study the quantum fluctuations and which is determined by the equations of motion Equation (14).

The “all backgrounds is no background” proposal provides an extension of “background independence” to finite values of k . The underlying idea is to describe (one single) background-independent quantum field theory of the metric through the (infinite) family of “all possible” background-dependent field theories that live on a non-dynamical classical spacetime. Each family member has its own classical metric $\bar{g}_{\mu\nu}$ rigidly attached to the spacetime manifold. For each given background $\bar{g}_{\mu\nu}$, standard methods can be used to quantize the fluctuation fields Φ . Repeating this procedure for all $\bar{g}_{\mu\nu}$ yields expectation values $\langle \mathcal{O} \rangle_{\bar{g}}$ which are manifestly $\bar{g}$ dependent in general. Loosely speaking, the family of backgrounds, which is at the heart of background independence in the abstract sense of the word, should be regarded as the set of all possible ground states, one of which will be picked dynamically.

Ultimately, the physical background metric that is present in the geometric phase of quantum gravity, is determined by the dynamics of the system in a self-consistent fashion by solving the

quantum equations of motion at finite k

$$\left. \frac{\delta \Gamma_k[h; \bar{g}]}{\delta h_{\mu\nu}} \right|_{\bar{g}=\bar{g}_k^{\text{sc}}, h=0} = 0, \quad (15)$$

where the self-consistent background metric $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ is inserted. Hence, the expectation value of the metric is a prediction rather than an input. Notably, setting $(h = 0, \bar{g} = \bar{g}_k^{\text{sc}})$ is a particular way of going “on-shell” (but not the only one). We refer to [3, 156] for further details.

Given these remarks, it is clear that future work must address the following challenges:

- (1) The different functional dependence of Γ_k on $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ induces differences in the propagators for the fluctuation field and the background field. Thus, the functional dependence of Γ_k on $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ separately should be computed for a class of background metrics as broad as possible, as ultimately background independence can only be achieved if the dependence on the two distinct arguments of Γ_k is disentangled cleanly.
- (2) For computational feasibility the existing calculations mainly employ either highly symmetric background geometries or the Seeley-DeWitt (early time) expansion of the heat-kernel which encapsulates only local (albeit universal) information [207]. It is important to highlight that computations evaluating the left-hand side of the Wetterich equation at $h = 0$ (i.e., equating fluctuation propagator and background propagator) can deform and/or remove fixed points and introduce unphysical zeros of beta functions [163].
- (3) The difference between the $g_{\mu\nu}$ dependence of Γ_k and its $\bar{g}_{\mu\nu}$ dependence, driven by the distinct dependence of regulator and gauge fixing on the two metrics, is encoded in the modified split Ward or Nielsen identity resulting from Equation (12). In principle, by solving the flow equation together with this Ward identity, one would obtain a flow for a functional of a single metric. In practice, the solutions of the Ward identity has only been possible for the simplest approximations [108, 155, 203–206].
- (4) When Γ_k and with it $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ show a strong k dependence, the effective spacetime is likely to possess multi-fractal properties which were argued to lead to a dimensional reduction in the ultraviolet [133, 208–210] and to a “fuzzy” spacetime structure at even lower scales [198, 211, 212]. In the existing analyses the fractal-like properties were characterized in terms of ordinary, i.e., smooth classical metrics, the trick being that one and the same spacetime manifold was equipped not with one but rather the one-parameter family of classical metrics, $\{(\bar{g}_k^{\text{sc}})_{\mu\nu}\}$. As these fractal-like properties relate to the k dependence of Γ_k , it is at present unclear whether an “echo” of this behavior exists in the physical limit $k \rightarrow 0$. Investigating the full momentum dependence of $\Gamma_{k \rightarrow 0}$ can provide an answer to this question. If there is, it should be a mostly negligible effect at scales relevant for current experiments.

In conclusion, the issue of the background dependence is a main obstacle to progress in the application of the FRG to quantum gravity, both at the conceptual and technical level.

4. ADDITIONAL METHODS FOR ASYMPTOTIC SAFETY

... where we review other techniques used to search for asymptotic safety in gravity, including the ϵ expansion, numerical simulations, tensor models, and stress the benefits of using multiple methods.

Sections 3.4 and 3.5 have highlighted the technical challenges one faces when employing the FRG to study asymptotically safe gravity. Therefore, there is a strong case for the use of complementary methods, especially those where background independence can be implemented, such as Regge calculus or random lattice techniques, as well as specific tensor models. Due to the rather different nature of the systematic errors in these approaches, this simultaneously addresses the challenge linked to the convergence of truncations. Furthermore, other techniques may be better suited to explore the complete phase diagram of quantum gravity potentially including pre-geometric phases.

Historically, the starting point for studies of asymptotically safe gravity has been the ϵ expansion around $d = 2$, [213–217], which has been pushed to two-loop order in [218], showing indications for an asymptotically safe fixed point. It has been shown that the Reuter fixed point in $d = 4$ dimensions is continuously connected to the perturbative fixed point seen in $2 + \epsilon$ spacetime dimensions [7, 125]. The connection between Asymptotic Safety and Liouville gravity in $d = 2$ dimensions has been made in [197]. An (off-shell) gauge and parameterization dependence, as exhibited by truncated FRG studies, is also present in the ϵ expansion. Higher-loop terms are required in order to resum the ϵ expansion for the critical exponent to learn about the $d = 4$ -dimensional case. This appears to be merely a technical challenge, to which the advanced techniques developed in the context of supergravity [219] might potentially be adapted.

In line with the near-perturbative nature of the fixed point in $d = 4$, expected from FRG studies [57, 58, 142, 143, 173], a Padé resummation might yield a fixed point that is continuously connected to the fixed point in the vicinity of two dimensions.

Lattice approaches provide access to a statistical theory of random spatial geometries, thereby being in a position to provide evidence for or against asymptotic safety in the Euclidean regime. There are two main ways in which discrete random geometries are explored: One can hold a triangulation fixed and vary the edge lengths, as in Regge calculus, or hold the edge lengths fixed but vary the triangulation, as in dynamical triangulations. The latter have developed in two research branches: Euclidean Dynamical Triangulations [220], and Causal Dynamical Triangulations [221, 222].

Regge calculus (see [223] for a review) based on the Einstein-Hilbert action is subject to the well-known conformal factor instability, which requires an extrapolation in order to extract information about a critical point, see the discussion in [224]. With this caveat in mind, indications for asymptotic safety are found in Monte Carlo simulations of Regge gravity [224] based on the Einstein-Hilbert action. Testing the effect of additional, e.g., curvature-squared operators, which could correspond to additional relevant directions and have an important impact on

the phase structure, is an outstanding challenge in Regge gravity. A first comparison of scaling exponents obtained with the FRG to the leading-order exponent in Regge gravity can be found in [130, 225, 226].

In the case of Causal Dynamical Triangulations (CDT), the configuration space includes only configurations that admit a Wick rotation, see [222] for a review. Therefore, an analytical continuation to a Lorentzian path integral is in principle possible. In two dimensions, one can solve CDT analytically. Owing to the fact that in this case there are no local degrees of freedom, it has been shown in [227] and [228, 229] that the Hamiltonian appearing in the continuum limit agrees with the one for two-dimensional continuum quantum gravity and Horava-Lifshitz gravity [230], respectively. Moreover, Liouville gravity can be recovered by allowing for topology change of the spatial slices [227]. It has been stressed in [222] though that the equivalence of CDT and Horava-Lifshitz gravity may not extend beyond the two-dimensional case.

In higher dimensions, one searches for the continuum limit numerically. In practice, evidence for several [231, 232] second-order phase transition lines/points exists in numerical simulations, both in spherical and toroidal spatial topology. The large-scale spatial topology does not appear to impact the phase structure [233], but can actually improve the numerical efficiency of the studies, as observed in [234]. The higher-order transition can be approached from a phase in which several geometric indicators (spatial volume of the geometry as a function of time [235]; Hausdorff dimension and spectral dimension [236]) signal the emergence of a spacetime with semi-classical geometric properties. The properties of the continuum limit remain to be established, as the process of following RG trajectories along lines of constant physics toward the phase transition has not yet led to conclusive results regarding asymptotic safety [237, 238].

In Euclidean Dynamical Triangulations (EDTs), the configuration space differs from CDTs, as configurations do not in general admit a Wick rotation. This gives rise to spatial topology change and the proliferation of so-called “baby universes.” Early work [239–242] has not shown a higher-order phase transition [243–245]. The inclusion of a measure-term has led to the hypothesis that the first-order transition line could feature a second-order endpoint, and some evidence exists that the volume profile of the “emergent universe” approaches that of Euclidean de Sitter, i.e., a sphere, as one tunes toward the tentative critical point [246, 247]. This measure term could be reinterpreted as a sum of higher-order curvature invariants [246, 247] contributing to the action. The investigation [247] was unable to corroborate the appearance of a second-order endpoint though. In summary, solid evidence for a second-order phase transition exists in the CDT case, while investigations are ongoing in the EDT case.

Finally, dynamical triangulations can be encoded in a purely combinatorial, “pre-geometric” class of models, so-called tensor models [248–255], that attempt to generalize matrix models [256] for two-dimensional gravity to the higher-dimensional case. FRG tools which interpret the tensor size N as an appropriate notion of “pre-geometric” (i.e., background-independent) coarse-graining scale [257], allow to recover the well-known continuum

limit in two-dimensional quantum gravity within systematic uncertainties related to truncations [258]. First tentative hints for universal critical behavior in models with 3- and 4-dimensional building blocks have been found [259, 260]. The importance of symmetry-identities has been emphasized in [261]. This method could in the future provide further evidence for asymptotic safety, see [262] for a discussion, once the systematic uncertainties are reduced by suitable extensions of the truncation, and an understanding of the emergent geometries has been developed.

More broadly, the framework of the Renormalization Group and the notion of a universal continuum limit linked to a fixed point have recently been gaining traction in several approaches to quantum gravity, including group field theories [263–265] as well as spin foam models [266]. Accordingly, the concept of asymptotic safety might play an important role in several distinct approaches to quantum gravity. In particular, in spin foams, a search for interacting fixed points in numerical simulations has started recently in reduced configuration spaces, see, e.g., [267–269]. In causal sets, investigating the phase diagram and the order of potential phase transitions has only shifted into focus more recently, with indications for first-order phase transitions in restricted configuration spaces for lower-dimensional causal set quantum gravity [270–273].

In summary, the further development and application of a broad range of tools to explore asymptotic safety could be key to gain quantitative control over a potential fixed point, establish its existence and to develop robust links to phenomenology, which rely on a good understanding and control over systematic errors within various techniques. It is encouraging, that indications for a second-order phase transition have already been found with such techniques.

5. RUNNING COUPLINGS

5.1. A Clarification of Semantics

...where we clarify that the term “running coupling” is used with different meanings in different contexts.

Much of the current work on asymptotic safety of gravity uses techniques and jargon that are more common in statistical than in particle physics. This concerns even basic notions such as the RG. If one aims at detecting asymptotic safety by means of standard perturbative particle physics observables, there is thus much room for misunderstanding.

The RG was used in particle physics largely as a tool to resum “large logarithms,” terms in the loop corrections to physical observables of the form $\log(p/\mu) = \log(p/\Lambda) + \log(\Lambda/\mu)$, where p is a momentum, μ a reference scale and Λ a UV cutoff. From the way they emerge, the beta functions that resum the large logs are just the coefficients of the logarithmic divergences $\log(\Lambda/\mu)$. One important feature of these logarithmic terms is that their coefficients are “universal,” up to next-to-leading non-trivial order (NLO) in the coupling expansion. This entails two things: on the one hand, it means that, up to NLO, they are

independent of the way one computes them⁹. On the other hand, one can use them to “RG improve” *any* tree level observable, and one is guaranteed to obtain the correct result (not the full result, of course, but the part that comes from calculating and then resumming the logs). Here by “RG improvement” we mean the substitution of the running coupling into a tree-level expression, and the subsequent identification of the RG scale with an appropriate physical scale of the system¹⁰. If one demands these properties of a running coupling, then one would say that *only dimensionless couplings can run*. Dimensionful couplings have power divergences that are simply subtracted in perturbation theory. In line with these arguments, it has been pointed out in [274, 275], that the one-loop corrections to gravity-mediated scattering amplitudes cannot be obtained from applying the RG improvement to Newton’s coupling.

In Wilson’s non-perturbative approach to renormalization, all possible terms consistent with symmetries are present in the action. Quite often, the Wilsonian momentum cutoff has a direct physics interpretation, e.g., as lattice spacing in condensed-matter applications (with a relation to the Kadanoff block-spinning [276] underlying Wilson’s renormalization idea), and as the mass of states that are “integrated out” in effective field theories. In lattice gauge theories the Wilsonian momentum cutoff is finally removed (in the continuum limit), but keeps its physics interpretation similar to the condensed-matter applications at intermediate stages. Nevertheless, the momentum cutoff is treated mathematically as an independent variable, and all couplings in the Wilsonian action depend on it. Apart from a few relevant parameters to be tuned to criticality, the remaining set of “running couplings” is not constrained by the demands of universality; still, this notion of running couplings remains also valid at the non-perturbative level.

The relation between the two definitions of the RG is this: At energy scales much higher than all the masses, the leading- and next to leading-order terms of the perturbative beta functions, that are independent of the renormalization scheme, can also be obtained from the Wilsonian RG and are independent of details of the coarse-graining scheme. In particular, the one-loop terms can be easily found from Equation (7). The recovery of 2-loop terms from the FRG has been addressed, e.g., in [277–283]. At energies comparable to the masses, the beta functions extracted from the Wilsonian RG include threshold effects which encode the automatic decoupling of massive modes from the flow at scales below the mass. This is an advantage over setups in which this decoupling is not accounted for automatically and must instead be done by hand.

If one accepts the more general Wilsonian definition of running coupling, then the statement “dimensionful couplings cannot run” translates into the statement that the Wilsonian running of dimensionful couplings does not carry the same

⁹They are almost always derived in dimensional regularization, which for technical reasons is the most convenient method, e.g., it respects gauge symmetries.

¹⁰For example in a process $e^+e^- \rightarrow e^+e^-$ at center of mass energy $\sqrt{s} \gg m_e$ at n -loop order the renormalized leading contribution with subtraction scale μ is proportional to the tree level cross-section (at scale μ) times $\sum_{l \leq n} [\alpha(\mu) c \log \frac{s}{\mu^2}]^l$, which shows that the most convenient choice is $\mu = \sqrt{s}$.

direct physical meaning as the running of dimensionless ones. Nevertheless, to encode physics correctly within a Wilsonian setup, the running of dimensionful couplings is critical and cannot be neglected, since in a massive scheme operators mix non-trivially.

To be more specific, one can consider what happens at second order phase transitions. The generic power-like running of the Wilsonian couplings in the FRG approach is in general non-perturbative and its calculation is limited only by the approximations. At the fixed point, the couplings have non-universal values (depending on the details of the microscopic theory), but there are also universal quantities which can be extracted from the flow close to criticality. These are the same for very different physical systems belonging to the same universality class. The power-like divergences are associated to non-universal features such as the position of the fixed point and of the critical surface (see, e.g., [284]). For example, the power quadratic divergence in systems belonging to the Ising universality class is related to the critical temperature T_c , which varies from one material to another. If one is interested in this physical information, the accurate scaling of the corresponding quadratic composite operator or the behavior of the two-point function should be determined.

Similar considerations may apply in quantum gravity, where the running Planck mass (the coefficient of the “ R ” operator in the effective Lagrangian) is a non-universal quantity which is just one of the parameters defining the position of a possible UV fixed point and of the critical surface containing it. Note that in an asymptotically safe theory of quantum gravity, the physics is related not just to the UV fixed point, but to the particular renormalized trajectory flowing away from it toward lower energy scales. Therefore it depends indirectly on all such Wilsonian (dimensionful) couplings. Observables, as already discussed, are computed at $k = 0$ on the on-shell configurations and are mostly sensible to a number of non-universal parameters related to the finite number of relevant directions, including the (flowing) Planck mass. We shall discuss in section 5.3 how one could define the effective couplings.

5.2. Remarks on Dimensional Regularization

...where we explain in which cases some care is required for the correct interpretation of results achieved within dimensional regularization.

A seemingly technical point where the Wilsonian RG approach differs from a perturbative particle-physics perspective is the regularization of quantum modes. While the FRG works with explicit momentum-space regulators (or spectral regulators of curved-spacetime Laplacians), conventional perturbation theory mostly uses dimensional regularization for reasons of convenience. Physics must not depend on the choice of the regularization scheme, hence it is an obvious question as to whether dimensional regularization can also be brought to work in a FRG context and for the asymptotic-safety scenario of gravity.

In fact, one-loop results for power-counting marginal operators quadratic in the curvature with dimensionless couplings exhibit the expected universality [120, 285, 286]. However, this is no longer true for the RG running of power-counting relevant and irrelevant operators, simply because they do not feature the same degree of universality. Even worse, dimensional regularization is blind to power divergencies and hence acts as a projection onto logarithmic divergences appearing as $1/\epsilon$ poles. For such reasons, Weinberg calls dimensional regularization “a bit misleading” in the context of asymptotically safe theories [5].

Dimensional regularization relies on the virtues of analytic continuation. Hence, its application requires to pay attention to the analytic structure of a problem at hand. This is well-known, for instance, from non-relativistic scattering problems where a naive application of dimensional regularization fails because of a different analytic structure of the propagators and more care is needed to apply analytic continuation methods to regularize and compute observables [287, 288]. The same is true for computations in large background fields where a naive straightforward application of dimensional regularization is not possible, but requires a careful definition in terms of a dimensionally continued proptime or ζ function regularization [289, 290]. The latter techniques can be linked to heat-kernel methods and allow to access information related to power divergences [291].

As most computations for asymptotically safe gravity are performed in “large backgrounds,” i.e., in a fiducial background spacetime, a proper use of dimensional regularization would similarly require a definition in terms of, e.g., a proptime or ζ function definition based on the heat kernel. In fact, approximations of the FRG have been mapped onto a proptime representation (proptime RG). Applications to gravity do lend further support to the existence of the Reuter fixed point and the asymptotic-safety scenario [188].

5.3. Correlation Functions and Form Factors

...where we clarify the distinction between RG scale dependence and physical scale dependence within the FRG context. We further detail how the physics of asymptotically safe theories is encoded in momentum-dependent correlation functions and form factors, discuss the definition of non-perturbative running couplings and the construction of observables from these objects.

The idea of the Wilsonian renormalization group is to solve the theory by integrating out quantum fluctuations, one (covariant) momentum shell at the time. It is crucial to distinguish the k -dependence from the dependence on physical scales. In the FRG approach governed by the Wetterich Equation (5) with an infrared cutoff, general correlation functions

$$C_k(p_1, \dots, p_n) = \langle \Phi_{i_1}(p_1) \cdots \Phi_{i_n}(p_n) \rangle_k \quad (16)$$

are trivial for all (covariant) momentum scales $p_i^2/k^2 \ll 1$, and carry the momentum dependence of the full theory for all

momentum scales $p_i^2/k^2 \gg 1$ and large scattering angles. Here, trivial means that for $p_i^2/k^2 \ll 1$ the correlation functions are that of a theory with a mass gap $m_{\text{gap}}^2 \gtrsim k^2$: the quantum dynamics dies off with powers of p_i^2/k^2 . In asymptotically safe theories, these correlation functions will exhibit indications of quantum scale invariance at large p_i^2/k^2 .

Evidently, the correlation function $C_k(p_1, \dots, p_n)$ is a highly non-trivial function of all p_i , other physics scales in the theory, such as mass scales, and the cutoff scale k . The latter is instrumental for the transition from the full quantum dynamics of the theory to the trivial one in the gapped regime. This non-trivial behavior is complicated by the fact that the n -point correlation functions carry n momenta p_i with $i = 1, \dots, n$. This results in a multiscale problem, unless we restrict ourselves to a symmetric point with $p_i^2 = p^2$. We also remark that both UV and IR regimes may exhibit asymptotic power-law momentum scaling or anomalous scaling and the momentum and cutoff dependence in the transition regime at $p_i^2/k^2 \approx 1$ is in general highly non-trivial. In particular, the momentum dependence is typically more general than a logarithmic one.

In section 6, we introduce diffeomorphism invariant observables as spacetime integrals over correlators akin to the one in Equation (16), cf. Equation (18), or S matrix elements via the proper background vertices $\bar{\Gamma}_k^{(\tilde{\Phi}_{i_1} \dots \tilde{\Phi}_{i_n})}$. To compute such observables, in the FRG approach to asymptotically safe gravity we first have to compute the proper vertices of the fluctuation fields, $\Gamma_k^{(\Phi_{i_1} \dots \Phi_{i_n})}$. Their scale- and (covariant) momentum dependence indirectly encode the physics of asymptotically safe gravity despite not being observables themselves. An important step toward observables is made by considering running couplings, that are renormalization group invariant combinations of the *form factors* or *dressings* of these vertices as defined in standard gauge theories and scalar and fermionic QFTs. These are defined from the k - and momentum-dependent vertices together with appropriate factors of the wave-function renormalizations. For instance, in the case of scalar and fermionic QFTs, these are directly related to S matrix elements. In turn, in gauge theories such as QCD they lack gauge invariance but nonetheless carry important physics information: In QCD these running couplings derived from the proper vertices of the fluctuating or background fields give direct access to the momentum scaling in the perturbative regime as measured by high-energy experiments, see, e.g., [292]. Further, non-perturbative physics, such as the emergence of the confinement mass gap, is also captured by these running couplings, see, e.g., [293–295].

This implies that the momentum dependence of these running couplings at $k = 0$ provides rather non-trivial physics information. In asymptotically safe gravity, it can in particular be used to identify scaling regimes in the UV and the IR as well as the transition scale: In [164, 166, 168, 169], non-perturbative generalizations of the Newton coupling $G_k^{(n)}(p^2)$ with $n = 3, 4$, defined from the n -point functions, have been computed from combinations of the proper two-, three- and four-point functions of the fluctuation fields in a flat background and all cutoff scales. For the generalization to the case with matter,

see [170, 172–174]. In these calculations, the dependence on the $n - 1$ momenta of an n -point vertex has been simplified by going to the momentum-symmetric point, allowing the definition of a running coupling that depends on a single momentum. A flat background, as used in the above studies is of course a first step toward a comprehensive understanding of the physical scale dependence of quantum gravity. For first steps toward an extension to generic background, see [156, 159].

On a generic background, the dependence on physical scales can also be captured in the language of form factors. In the background effective action these form factors appear naturally, see [296],

$$\bar{\Gamma}[g_{\mu\nu}] = \int d^4x \sqrt{g} [f(R) + f_1(R_{\mu\nu}R_{\mu\nu}) + C_{\mu\nu\rho\sigma} W^T(\Delta) C^{\mu\nu\rho\sigma} - R W^R(\Delta) R + \dots], \quad (17)$$

Equation (17) also summarizes concisely the approximation considered so far for the background effective action. The corresponding form factors W^R and W^T have been computed in [175]. Note that Equation (17) can also be understood as the dynamical effective action in the diffeomorphism invariant single metric approach put forward in [284, 297–300]. There it has been argued that the physical gauge there facilitates the direct physics interpretation of form factor such as W^T and W^R .

Both within the language of momentum-dependent correlation functions as well as with form factors, the asymptotically safe regime, the transition regime and a long infrared regime with classical scaling have been identified. The results are rather promising and open a path toward the computation of observables or their local integral kernels. Still, the approximations used so far do in particular not sustain large curvatures and have to be upgraded significantly.

6. OBSERVABLES

...where we emphasize the necessity to investigate observables in order to make quantum gravity testable, and discuss three possible classes of observables.

The physical behavior of a system is probed through observables. While their definition and construction is not a problem in many interesting cases of quantum and statistical field theories in flat, and possibly some specific classical curved spacetimes, it is in general very difficult to define meaningful observables in quantum gravity. To begin with, already in classical gravity diffeomorphism invariance makes the notion of a spacetime point unphysical and hence implies that there cannot exist any local observable: any gauge invariant observable must be the integral of a scalar density over all spacetime. The situation is somewhat better in the presence of matter, for example it makes sense to define the value of the scalar curvature at the position of a particle, or at a point where certain matter fields have predetermined values [301]. These observables are however difficult to work with in practice. These problems persist in

quantum gravity, see, e.g., [302]. Nevertheless the construction of observables remains a crucial task.

In the following, we will focus on observables in the sense of quantities that are of direct phenomenological relevance. These often rely on introducing a (dynamically generated) background that provides a suitable notion of locality. The type of observables that one will consider depends very strongly on the type of observations that one has in mind. We will distinguish three possible classes of observations that could be used to test asymptotic safety.

6.1. Particle Physics at the Planck Scale

The first is appropriate when we imagine living in a macroscopic classical spacetime and probing its short distance structure by some “microscope” of the kind that is used in particle physics. For example, we could try to directly measure scattering cross-sections and decay rates at Planckian scale or beyond. In this case the issue of diffeomorphism invariance is circumvented by postulating the existence of an asymptotically flat background, which is necessary in order to define the appropriate notions of particles and asymptotic states. The validity of this postulate remains to be investigated in a given quantum theory of gravity. In principle, the integral kernels of these particle-physics observables can be constructed from the proper vertices of the background effective action $\bar{\Gamma}_k^{(\bar{\Phi}_{i_1} \dots \bar{\Phi}_{i_n})}$ for an asymptotically flat spacetime, see [303]. We provide some details on the calculation of these quantities in section 5.3. Indeed, the original formulation of Asymptotic Safety by Weinberg was formulated in these terms: as stated in [4, 5], ideally, the couplings whose running one wants to study should be defined directly in terms of such observables. However, most of the actual work on Asymptotic Safety is based on the running of parameters in the Lagrangian, that are not directly observable or not even directly related to observables. Assuming that this notion makes sense, measurement of the S matrix at the Planck scale and beyond would give the most direct and unambiguous test of Asymptotic Safety. Unfortunately, neither the theoretical nor the experimental sides of the comparison are available. In settings with extra dimensions, scattering cross sections have been calculated within the framework of RG improvement [304–306], see section 7.4 for a discussion of the potential pitfalls of this procedure. With current technology, these observables are also unlikely to ever be measured. Furthermore, the postulate of an asymptotically flat background leaves out many situations that are of interest in the context of quantum gravity.

6.2. Low-Energy Imprints

A second possibility, still closely related to the world of particle physics, but not requiring Planckian energy, is the observation of properties of the low-energy world that could carry an imprint of asymptotically safe quantum gravity. One can distinguish two sub-cases, that we shall refer to as “higher-order observables” and “marginal observables.” Both sets of observables are most directly calculable if one assumes a “great desert” between the Planck and the Fermi scale. Else, one requires a specific model for the intervening physics.

- (i) The high energy theory will leave traces in the low energy effective field theory in the form of higher order operators that are suppressed by inverse powers of the high scale. In particular, higher-order matter self-interactions are very likely both non-vanishing and irrelevant in the UV, if an asymptotically safe matter-gravity fixed point exists [307–312]. This results in predictions for these higher-order couplings in the IR. The separation of scales between the Planck scale and IR scales is so large that, typically, these quantum-gravity effects are unmeasurably tiny. Still, one may hope that there exists a signature that is forbidden in any non-gravitational process and that becomes detectable under rather unexpectedly favorable circumstances.
- (ii) The other, significantly more promising, possibility is that some gross features of the low energy world, probed at present or future colliders and linked to canonically marginal couplings, i.e., dimensionless operators, could be directly “explained” by properties of a UV-complete quantum theory of gravity and matter. This is due to the fact, explained in section 3, that Asymptotic Safety may yield more predictions than a perturbatively renormalizable model. In the gravitational sector, this mechanism may not lead to testable predictions: here only a handful of parameters are experimentally accessible and there are essentially no constraints on the value of the curvature-squared couplings. In the matter sector, this picture changes completely. In this case literally thousands of observables are available, depending on at least two dozen free parameters. Some of these canonically marginal couplings could become irrelevant directions of an asymptotically safe gravity-matter model. First tentative hints have been obtained in this direction, for example a proposed scenario for a prediction of the Higgs mass [313] and a calculation of the top mass [178], the Abelian gauge coupling [194, 314] and the bottom mass [315]. These are obtained in comparatively small truncations and are subject to the assumption that Euclidean results carry over to Lorentzian gravity-matter systems. These are of course not “smoking guns” for Asymptotic Safety, but it is not unreasonable to expect that a microscopic description of quantum gravity constrains the features of a matter sector that can consistently be coupled to it. Indeed, the swampland program in string theory is based upon the same assumption. Ultimately, one could hope to arrive at an extended list of calculable properties of matter models from various quantum gravity theories, allowing to rule out some of the latter observationally without the need to probe Planck-scale physics directly. It therefore seems worthwhile to more systematically develop the predictions that an asymptotically safe theory of gravity and matter can make for low-energy observables. In particular, the dark-matter sector could allow to make genuine predictions [316–318], in contrast to the consistency tests that the already measured properties of the Standard Model provide. We shall discuss in section 7.1 how such effects could be calculated.

6.3. Asymptotically Safe Cosmology

The third class of observations is related to cosmology. As long as a Friedmann-Robertson-Walker (or some other) background

is a good approximation, there is a well-developed machinery for the treatment of fluctuation correlators [319]. At the formal level, observables in quantum gravity are given by integrated correlators, for example spacetime integrals of n -point correlations of the Ricci scalar

$$\mathcal{O}_R^{(n)} = \langle \prod_i \int d^4x_i \sqrt{g(x_i)} R(x_1) \cdots R(x_n) \rangle. \quad (18)$$

Naturally, while the spacetime dependent curvature fluctuations in Equation (18) are not observables themselves, they carry the physics information encoded in its spacetime or momentum-scale dependence.

Inflation is believed to occur at sub-Planckian energies, but it may be close enough to a fixed-point regime to be directly influenced by it. Further, in settings like Starobinsky inflation, higher-order operators in the gravitational theory actually drive inflation. Along this line, it was explored in [177] whether the freedom in the R^2 coupling offered by Asymptotic Safety can be used to realize Starobinsky inflation giving power spectra compatible with present observations. Moreover, there are some tentative hints that quantum-gravity effects typically drive scalar potentials toward flatness, see, e.g., [182, 299, 316], and generally impose strong constraints on the inflationary potential that is usually introduced in a rather *ad-hoc* manner, see also [320]. In a more unorthodox approach to early cosmology, the idea is being explored that quantum gravity directly solves the horizon, flatness and monopole problems and generates the appropriate spectrum of fluctuations without the need for additional degrees of freedom together with an *ad-hoc* potential. In particular, in [321] it has been demonstrated that an action including all gravitational four-derivative invariants leads to the suppression of spacetime configurations with an initial singularity as well as anisotropies and inhomogeneities. In the early universe the usual flat space QFT machinery is not available and one has to use different observables that are geared to high temperature/high curvature situations. Then one may hope that features of the fixed point such as scaling exponents and OPE coefficients - that in statistical physics are generally considered measurable physical quantities - could leave an imprint in these cosmological observables.

Similar to quantum effects in QED encoded in the Euler-Heisenberg Lagrangian and its higher-loop extensions, quantum effects in gravity are encoded in the full effective action, including its non-local parts. The potential dynamical importance of non-localities for cosmology, e.g., in the context of dynamical dark energy, has been emphasized in [201].

6.4. Remarks

As with other situations where non-perturbative physics is involved, one could try to cross-check results obtained with continuum QFT methods with lattice studies. It is worth mentioning that also in lattice approaches to quantum gravity, observables are very hard to define and especially to implement in the simulations, see, e.g., [322] for encouraging recent results. This is in stark contrast to the large number of observables

that can be defined in the presence of an asymptotically flat background.

Finally, let us recall that in other approaches to quantum gravity such as LQG, “geometrical” observables such as lengths, areas, volumes, and curvatures have played an important role. These have also been discussed to some extent in Asymptotic Safety, [323], and can be computed with a flow equation for composite operators [90, 324–329]. While presently it is not clear what type of measurement is required to access such observables, they can be used to explore whether different approaches to quantum gravity give rise to universal physical results. Further, such geometrical observables have been used in [330] to set up a physical renormalization scheme.

7. RELATION OF ASYMPTOTIC SAFETY TO THE EFFECTIVE-FIELD THEORY APPROACH

7.1. Asymptotic Safety and Effective Field Theory

...where we discuss the relation of the EFT framework to Asymptotic Safety and also outline a strategy how to devise approximations in which the link between the two descriptions can be established in practice.

The framework of EFT is pervasive in modern particle physics. EFT is based on an expansion in E/M , where E is the typical energy scale of the experiment, and M is the scale above which the EFT description may no longer be meaningful. In EFT, one finds that higher loop corrections are suppressed by higher powers of E/M , so that the tree level and one loop are usually enough to explain most of the phenomenology, provided the system is indeed perturbative in nature, as happens to be the case in many particle-physics applications. Physical predictions are possible even when the theory is not perturbatively renormalizable, as long as one considers only low-energy observables and assumes that the dimensionless counterparts of all couplings are roughly of $\mathcal{O}(1)$.

Einstein gravity is a paradigmatic example of this point of view. It is perturbatively non-renormalizable, but one can still reliably compute observables in perturbation theory, as long as they are not affected by the higher-derivative terms in the action, whose coefficients are not calculable. This is the case for some non-analytic parts of scattering amplitudes. The calculation of the quantum corrections to the Newtonian potential is the most reliable calculation ever performed in quantum gravity [331]. It is also the most accurate, since the separation of scales between the characteristic scale of the theory (the Planck scale) and the scale where one performs experiments (even at the LHC) is the largest of any EFT, so that loop corrections are suppressed by enormous factors. In this way, every test of Einstein’s GR is also a test of this EFT of gravity.

In view of this, the motivation for asymptotic safety is two-fold: first, to have predictions for what happens at and beyond the Planck scale and second, the promise of increased predictivity,

in particular also at lower energies. This is especially desirable in the presence of standard-model or beyond-standard-model matter which is or might be detected in present and future colliders or, e.g., in dark-matter detection experiments. We stress again that the enhanced predictivity comes from the fact that asymptotic safety selects a class of RG trajectories which are expected to be parametrized by only a few free parameters. In principle, all the remaining coefficients in the effective action are calculable, including the coefficients of local higher dimensional operators that appear perturbatively divergent and are therefore not calculable in the EFT.

When one follows a realistic RG trajectory from the UV fixed point, crossing the Planck scale and moving toward the IR, one must eventually arrive in the immediate neighborhood of the free-theory fixed point of Einstein theory, which is the domain where EFT is applicable. In this regime, all the predictions of EFT must still hold true. Indeed, in the FRG formalism, the loop expansion can be reconstructed systematically by expanding the right-hand side of the equation in powers of $\hbar$, cf. Equation (7). This is usually not done, because there are already other methods that are perhaps better suited for this task; but in principle, the FRG can reproduce all the results of the EFT in this way.

In practice, constructing a flow that links the description of the fixed point, which might or might not be near-perturbative, to the perturbative low-energy regime after potentially passing through a more strongly-coupled transition regime, is a challenge. A possible strategy to deal with this complex problem is to figure out which parts of the flow can be captured by perturbation theory, and then use different tools (perturbation theory, one's favorite FRG approach) in the respective regime so as to obtain maximally reliable predictions of the observables. In order to link the description in terms of the FRG for the effective average action Γ_k to the perturbative EFT setup, one needs to calculate $\Gamma_{k=M}$, where M is the scale at which a perturbative description becomes possible. This procedure has been performed and carefully checked in QCD, where we flow from an asymptotically free theory of quarks and hadrons in the ultraviolet to chiral perturbation theory and low energy effective models in the infrared.

First steps toward using the FRG to derive the effective action of quantum gravity and matter systems have been taken in [332, 333]. Such calculations overlap significantly with EFT calculations.

Investigations in low energy effective theories are typically based on the Wilsonian action $S_{\text{eff},\Lambda}$ (regularized with a UV cutoff) both in QCD and in standard perturbative low energy EFT approach which is used in collider physics. The Wilsonian action is the generalized Legendre transform of the effective average action [88, 89] and obeys the Polchinski equation [334]. So far, the Wilsonian action has been less used in asymptotic-safety investigations, but it could help to compare to results obtained from collider measurements of scattering observables for its closeness to low energy effective theories. These works can be based on recent proposals in [335, 336] for the flow of the Wilsonian action based on proper time regulator schemes [336].

The choice of truncation used for the effective average action down to the scale M might be crucial to correctly encode the various consequences of the UV fixed point, both in the matter

and gravity sector. $\Gamma_{k=M}$ or $S_{\text{eff},\Lambda=M}$ provide the initial condition for a subsequent perturbative calculation at one or two loops; of course, also RG schemes would need to be matched for precision calculations. The perturbative part of the RG evolution gives rise to the non-localities in $\Gamma_{k \rightarrow 0}$ and all IR effects which are necessary to include to correctly describe observables. In this way, the FRG and perturbation theory can be used concertedly in order to link the UV fixed point to observable physics in the IR (see also section 5.3), and the use of different RG equations (Wetterich and Polchinski) would offer non-trivial consistency checks.

7.2. Effective vs. Fundamental Asymptotic Safety

...where we discuss why an asymptotically safe fixed point could matter even if the deep UV of quantum gravity is described by a completely different theory.

The RG fixed point underlying asymptotic safety features infinitely many infrared attractive directions. Therefore, a fixed point can serve various purposes in different scenarios: (1) it can be the UV starting point of an RG trajectory, (2) it can be the IR endpoint of an RG trajectory, (3) it can generate an intermediate scaling regime at finite scales. The latter option can play a role in settings where a more “fundamental” description of quantum gravity holds at small distance scales, i.e., beyond a finite momentum cutoff k_{UV} . Indeed, for $k < k_{\text{UV}}$, an effective description (with the metric as the effective gravitational field—not necessarily in the sense of perturbative EFT) holds, i.e., we are in the theory space of asymptotically safe gravity. The more fundamental description provides the initial condition for the RG flow at k_{UV} . If the initial condition satisfies a finite number of conditions related to the relevant directions of the fixed point, the flow will pass close by the fixed point and exhibit an approximate scaling regime over a finite range of scales. The flow toward the deep IR will then closely resemble that of an actual fixed-point trajectory, resulting in essentially the same predictivity [337], see [338] for a general discussion and [339] for a discussion in the context of string theory.

In this sense, an asymptotically safe fixed point can play a role in an EFT setup for gravity, and serve as a way to extend the regime of validity of the standard perturbative EFT framework.

7.3. The Structure of the Vacuum

...where we caution that the true ground state of gravity might not be a flat background, making the bridge to the EFT setting potentially more intricate. This question has so far only been addressed within a severe approximation of the dynamics and degrees of freedom.

The EFT approach to quantum gravity typically quantizes (small) fluctuations about a flat background. To link asymptotic safety to the EFT regime, one must therefore explore whether a flat background is a self-consistent choice, i.e., whether the flat background corresponds to the ground state of the theory. Here,

we should highlight that the ground state should of course be determined from the full gravity-matter system.

To date the only explicit investigation of the vacuum structure of asymptotically safe gravity based on the effective action $\Gamma_{k=0}$ has been performed within the conformally reduced $R + R^2$ -approximation¹¹ and a layered structure of the effective spacetime has been found within this simple truncation (borrowing terminology from a vacuum model of Yang Mills theory, it has been termed “lasagna vacuum”) [340]. Thereby the spatial modulation of the metric cures the notorious conformal factor instability generating a phase similar to those present also in higher-derivative low-dimensional condensed matter systems.

While this proposed vacuum structure has only been found in a severe approximation of the dynamics and degrees of freedom, this can be read as a firm warning regarding all backgrounds that are not shown to be solutions of the effective field equations. They are of no physical relevance and might convey an incorrect general picture. In particular, one typically expects truncations to converge faster when the field configurations are expanded about the true ground state of the theory—an expectation that can be tested within, e.g., the $O(N)$ model. On the other hand, it is crucial to remark that a spatially modulated ground state appears to be difficult to reconcile with stringent tests of Lorentz symmetry in the gravitational and the matter sector. Further, while the conformal approximation could suffice to capture the presence of a fixed point, it is to be expected that the inclusion of spin-2 modes will have a strong impact on such studies.

Moreover, the importance of properly accounting for the (k -dependent) ground state in studies of the flow is emphasized in a recent background-independent re-analysis of the cosmological constant problem allegedly caused by quantum vacuum fluctuations. Paying careful attention to identifying the correct ground state, the often discussed naturalness problem disappears, see [198].

Understanding the ground state of the theory at $k = 0$ is important. It is expected that since $\Lambda_k = \lambda_* k^2 \rightarrow \infty$ in the quantum regime governed by the Reuter fixed point, the self-consistent metrics (cf. section 3.5) $\bar{g}_{k \rightarrow \infty}^{\text{sc}} \propto k^{-2}$ will display increasing and ultimately diverging curvature. It is an open question how this manifests itself at the level of $\Gamma_{k \rightarrow 0}$ and its effective field equations. Whether this is an unphysical effect and only present at large k or whether it translates into a physical scale dependence is presumably important for questions of singularity resolution in black-hole spacetimes and the early universe. More generally, accounting for true vacuum of the theory, with the help of the self-consistent background is important for a quantitatively precise exploration of the phenomenological implications of the quantum-gravity effects.

7.4. RG Improvement

...where we critically review and discuss the procedure of RG improvement, discuss its interpretation as “quantum-gravity

inspired” phenomenology, and caution regarding the quantitative reliability of this tool.

Since the task of calculating the effective action $\Gamma_{k \rightarrow 0}$, including its non-local contributions, is an extremely challenging one, one may hope to extract qualitative information on the effects of quantum fluctuations by applying the procedure of “RG improvement” in gravity. In section VI.A, we have already defined what is meant by RG improvement in a perturbative context. Proceeding in a similar way in a gravitational context, it has been a common strategy to retain the dependence of some of the couplings, G_k and Λ_k say, on the RG scale k and identify the latter with a geometrical quantity or momentum. Based on such RG improvement ideas there is a substantial body of work investigating black-hole physics [341–355], gravitational collapse [356–362], and cosmological scenarios [320, 360, 363–376] inspired by Asymptotic Safety. One might expect that this procedure could be justified in some cases where the external scale in question acts as an IR cutoff for fluctuations.

The “improvement” could be applied at different stages, for instance, at the level of the action or the field equations, or of the solution of the field equation. This freedom already implies that this procedure could lead to ambiguous results. As an example, we may consider the RG-improvement procedure based on the effective average action approximated by the Einstein-Hilbert action,

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} [2\Lambda_k - R]. \quad (19)$$

Dimensional analysis implies that at the fixed point $G_k = g_* k^{-2}$, $\Lambda_k = \lambda_* k^2$. Identifying k^2 with the Ricci scalar and substituting the result back into Equation (19) leads to a higher-derivative R^2 action. This is precisely what one would expect for the fixed-point action for an $f(R)$ theory in the large R limit. Indeed RG improving any $f(R)$ theory in the same way results in an R^2 action, as expected from classical scale invariance. Thus the scale identification generates interactions that have a natural place in the effective action (Equation 17). However, this can lead at most to qualitative insights, as is made clear, for example, by the fact that even the simple identification $k^2 \sim R$ can only be made up to some arbitrary numerical factor.

To understand better whether an RG improvement is justified, let us consider some classic QFT examples, and contrast them with their gravitational counterparts. The Uehling potential in QED is probably the paradigmatic example: the correct form of the one-loop potential between two point charges can be obtained by inserting the one-loop form of the running coupling in place of the classical coupling and identifying the RG scale with the Fourier momentum of the static potential between the point charges. Conversely, one can read off the screening nature of the QED coupling from the one-loop effective action. Similarly, the Coleman-Weinberg effective potential is obtained, in a classically scale-invariant theory, by replacing the classical quartic scalar coupling by its one-loop counterpart, evaluated at a renormalization scale $k \sim \phi$. This is justified, insofar as the classical VEV of the scalar is the only scale in the problem.

¹¹In this approximation, only fluctuations of the conformal factor are taken into account. Quite surprisingly, this appears to suffice to generate an asymptotically safe fixed point in simple truncations [189], contrary to the expectation that the important degrees of freedom in gravity are the spin-2 ones.

Similar considerations have also been applied to non-Abelian gauge theories [377–379].

Coming closer to gravity, a recent example in curved space where RG improvement works, is the case of interacting conformally coupled fields in de Sitter spacetime. A correlator evaluated at the fixed point can be related to a CFT correlator in flat space by a Weyl transformation. Then, the late time power-like behavior of correlators can be obtained as a resummation of secular terms controlled by the anomalous dimensions in flat space, with an RG improvement at the renormalization scale $\mu = H$ [380], where the Hubble scale H of the de Sitter background is the only non-trivial scale in the problem.

Even more relevant for us, the running of G and the quantum corrections to the Newtonian potential due to a scalar field loop have been compared in [381]. They find that in general the RG improvement gives the expected qualitative behavior, and also reproduces the correct numerical coefficients for minimal coupling ($\xi = 0$) or conformal coupling ($\xi = 1/6$).

The reason why all these examples work (at the quantitative level) is the logarithmic running of the coupling. It is particularly instructive to compare the Uehling potential with the analogous calculation in gravity. In the calculation of [381], the running of G is logarithmic and proportional to the mass of the scalar field. This gives a result that is in agreement with the quantum correction to the potential. On the other hand, if one extracts the (quadratic) running of G from the FRG, and tries to derive the analog of the Uehling potential from there, one gets a term with the opposite sign of the quantum correction calculated in EFT [331]. This is a clear failure of the RG improvement: the EFT calculation gives a screening contribution, whereas the FRG seems to give an antiscreening one, as required by asymptotic safety. The situation has been clarified in part in [119]: due to the use of the background field method, there are different ways of defining Newton's coupling that have different types of behavior at low energy (where the EFT result holds) and at high energy, where one is assumed to approach a fixed point. However, this leads us back to the issue of the shift Ward identities, cf. section 3.5 that, as discussed earlier, does not currently have a satisfactory solution.

In conclusion, physical quantum effects in an asymptotically safe theory have to be calculated, as in any other QFT, from the effective action, where all fluctuations have been integrated out. We stress that the results one obtains from the RG improvement, e.g., for black holes or the early universe, cannot be viewed as actual derivations from a fundamental theory of quantum gravity, but should still be viewed as “quantum-gravity-inspired models,” providing qualitatively sensible, though not necessarily precise, answers in some cases where there is a clearly identifiable single scale in the problem.

8. SCALE SYMMETRY AND CONFORMAL SYMMETRY

8.1. The RG as Scale Anomaly

...where we clarify the meaning of scale symmetry in the context of asymptotic safety.

A point that tends to generate confusion concerns the interpretation of the RG flow as an anomalous breaking of scale invariance. It may seem puzzling that the asymptotic safety program claims (quantum) scale invariance even though Γ_k contains dimensionful couplings. The goal of this section is to clarify this point. We follow [382], see also [284], section 6.9¹².

Consider a perturbatively renormalizable QFT, with an interaction term $u\mathcal{O} \equiv u \int d^4x \mathcal{L}$, where $\mathcal{L}$ is a dimension-four operator and u a dimensionless coupling. If there is no mass term, the theory is scale invariant under the standard realization of scale transformations which act on the fields but not on the couplings. In the quantum theory, however, scale invariance is broken by the beta function

$$\delta_\epsilon \Gamma = -\mathcal{A}(\epsilon) \sim -\epsilon \beta_u \mathcal{O}. \quad (20)$$

Here ϵ is the infinitesimal parameter generating the transformation, $\delta_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}$, etc. and $\mathcal{A}(\epsilon)$ is the trace anomaly which can be formally seen as due to non-invariance of the functional integration measure. At a fixed point $\beta_u = 0$ and scale invariance is recovered.

Equation (20) can be generalized to the Wilsonian RG. In this case there is an additional term coming from the presence of an explicit momentum cutoff which is given by the “beta functional” defined in Equation (5):

$$\delta_\epsilon \Gamma_k = -\mathcal{A}(\epsilon) + \epsilon k \partial_k \Gamma_k. \quad (21)$$

For the effective average action given in Equation (1) one finds that the anomaly is given by [382]

$$\mathcal{A} \sim \epsilon \sum_i \beta_{u_i} k^{d_i} \mathcal{O}_i, \quad (22)$$

where d_i is the canonical mass dimension of $\mathcal{O}_i$. Again $\mathcal{A}$ vanishes at a fixed point. Nevertheless, the standard realization of scale invariance, acting on fields only, is broken due to the extra term in Equation (21)

$$\delta_\epsilon \Gamma_k \sim \epsilon \sum_i d_i \bar{u}_i \mathcal{O}_i. \quad (23)$$

There is however an alternative realization of scale invariance acting on both the fields and the cutoff. Here the transformation of the fields remaining unaltered $\hat{\delta}_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}$, etc. while the cutoff transforms as $\hat{\delta}_\epsilon k = -\epsilon k$. Under this alternative realization,

$$\hat{\delta}_\epsilon \Gamma_k \sim -\mathcal{A}(\epsilon), \quad (24)$$

which vanishes at a fixed point.

In conclusion, we see that in a “Wilsonian” formulation of the RG, quantum scale invariance is realized at a fixed point, albeit with respect to a different implementation of rescalings than the one generally used in particle physics.

¹²Here we discuss global scale invariance. It has been shown that local scale invariance can be maintained in the RG flow provided a dilaton field is present [383–385].

8.2. Black Hole Entropy

...where we discuss an argument against a QFT for gravity based on black-hole entropy and point out where assumptions are being made which require further investigation.

Aharony and Banks [386] and Shomer [387] presented a chain of arguments indicating that a quantum-field theoretic description of gravity in four dimensions cannot be UV complete. In short, this chain proceeds along the following lines. First, it is assumed that, at high energies, the density of states in quantum gravity is dominated by black holes, which also goes by the name of “asymptotic darkness.” Black hole thermodynamics, building on quantum field theory on a curved background, implies that the leading term in the entropy S of the black hole is proportional to the area A of its horizon. For a d -dimensional Schwarzschild black hole

$$S \propto A \propto M^{\frac{d-2}{d-3}} \quad (25)$$

where M is the ADM mass of the black hole. Identifying M with a typical energy scale E , the asymptotic darkness hypothesis then suggests that the number of states available at high energy should scale as

$$S^{\text{BH}} \propto E^{\frac{d-2}{d-3}} \quad (26)$$

In four dimensions this implies that $S^{\text{BH}} \propto E^2$. On the other hand, the degrees of freedom of a conformal field theory (CFT) living on a d -dimensional Minkowski space follow the scaling law

$$S^{\text{CFT}} \propto E^{\frac{d-1}{d}} \quad (27)$$

which in four dimensions becomes $S^{\text{CFT}} \propto E^{\frac{3}{4}}$. The mismatch between the density of available states Equations (26) and (27) is then taken as an indication that the high-energy completion of four-dimensional gravity cannot be given by a conformal field theory.

We now critically review the assumptions entering into this chain of arguments:

(1) Scales involved in the problem:

Seeing quantum-gravity effects in scattering events requires going to *large energies and small impact parameters* relative to the Planck scale. This is not the same as considering just trans-Planckian energies: the energy involved in the merger of two astrophysical black holes clearly exceeds the Planck mass $m_{\text{Pl}} \approx 10^{-5}g$ by many orders of magnitude. Nevertheless, classical general relativity provides a very accurate description of these events, for which the impact parameter is large compared to the Planck length.

(2) The asymptotic darkness hypothesis:

The idea of asymptotic darkness relies on the hoop conjecture [388] which states that scattering at sufficiently high energy results in black-hole formation. While numerical simulations confirm this expectation in classical gravity [389, 390], a corresponding study in the quantum case is lacking, see also the discussion in [391]. When phrased in terms of the effective action (Equation 17), it is expected that the form-factors

$W(\Delta)$ (or, more generally, the 1PI vertices) will play a central role in correctly describing scattering processes at trans-Planckian scales. Currently, little is known about these effects though, and it is an open question whether or not Planckian scattering in asymptotically safe gravity does or does not lead to black-hole formation. In [392], it has been proposed that black-hole formation in Planckian scattering is a key property of gravity that allows the theory to self-unitarize (classicalisation). Whether this has anything to do with asymptotic safety is an open question. See [393, 394] for related discussions in the context of non-linear sigma models.

(3) Corrections to the entropy formula:

The semi-classical area law (Equation 25) is a good approximation for large black holes. It receives further corrections from quantum gravity though. Logarithmic corrections were determined in [395], indicating that

$$S = \frac{A}{4G} - \frac{3}{2} \log \left(\frac{A}{4G} \right) + \dots \quad (28)$$

Clearly, these corrections become increasingly important for small (i.e., near-Planckian) black holes, see, e.g., [395–398]. Thus, it is a priori unclear if the simple scaling law (Equation 25) is applicable in the quantum gravity regime.

(4) Dimensional reduction of the momentum space:

A critical point in extending scaling arguments to quantum gravity is the identification of the correct notion of dimensionality which actually controls the scaling laws. While in flat Minkowski space there is just the dimension of spacetime d , fluctuating spacetimes are typically characterized by a whole set of “generalized dimensions” (spectral dimension, Hausdorff dimension, etc.) which do not necessarily agree. In particular, a rather universal result about quantum gravity [399, 400] indicates that the dimension of the theory’s momentum space (spectral dimension) undergoes a dimensional reduction to $d_s = 2$ at energies above the Planck scale. In [401], it was argued that such a mechanism could constitute a potential way to reconcile the semi-classical scaling in gravity with the scaling of states in the conformal field theory. In order to make such proposals robust, it is important to identify the proper notion of dimensionality which controls the scaling of the quantity of interest. In the context of black hole thermodynamics, it has been suggested that this could be achieved with the “Unruh dimension” [402] governing the scaling laws in the black-hole evaporation process.

(5) Entropy of asymptotically safe black holes:

The entropy of black holes in asymptotic safety has been investigated in [342, 345, 346, 351] based on RG improvement techniques (the cautionary remarks regarding RG improvement from section 7.4 apply in this case). One outcome of this investigation was that the entropy of Planck-size black holes follows the Cardy-Verlinde formula [351] indicating compatibility with a conformal field theory description. Concerning macroscopic black holes, the semi-classical result for the black-hole entropy can presumably be understood entirely in terms of the entanglement entropy of matter fields living on the black-hole background

geometry [403], see [404] for a comprehensive review, and [405, 406] for discussions in the context of the FRG and asymptotic safety.

In conclusion, combining semi-classical arguments based on the asymptotic darkness hypothesis and conformal field theory in flat space gives rise to results in tension with the asymptotic-safety conjecture. It is clear that much more work is needed in order to actually show that these arguments also apply in the framework of quantum gravity.

9. UNITARITY

9.1. General Remarks

...where we point out that the concept of unitarity in quantum gravity is way more subtle than for a quantum field theory on flat Minkowski space.

Conservation of probabilities is a cornerstone of quantum mechanics. For a QFT in a flat Lorentzian background, this feature is reflected by the S matrix, connecting the initial state and the final state of a physical system, being unitary. Starting from a QFT defined on a Euclidean signature spacetime the Osterwalder-Schrader axioms [407, 408], including the requirement of reflection positivity, guarantee that the theory has an analytic continuation to a unitary QFT.

Notably, it is highly non-trivial to generalize the concept of a unitary S matrix to more general backgrounds [409] or to the gravitational interactions [410, 411]. Examples for such generalizations are the local S matrix in de Sitter space studied in [412] or the one recently constructed in [413].

Along a different line, the existence of unphysical modes such as tachyons, negative norm states, etc., in a given background $\bar{g}_{\mu\nu}$ does not automatically signal the inconsistency of the theory. It may just indicate the instability of this particular background¹³. As an example, [340] highlights how a non-standard background removes the conformal-mode problem in the Euclidean path-integral. From a phenomenological point of view, a minimal requirement is to impose that cosmologically relevant backgrounds of Friedman-Lemaître-Robertson-Walker-type are stable on cosmic time-scales.

An important indicator that asymptotically safe gravity could indeed be unitary comes from the causal dynamical triangulations (CDT) program. Here one finds that the (two-step) transfer matrix connecting spacial slices at different time-steps is self-adjoint and bounded [414, 415], indicating that it satisfies the requirement of reflection positivity. Since the analytic continuation to Lorentzian signature is well-defined in CDT, the resulting Hamiltonian in the Lorentzian setting is self-adjoint. Under the preconditions that this feature survives in the continuum limit and that CDT indeed probes the Reuter fixed point, this indeed points toward Asymptotic Safety being a unitary theory.

¹³This is a well-known situation, for instance, in scalar theories, where an expansion about a saddle point of the potential leads to tachyonic instabilities, but does of course not signal an inconsistency of the theory. For instance, in inflationary scenarios these instabilities are key to the resulting physics.

These limitations should be kept in mind when discussing unitarity in a background-independent, quantum gravitational setting.

9.2. Flat-Space Propagators

...where we review Ostrogradsky's theorem and its loopholes.

With the above cautionary remarks in mind, let us discuss the gravitational propagator on a flat background. In the presence of a finite number of higher-derivative terms, a partial-fraction decomposition of the propagator reveals the presence of additional modes. For example, a propagator derived from a four-derivative theory yields

$$\frac{1}{p^2(p^2 + m^2)} = \frac{1}{m^2} \left(\frac{1}{p^2} - \frac{1}{p^2 + m^2} \right). \quad (29)$$

The terms in the partial-fraction decomposition come with alternating signs with the modes associated to the negative residues corresponding to ghosts. In the case of physical fields related to asymptotic states, this violates reflection positivity [416]. The latter signals the violation of unitarity in the Lorentzian theory and is related to a spectral function with negative parts. The violation of unitarity is already present at the classical level where it corresponds to an instability of the theory according to Ostrogradsky's theorem. Any non-degenerate Hamiltonian with higher time derivatives of finite order unavoidably features such an instability, see, e.g. the pedagogical discussion in [417]. This directly implies that truncations to finite order in momenta generically feature truncation-induced instabilities and are not suitable to investigate the unitarity of the theory.

There are three prominent ways to avoid the Ostrogradsky instability:

- 1) Propagators consisting of an entire function with a single zero at vanishing momentum may avoid the occurrence of negative residues. This is the path taken by non-local ghost-free gravity [418–421]. At the classical level, the well-posedness of the corresponding initial-value problem has been discussed in [422].
- 2) One can give up Lorentz invariance, introducing higher-order spatial derivative terms while keeping two time-derivatives. This is the idea underlying Hořava-Lifshitz gravity [230] which, by construction, is a power-counting renormalizable, unitary theory of gravity.
- 3) Accept that Nature allows for the violation of causality at microscopic levels [423–426]. In this case, the degrees of freedom associated with negative residues are interpreted as “particles propagating backward in time.” If these particles are sufficiently heavy this may not leave an experimentally detectable trace.

We stress that in any case unitarity should be assessed based on the propagators derived from the effective action $\Gamma_{k=0}$. Propagators derived from the effective average action Γ_k at intermediate k may feature artificial poles. Under the flow in

k , the mass of a ghost might diverge so that the corresponding degrees of freedom decouple, see [427].

9.3. Spectral Function of the Graviton

...where we discuss the consequences of potential negative spectral weights of the graviton.

The ghost mode discussed in the last section 9.2 is but one example for a spectral function that has negative spectral weights: Evidently, the second term in parenthesis in Equation (29) leads to a δ -function with negative prefactor in the spectral function of the graviton. In asymptotically safe gravity the graviton propagator is a general function of momentum. Consequently the spectral function more generally may simply have negative parts.

To begin with, negative spectral weights are a well-known feature of the gluon spectral function in Yang-Mills theory: upon the assumption of a spectral representation of the gluon, it can be shown that its total spectral sum is vanishing due to the Oehme-Zimmermann superconvergence relation [428, 429]. This relation already implies that in the asymptotically free regime of Yang-Mills theory, the spectral function of the gluon is negative for large spectral values. Indeed, the analytic form for large spectral values can be computed within perturbation theory. More recently it could also be shown by similar arguments that the spectral function is also negative for small spectral values [430].

These investigations highlight the fact that even the existence of spectral representations for gauge fields with non-linear gauge symmetries is an open issue. This is tied to the fact that these fields are not directly related to asymptotic states even in regimes where they heuristically can be interpreted as particles. In QCD this is manifest in gluon jets at colliders. In the context of gravity, this feature is intrinsic to the proposal made in point 3) of the previous subsection: owing to their large mass, the states associated with the negative residue terms do not correspond to asymptotic states, see [431] for a recent discussion.

In summary, even if the spectral representation of a gauge fields exists, it very well can—and in the case of the gluon must—contain negative parts. Evidently, this adds significantly to the already discussed intricacies of discussing unitarity and the interpretation of positivity violations in quantum gravity: negative spectral weights may be present without spoiling unitarity but clearly their presence casts doubts on unitarity. This situation asks for the investigation of the spectral representation of correlations of well-defined diffeomorphism-invariant observables, see section 6.

9.4. Interpretation of Potential Ghost Modes

...where we refer back to the concept of effective asymptotic safety discussed in section 7.2 and discuss the interpretation of the masses of unstable graviton modes in this context.

Future studies of unitarity may reveal that asymptotic safety features *physical* ghost modes and hence is not a unitary fundamental QFT. Even in this case, an asymptotically safe fixed point can still play a role within the setting described in section 6, and serve as an extension of the EFT regime for gravity. Then, the asymptotically safe description in this setting could inherit unitarity from the more “fundamental” description. In particular, the asymptotically safe setting can in this case exhibit unstable modes, with masses $m > k_{UV}$ —these signal the need for a more “fundamental” description. Conversely, the masses of ghost modes can be used as an estimate for the scale of new physics.

9.5. Remarks

In summary, it is currently unclear whether or not asymptotically safe gravity is unitary, it shares with other approaches to quantum gravity. The question combines both conceptual and technical challenges in quantum gravity: there is the conceptual question of the complex structure of correlation functions in the presence of a dynamical metric field, as well as the necessity of *non-perturbative* numerical computations in Lorentzian signature. As already emphasized in section 6, cross-checks between quantum-gravity approaches and the concerted use of more than one method are called for.

10. LORENTZIAN NATURE OF QUANTUM GRAVITY

...where we highlight the expected fundamental difference between Lorentzian and Euclidean quantum gravity and explain why the flow equation is typically set up in a Euclidean setting.

Hitherto, the bulk of the Asymptotic Safety literature employs background spacetimes carrying Euclidean signature metrics. This brings two technical advantages: Firstly, Euclidean signature entails that the squared momentum of the fluctuation fields is positive semi-definite. Thus it is straightforward to define the “direction of the RG flow,” first integrating out fluctuations with a large squared momentum before successively moving toward lower values. Secondly, the regulated propagators do not exhibit poles, as the particle cannot go on shell.

For a QFT defined on flat Euclidean space $\mathbb{R}^d$, one can carry out the computation and analytically continue the results to Lorentzian signature by a Wick rotation. In the context of quantum gravity, including Asymptotic Safety, this strategy is very challenging for several reasons listed below, part of which has been already discussed in detail in section 9.

1. A generic background metric may not admit a (global) Killing vector which lends itself to an analytic continuation to a well-defined Lorentzian time direction [432].
2. The complex structure of the full graviton propagators may obstruct the simple analytic continuation of the Euclidean propagator, for example there may very well be cuts touching the Euclidean axis. Within the FRG this is complicated further as a momentum regularization either breaks the underlying (global) spacetime symmetry or leads to additional poles

and/or cuts [433]. There has been much progress in the past years on this in standard QFTs, see e.g., [433–438], but the extension to asymptotically safe gravity has not been put forward yet.

3. At the structural level, there are solid arguments to expect that the effective actions obtained from integrating out fluctuations in a Lorentzian and Euclidean signature setting will be different. Firstly, the space of metrics of the two settings comes with different topological properties: while all Euclidean metrics can be connected by geodesics (defined with respect to a suitable connection) this property does not hold in the Lorentzian case [439]. Secondly, the heat kernels for differential operators constructed from a Euclidean and Lorentzian signature metric differ by non-local terms [440]. While the latter do not affect the singularity structure of the heat kernel underlying perturbative renormalization, they may lead to differences in Γ .

A way to address the first point comes from studying Asymptotic Safety in the Arnowitt-Deser-Misner (ADM) formalism. In this case, spacetime has a built-in foliation structure which defines a natural time direction. A first investigation of Asymptotic Safety in this framework has been performed in [441] and further developed in a series of works [226, 442–448]. This provided first-hand indications that the asymptotic-safety mechanism remains operative for Lorentzian signature metrics as well, at least within very small truncations. At this stage the computations in the Lorentzian signature framework have not reached a level of sophistication where the structural differences outlined in point (3) can be resolved. In general, the systematic development of the FRG applicable to Lorentzian signature spacetimes is a research area to be developed in the future.

This point could in the future become another example for the progress that can become possible if tools and concepts from various quantum-gravity approaches are brought together.

Specifically, causal set theory [see [449] for a review], at least when restricted to so-called “sprinklings,” can be viewed as a discretization of the Lorentzian path integral over geometries. See also [450]. This motivates the search for a universal continuum limit, linked to a second-order phase transition in the phase diagram for causal sets. Monte Carlo studies of the phase diagram for restricted configuration spaces in low dimensionalities can be found in [270–273].

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Coarse Graining Spin Foam Quantum Gravity—A Review

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In quantum gravity, we envision renormalization as the key tool for bridging the gap between microscopic models and observable scales. For spin foam quantum gravity, which is defined on a discretization akin to lattice gauge theories, the goal is to derive an effective theory on a coarser discretization from the dynamics on the finer one, coarse graining the system in the process and thus relating physics at different scales. In this review I will discuss the motivation for studying renormalization in spin foam quantum gravity, e.g., to restore diffeomorphism symmetry, and explain how to define renormalization in a background independent setting by formulating it in terms of boundary data. I will motivate the importance of the boundary data by studying coarse graining of a concrete example and extending this to the spin foam setting. This will naturally lead me to the methods currently used for renormalizing spin foam quantum gravity, such as tensor network renormalization, and a discussion of recent results. I will conclude with an overview of future prospects and research directions.

Keywords: quantum gravity, spin foam models, renormalization, coarse graining, numerical methods

1. A BRIEF INTRODUCTION TO SPIN FOAM QUANTUM GRAVITY

Spin foam quantum gravity [1, 2] is a promising approach to quantum gravity closely related to loop quantum gravity [3]. The aim is to define the path integral for gravity in a non-perturbative and background independent fashion, that is without any reference to a fixed background space-time or structure.

The starting point of spin foam models is the Plebanski-Holst formulation of general relativity [4], in which gravity is formulated as constrained topological BF theory [5]. To formulate this theory as a path integral, one introduces a lattice as a regulator, more precisely a 2-complex, in order to truncate the number of degrees of freedom. On this 2-complex, which is a collection of vertices, edges and faces, the topological theory is first discretized and quantized. This is in close analogy to 3D (topological) gravity, where this formulation gives rise to the Ponzano-Regge model [6–8], a well-defined model of 3D quantum gravity defined on a triangulation.

However, gravity in 4D is not topological, which requires the implementation of so-called simplicity constraints. In the continuum they serve the role to break the too many symmetries of the theory and reduce the B -field in BF theory to a simple 2-form, reducing the action to the familiar Holst action [9]. In spin foam quantum gravity, one derives such constraints for the discretization of the classical B -field, so-called bivectors. In 4D, bivectors are assigned to 2D faces, e.g., triangles, and encode their geometry. The constraints ensure that these bivectors are simple, i.e., they can always be written as a wedge product of two vectors. Geometrically these vectors span two edges of a triangle. Different versions of these discrete constraints agree for single, classical building blocks, e.g., a 4-simplex, such that they correspond to different discretizations.

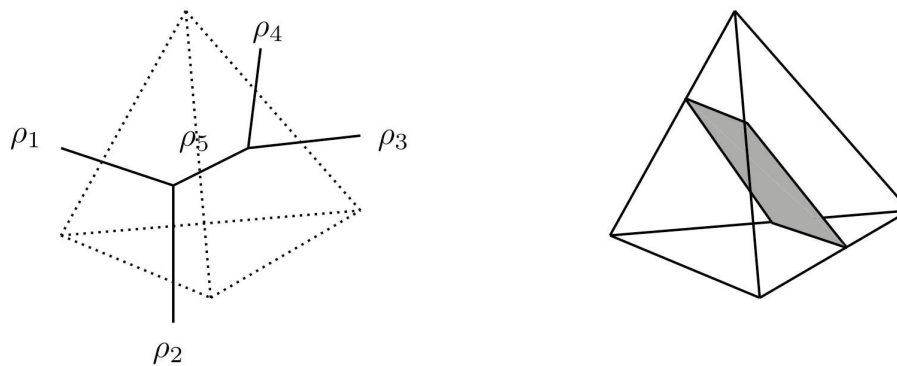


FIGURE 1 | (Left) A 4-valent intertwiner, dual to a tetrahedron, expanded in an orthonormal basis. The shape of a tetrahedron is not determined by the areas of its four triangles. **(Right)** The intermediate representation gives the area of a parallelogram in the center of the tetrahedron. Its corners are located on the center points of the edges of the tetrahedron according to the split of the intertwiner.

However, their implementation in the quantum theory, which leads to restrictions on the variables of the theory, generically results in different models with starkly different dynamics. Two examples are the Barrett-Crane (BC) model [10, 11] and the Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK) model [12–14]. The former strongly implements a condition on bivectors, which significantly reduces the degrees of freedom of the model. This was criticized [15, 16] and motivated the development of the EPRL/FK model, in which constraints are implemented weakly, i.e., at the level of expectation values. Despite these insights, it remains an open question whether these constraints are sufficient to recover general relativity in a continuum limit. The hope is that coarse graining/renormalization can shed a light on this intriguing question.

Despite these differences, all spin foam models are written in a similar form. The 2-complex, which is frequently dual to a triangulation, is colored with group theoretic data: each face f carries an irreducible representation ρ_f of the underlying symmetry group [Spin(4) for Riemannian, $SL(2, \mathbb{C})$ for Lorentzian signature], while each edge e carries an intertwiner ι_e , an invariant tensor in the tensor product of representation spaces associated to faces meeting at an edge. These data encode the geometry of the spin foam: in 4D, each edge is dual to a 3D polyhedron, which has as many faces as (dual) faces that share this edge. Then, the areas of these faces are given by the associated representations ρ_f . However, this does not determine the shape of the polyhedron uniquely, see e.g., a tetrahedron. A flat tetrahedron is uniquely determined by specifying its six edge lengths, whereas it only has four faces. Thus, the areas of these faces alone do not fix the shape of the tetrahedron. Part of the information on the shape is stored in the intertwiner, which can be expanded into an orthonormal basis using group representation theory. For a tetrahedron, its dual 4-valent intertwiner can be split into two 3-valent ones, where the new link carries again a group representation labeling the basis element. Geometrically this representation gives the area of a parallelogram spanned by the midpoints of the edges of the tetrahedron [5, 17], see also **Figure 1**. However, due to

the uncertainty principle the shape cannot be fully specified since area operators associated with intersecting faces do not commute and thus cannot be diagonalized simultaneously. Note that coherent intertwiners can be defined that are sharply peaked on the geometry of a classical polyhedron¹.

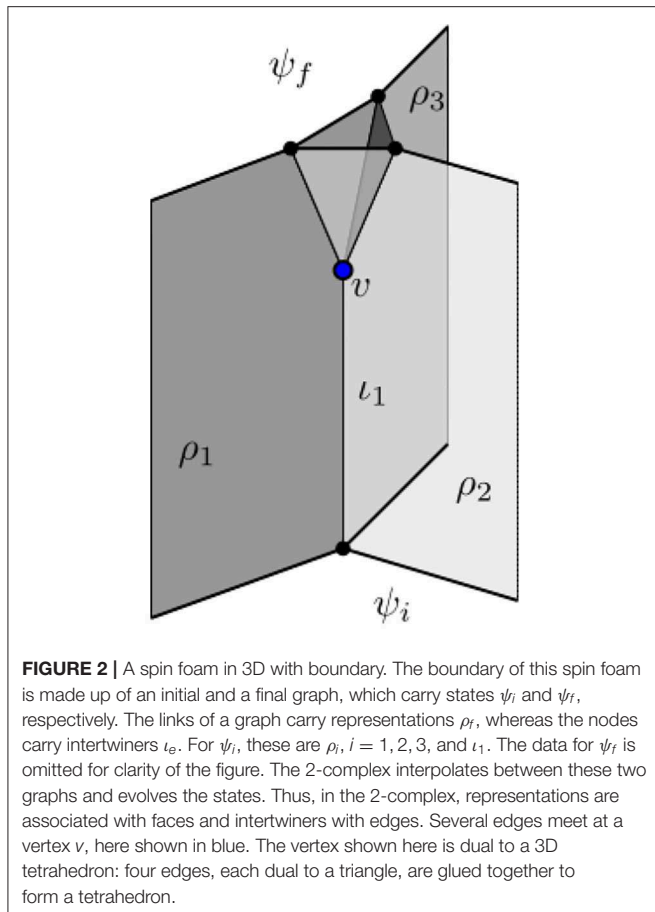
From these 3D polyhedra a 4D geometry is built at the vertices of a 2-complex. At such a vertex, several edges and faces meet, indicating how 3D polyhedra are glued together to form a 4D geometry. If two edges meet at the same vertex and are part of the same face, their dual 3D polyhedra are glued along the shared face. Crucially, since the representation associated to the face determines its area, it is ensured that the face has the same area in both polyhedra. From the group theoretic data, this “gluing” is performed by contracting the intertwiners according to the combinatorics of the 2-complex, which essentially amounts to a spin network evaluation². The resulting number is known as the vertex amplitude $\mathcal{A}_v$, i.e., the amplitude of the spin foam model assigned to the discrete 4D geometry dual to the vertex v with configuration $\{\rho_f, \iota_e\}$. Similarly, we assign local amplitudes to the $\mathcal{A}_e$ and $\mathcal{A}_f$ to the edges e and faces f , respectively. The former ensures that intertwiners are normalized, while the latter corresponds to the dimension of the representation ρ_f . See Perez [1] for more details of the derivation. Eventually, the path integral is defined as a sum over all these configurations:

$$Z = \sum_{\{\rho_f, \iota_e\}} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v. \quad (1)$$

Crucially, these geometric building blocks and amplitudes are derived from general relativity formulated as a constrained

¹These coherent intertwiners, called Livine-Speziale intertwiners [18] (see also section 5.2), are given by a tensor product of coherent SU(2) states, which are sharply peaked on the outward pointing normals of the faces of the polyhedron. If these normals times their respective areas sum up to zero, these data uniquely define a convex polyhedron by Minkowski’s theorem [19, 20].

²This spin network can be obtained by drawing a 3-sphere around a vertex in the 2-complex. Edges, corresponding to intertwiners, intersect the sphere at nodes. Faces, intersect the sphere at links, connecting the nodes, determining how to contract the intertwiners.



topological field theory. In case the spin foam has a boundary (see **Figure 2**), it serves as an amplitude functional mapping states from its boundary Hilbert space into the complex numbers. This concept will be crucial in this review.

At the level of a few simplices, these models are well-explored. A well-understood result across models, which furthermore underlines the relation to general relativity, is the asymptotic expansion of the vertex amplitude dual to a 4-simplex [21–25]. In these works the vertex amplitude is investigated for coherent intertwiners, which are sharply peaked on the geometry of classical polyhedra. Then the vertex amplitude can be written as an integral over several copies of the symmetry group. This integral is then evaluated via a stationary phase approximation by uniformly scaling up all representations. Hence it is commonly referred to as large- j limit, in which the amplitude is generically proportional to the cosine of the Regge action [26, 27], a discretization of classical gravity. Physically, this amplitude should be valid for 4-simplices of mesoscopic and even macroscopic size. In recent years, numerical calculations of the vertex amplitude beyond the asymptotic expansion have seen promising progress [28, 29].

Beyond a single building block, the dynamics is less explored, in particular how the choice of the 2-complex impacts the results of the theory. Indeed, a priori the theory itself does not

specify how the 2-complex should be chosen. In this review we take a viewpoint that is akin to lattice gauge theory, and regard the 2-complex as a regulator, a particular choice to truncate the number of degrees of freedom of the theory. As such, physics must not depend on this choice and it must be removed eventually, e.g., in a continuum limit, in order to derive consistent results. One route toward such a limit lies in coarse graining: By coarse graining, i.e., defining an effective coarse amplitude from a collection of fine ones, we readily relate two theories defined on two different regulators. Moreover, by coarse graining we gain insight into the dynamics of a collection of building blocks and learn which configurations are more relevant on a coarser scale. In short, the aim is to derive a family of amplitudes to assign to different regulators, which reproduce the same physics (at least approximately). This defines a renormalization group flow of amplitudes [30, 31]. It is the purpose of this article to review the progress of this approach and outline how it helps turning spin foam quantum gravity into a computational formalism.

This review is structured as follows: in section 2 we start by outlining the most pressing challenges faced by spin foam quantum gravity and how these are addressed by coarse graining. Section 3 discusses the issue of restoring diffeomorphism invariance in the discrete as well as the typical appearance of non-local interactions under coarse graining, which is one motivation for the consistent boundary formulation outlined in section 4. Section 5 reviews two numerical methods to perform such coarse graining algorithms, namely tensor network renormalization and restricted path integral models. In section 6 we conclude with several interesting future research directions.

2. KEY CHALLENGES IN SPIN FOAM QUANTUM GRAVITY

Before explaining renormalization in spin foam models and its progress over the last decade, it is crucial to first discuss the key challenges spin foam quantum gravity is facing and how renormalization plays a vital role in overcoming them.

2.1. Fate of Diffeomorphism Symmetry

Diffeomorphism symmetry, as the fundamental symmetry of general relativity, is deeply intertwined with the dynamics of gravity. It implies that physics must not depend on the choice of coordinates and only diffeomorphism invariant (Dirac) observables are physically meaningful [32, 33]. Moreover, this symmetry forbids a choice of a preferred or fixed background space-time. Conversely, the complexity of this symmetry is a root of the difficulty for defining a theory of quantum gravity; spin foam models are no exception.

While spin foam quantum gravity embraces the concept of background independence, the introduced regulator, frequently a 2-complex dual to a triangulation, generically breaks (a discrete remnant of) diffeomorphism symmetry [34–36], often called a vertex translation symmetry [37]. There exist instances, where this symmetry is preserved

in the discrete, where the discretization perfectly reflects the continuum dynamics, or the symmetry can be restored iteratively via coarse graining. We explain this in detail in section 3. For spin foams to be a viable theory of quantum gravity, diffeomorphism symmetry must be restored, at least approximately, in order to derive reliable physical predictions. There exists strong evidence that the amplitudes of the system can be systematically improved via coarse graining [38–40], such that the symmetry is broken less. Naturally the question arises whether this procedure converges to a fixed point, which would automatically imply an independence of the chosen regulator. Due to the non-local nature of diffeomorphism symmetry and in order to find a theory with propagating degrees of freedom, we conjecture such a fixed point to lie on a phase transition of second order. There it would be possible to take the continuum (or rather refinement limit) of the theory.

2.2. Discretization (In)dependence

Closely related to diffeomorphism symmetry is the question of discretization (in)dependence. Generically the results computed in spin foam models will depend sensitively on the chosen regulator, e.g., the number of simplices and subsequently the number of degrees of freedom. Moreover, there is no input from the theory itself which regulator to choose. However, in order to have a viable theory, it is imperative to find the same results no matter which discretization is chosen, at least to an approximation.

In the research community, there exist two complementary paths addressing this question [41]. On the one hand, there is the approach to solve discretization dependence by summing over all possible regulators, e.g., triangulations. This summation over triangulations (and topologies) is most holistically formulated in terms of group field theories [42, 43], which are quantum field theories formulated on several copies of a Lie group. The fields represent atoms of space-time, e.g., tetrahedra, whose interaction terms describe how 4D objects are formed, e.g., five tetrahedra glued together to form a 4-simplex. From this formalism, spin foam amplitudes arise as Feynman diagrams in a perturbative expansion. As for all quantum field theories, it must be shown that this theory is renormalizable, e.g., via perturbative or non-perturbative methods, see [44] for a review.

On the other hand, we discuss the refinement approach [30, 31] in this review, where we interpret the triangulation as a regulator to truncate the number of degrees of freedom, similar to the lattice in lattice field theories. The idea to overcome discretization dependence is by assigning different amplitudes to different discretizations in such a way that the results agree. One example is to derive coarse amplitudes from fine ones via coarse graining. In this way, we are relating theories across different discretizations. The goal is to derive such relations for all possible discretizations, which is equivalent to a complete renormalization group trajectory. Again, this is similar to lattice field theory, where one also assigns different theories to different lattices, parameterized by coupling constants.

2.3. Computability

The choice of a discretization (and thus appropriate) amplitude also enters, at least partially, in another key challenge for spin foams, namely their computability. To be more precise, by computability we refer to two interconnected issues. On the one hand, there is the challenge to compute the fundamental spin foam amplitudes for a single building block, e.g., a 4-simplex. While this is well-studied and explored in the semi-classical regime [21–25], in particular using coherent states and stationary phase approximation, computing a vertex amplitude in the quantum regime, e.g., for small spins, can only be done numerically. However, in recent years there has been significant progress in computing these amplitudes, e.g., for the EPRL/FK in Euclidean and the more challenging Lorentzian signature [28, 29, 45].

Renormalization and coarse graining become important at the stages when we calculate amplitudes or observables for multiple vertices/larger triangulations. Even if we have an efficient way of calculating spin foam amplitudes (or can access the relevant amplitudes from a database), summing over the various degrees of freedom remains a difficult task for such a high dimensional configuration space³. However, if we assume that the full RG trajectory of the system is known, we can use the discretization to our advantage and perform the same calculation on a much coarser spin foam with appropriately adapted amplitudes. Alternatively and more realistically, one can envision coarse graining the system first, essentially evaluating it in parts, deriving an effective theory on a coarser regulator from a finer one. On this coarse theory, expectation values of coarse observables can efficiently be computed. This method is already realized nowadays in tensor network renormalization techniques [46–48], see e.g., [49]. Note that the existence of a continuum limit is not assumed, rather we assume that coarse graining can be performed without severe truncations.

2.4. Uniqueness, Phase Diagram, and Continuum Limit

Discretizing a continuum theory is generically not a unique process, take the 1D non-relativistic particle in a non-trivial potential as an example. There exist many choices how to discretize the potential, which all result in different dynamics. However, the expectation is that, no matter the choice, we reobtain the original continuum physics in a suitable continuum limit (or approximate it well in a fine discretization). This is even more severe in the case where the continuum theory possesses a symmetry, like reparametrization or diffeomorphism invariance, which is broken in general in the discrete [37, 38].

These topics, uniqueness of the theory, universality and the continuum limit remain open questions in spin foam quantum gravity. Modern spin foam models are frequently derived by starting from topological BF theory and then imposing simplicity constraints in the discrete [1]. The latter procedure is not unique, where, e.g., the well-developed EPRL/FK model imposes the linear simplicity constraints weakly [13, 14]. Some effects on

³Monte Carlo methods are only of limited use, since spin foam models are proper quantum amplitudes, i.e., complex-valued and highly oscillatory.

different choices of (implementations of) simplicity constraints can be found in the literature [15], however a phase diagram differentiating different universal dynamics is missing, and with it potential hints for a continuum limit and UV-completion of the theory.

These key challenges are deeply intertwined with one another and can be addressed by a coarse graining/renormalization scheme. In the following we review how our understanding of these connections developed over time and what the role of coarse graining is.

3. RESTORING DIFFEOMORPHISM SYMMETRY IN THE DISCRETE

Regge calculus [26] is a discretization of general relativity. In it the differentiable manifold is replaced by a D -dimensional triangulation, whose edge lengths are the dynamical degrees of freedom. Crucially, Regge calculus does not refer to coordinates of vertices of the triangulation and is solely formulated in terms of their distances. Hence it is manifestly coordinate free. Each of the D -simplices is internally flat, i.e., its $D + 1$ vertices can be embedded into $\mathbb{R}^D$. Curvature is distributional and located on $(D - 2)$ -subsimplices, so-called hinges. To each of these hinges in the bulk one associates a deficit angle ϵ_h , which is the difference between the sum all dihedral angles of simplices meeting at this hinge and 2π . This is nicely visualized in $d = 2$: Several triangles meet at a single vertex. If their angles located at this vertex sum up to 2π , it is flat and can be drawn on a piece of paper. However, if the deficit angle differs from 0, e.g., if $\epsilon_h > 0$, we can no longer embed this collection of triangles into $\mathbb{R}^2$ and observe positive curvature around that vertex. Note that the edge lengths are the only dynamical variables, as the dihedral angles are given as functions of the edge lengths.

In addition to making no reference to coordinates, in some instances Regge calculus possesses additional symmetries in the discrete linked to diffeomorphism invariance [50]. One such example is 3D Regge calculus for $\Lambda = 0$: its equations of motion state that all deficit angles $\epsilon_e = 0$ in the bulk, for all boundary data, describing a theory that glues piecewise flat tetrahedra in a flat way. Thus, it perfectly matches the continuum solution. Moreover, the Regge action is invariant under vertex translations, i.e., moving a vertex and accordingly changing the edge lengths it is connected to. One such example is the 4-1 Pachner move: If we place an additional vertex in the center of a tetrahedron, we can freely choose three edge lengths connecting it to the vertices of the coarse tetrahedron. The fourth is then fixed uniquely by the equations of motion. This symmetry is reflected by nulleigenvalues of the matrix of second derivatives of the action.

Moreover, the 3D Regge action itself is invariant under such Pachner moves, i.e., local changes of the triangulation. See **Figure 3** for one such Pachner move in 3D. This renders it triangulation independent, since any triangulation of a manifold can be related to any other triangulation of the same manifold by a consecutive application of Pachner moves [51, 52]. This is not surprising, since 3D gravity is topological, i.e., has no local degrees of freedom. Nevertheless, we are convinced that

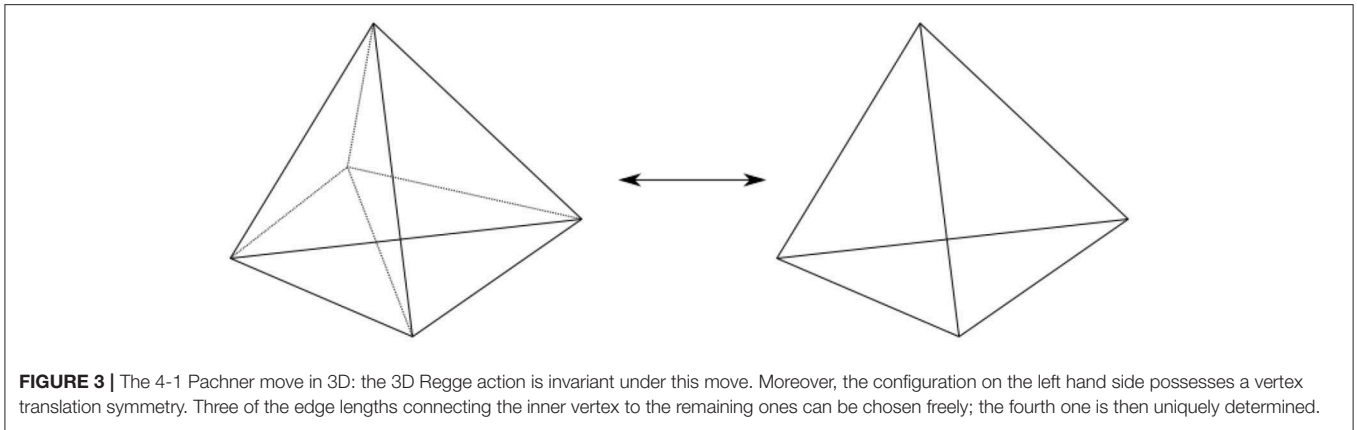
triangulation independence and diffeomorphism symmetry in the discrete, in the form of a vertex translation symmetry, are closely related also beyond topological theories. Diffeomorphism symmetry is deeply entangled with the dynamics of general relativity. When perfectly realized in a discrete system, by fully capturing the continuum dynamics, it is irrelevant whether we consider a coarse or a fine discretization. Thus, the theory is discretization independent. Invoking the invariance under vertex displacements, we can imagine this by moving vertices on top of each other, effectively removing them. Conversely, achieving discretization independence by finding a fixed point of a coarse graining flow, e.g., on a second order phase transition, does not necessarily imply that diffeomorphism symmetry is restored, yet this conjecture is supported by several examples that we outline below.

A nice example how coarse graining can improve an action (or amplitudes in the quantum case) is again 3D Regge calculus with a non-vanishing cosmological constant. Due to the cosmological constant, the equations of motion state that deficit angles do not vanish. Moreover, the theory is not triangulation independent and the vertex translation symmetry is broken. In [38], Bahr and Dittrich devise a coarse graining scheme for the triangulation: On a refined triangulation, subdividing large edges into small ones, they solve the equations of motion for the small edges and define an effective action for the remaining large ones. This procedure converges to a fixed point action, which describes Regge calculus for constantly curved tetrahedra. On this fixed point, deficit angles vanish, the theory is triangulation independent (by definition) and the vertex displacement symmetry is restored. Indeed, this improved discretization/action encodes the continuum solution in the discrete, thus implementing a discrete remnant of diffeomorphism symmetry. Moreover, since it correctly captures the continuum dynamics, no information or accuracy is lost when using coarse triangulations. An analogous quantum version is the Turaev-Viro spin foam model [53], defined as a quantum deformed Ponzano-Regge model [6].

There exist several instances where the continuum solution is pulled back to the discrete setting, where the discrete theory possesses a vertex translation symmetry. One example is 4D Regge calculus [50], when the boundary data allow for flat solutions in the bulk, or the 1D quantum parameterized (an)harmonic oscillator [37]. In general we cannot guess these solutions, but with coarse graining methods we can construct or at least approximate them well. However, the examples that we discuss here are either topological or one-dimensional, and hence it is possible to retain a local description. For higher dimensional, interacting theories, non-local interactions appear, which can be a stumbling stone for coarse graining methods.

3.1. Non-localities

Before explaining non-localities or non-local interactions, we first need to state what a local theory is in this context. In most discrete theories, we associate variables to parts of the discretization, e.g., in spin foam models we assign irreducible representations ρ_f to faces f of the dual complex and intertwiners ι_e to edges e . In Regge calculus, we assign edge lengths l_e to the edges e of a triangulation. We define this theory to be local if the



partition functions is given by a product of amplitudes assigned to (sub)simplices or if the action is given as a sum over actions assigned to (sub)simplices. Moreover, the action and amplitude for each (sub)simplex only depend on those variables attached to (sub)simplices contained in the (sub)simplex. Spin foam models are an example for such local theories, since the partition function is given via a local assignment of vertex, edge and face amplitudes, see Equation (1). Similarly, the Regge action can be written as a sum over contributions associated to the D -simplices of the triangulation.

When we apply coarse graining methods to such interacting, i.e., non-topological, theories, it is highly unlikely that this local form of the theory can be preserved. There exist several examples in the literature where this has been shown in the past. In [54], 4D Regge calculus was linearized around a flat background solution and the perturbations of the edge length integrated over. The question is whether it is possible to find a path integral measure that is invariant under Pachner moves. However, when integrating out these degrees of freedom, one picks up a non-local factor that cannot be written as a local product. In [55] it is shown that said factor is related to a condition whether the six vertices involved in the Pachner move lie on a 3-sphere. Moreover, these articles reveal that the 4D Regge action itself is not invariant under Pachner moves. In a similar vein, [56] studies Pachner moves in 4D holomorphic spin foam models [57]. The advantage of these models is that Pachner moves can be explicitly computed. Again, the resulting amplitude is non-local, in the sense that the resulting expression cannot be written as an assignment of local amplitudes.

To illustrate this point further let us consider the concrete example of the 2D Ising model.

3.1.1. Ising Model as an Example

There exist plenty of ways to coarse grain discrete systems. A straightforward example is the 2D Ising model subject to a simple decimation procedure, where one simply sums over “every other” spin to derive an effective model on a larger scale.

We consider the Ising model defined on a regular 2D lattice with vanishing external magnetic field. There are only nearest neighbor interactions, i.e., an Ising spin $\sigma_i \in \{-1, 1\}$ only interacts with its direct neighbors. Then we can write the

partition function as product of weights associated to the edges of the lattice:

$$Z_{\text{Ising}} = \sum_{\{\sigma_i\}} \prod_e \exp(\beta \sigma_{s(e)} \sigma_{t(e)}), \quad (2)$$

where β is the inverse temperature, and $s(e)/t(e)$ denote the source and target of the edge e^4 . Note that the system has a global $\mathbb{Z}_2$ symmetry, it remains invariant if all spins are flipped.

We implement a decimation procedure by summing over every other spin, essentially evaluating the partition function in parts. In order to derive the new effective amplitude of the system, it is sufficient to consider four Ising spins $\sigma_1, \dots, \sigma_4$ that all connect to another Ising spin $\tilde{\sigma}$, see Figure 4. The four coarse spins sit on the corner of a coarser square rotated by 45° with spin $\tilde{\sigma}$ in the center of the square. We obtain:

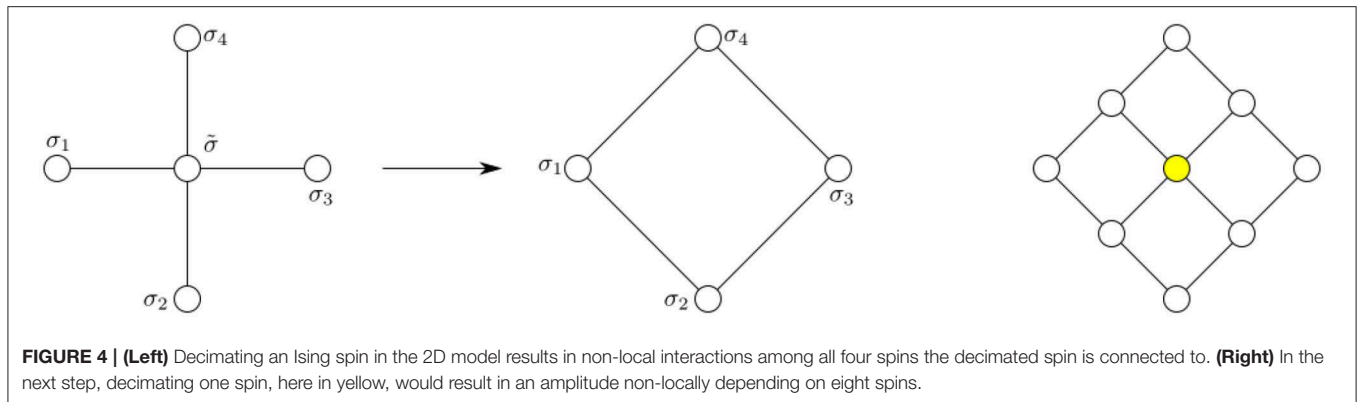
$$\sum_{\tilde{\sigma}} \exp(\beta \tilde{\sigma} (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)) = 2 \cosh(\beta (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)). \quad (3)$$

Clearly this expression is not of the same form as the original action, in particular is not written in terms of $\mathbb{Z}_2$ group multiplications. Nevertheless, the remaining spins still satisfy the global $\mathbb{Z}_2$ symmetry. Thanks to this global symmetry, this expression can only take three different values depending on the configurations of the four spins $\{\sigma_i\}$; either all spins are aligned, one is not aligned with the others or we have two pairs of aligned spins. To express this again in terms of spin interactions, we make the most general ansatz of four spin interactions compatible with the global $\mathbb{Z}_2$ symmetry:

$$\mathcal{A}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) := \exp(a(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_1\sigma_4) + b(\sigma_1\sigma_3 + \sigma_2\sigma_4) + c\sigma_1\sigma_2\sigma_3\sigma_4 + d). \quad (4)$$

a is the parameter for nearest neighbor interactions, b for next-to-nearest neighbor interactions, c for a four spin interaction and d is a constant. We can compare these equations directly for each

⁴The choice of orientation is fiducial, but allows for a short-hand notation.



configuration:

$$\begin{aligned}
 \exp(4a + 2b + c + d) &= 2 \cosh(4\beta) & \text{all } \sigma_i &= \pm 1 \\
 \exp(-c + d) &= 2 \cosh(2\beta) & \sigma_i &= \sigma_j = \sigma_k = -\sigma_l \\
 \exp(-2b + c + d) &= 2 & \sigma_i &= \sigma_j = -\sigma_k = -\sigma_l \\
 \exp(-4a + 2b + c + d) &= 2 & \sigma_i &= -\sigma_j = \sigma_k = -\sigma_l,
 \end{aligned} \tag{5}$$

where we denote a cyclic order i, j, k, l around the square. Here we have four equations for four unknown parameters, which we can straightforwardly solve. We leave deriving the solution to the interested reader.

The coarse grained amplitude is notably different than the initial one. While we find again nearest neighbor interactions, new non-local interactions appear as well. From this new form it is not obvious how to return to the original expression. Moreover, it is not clear how to iterate the procedure without approximations: decimating one spin alone results in non-local interactions among eight spins, some of which ought to be decimated as well, see again **Figure 4**. Nevertheless, already this simple example hints toward a resolution: the non-localities arise since we attempt to express the coarse grained dynamics in terms of the old degrees of freedom and building blocks. Yet we can still write the partition function as a local product of amplitudes associated to rotated squares, where the non-local interactions are completely contained within these locally assigned amplitudes. In the next section we introduce this change of perspective more concretely and discuss the concept of generalized boundary data⁵.

4. CHANGE OF CONCEPT: GENERALIZED BOUNDARIES AND AMPLITUDE MAPS

The vital insight to arrive at a practical coarse graining scheme for spin foams is the following: instead of pertaining the original degrees of freedom and building blocks, e.g., simplices, and

allowing more and more complicated, non-local interactions among them, we work with locally interacting amplitudes, which allow for more general and complex boundary data. The non-locality is still present, yet contained within the amplitudes and expressed as interactions of these boundary data. Thus, the complexity of the boundary data controls the non-locality preserved under coarse graining and the complexity of the amplitude. Truncating the boundary data allows us to introduce controllable approximations, while the partition function is still written as a local assignment of amplitudes. This we can iterate a coarse graining procedure that only needs to consider few building blocks at a time.

As a path integral approach, spin foam quantum gravity is already phrased in this language, as amplitude functionals for certain boundary states. Take a spin foam on a 2-complex Γ_b with boundary b . Since the 2-complex is discrete, namely a collection of faces, edges and vertices, its boundary b is also discrete, namely a graph, with nodes and links. The complexity of this boundary depends on the number of nodes and links. To each of these boundaries b one associates a boundary Hilbert space $\mathcal{H}_b$, whose complexity again depends on the complexity of the boundary. A spin foam model for said two complex then acts as an amplitude functional $\mathcal{A}_b$ mapping states $\psi_b \in \mathcal{H}_b \rightarrow \mathbb{C}$.

The vital difference is that we allow for more general building blocks, in particular with more complex boundary data and thus boundary Hilbert spaces. When using Pachner moves, one integrates out bulk degrees of freedom while keeping the boundary unchanged. In 4D, when performing a $4-2$ move, one integrates out bulk variables and derives one effective amplitude for two glued 4-simplices, prescribed by the same boundary data. However, splitting this effective amplitude into two, one assigned to each building block, is not straightforward due to the previously mentioned non-local interactions. Instead, we allow for more general building blocks with more complicated boundary data. That way, we still have local assignments of amplitudes to building blocks, which in turn interact locally with neighboring ones. In turn, non-localities still arise, yet they are contained in each building block and captured by more complex boundary data.

While this picture recasts the problem of arising non-localities, three immediate challenges arise. Firstly, iterating this procedure leads to more and more complicated building blocks,

⁵Note that the discussion of the Ising model is primarily to give an intuitive example on how non-local interactions arise under coarse graining. Since the 2D Ising model is solved analytically [58], it is ideal to test the capabilities and feasibility of new coarse graining approaches. See e.g., [59] for an overview of real-space renormalization techniques in statistical physics.

whose amplitudes are still given by the fine boundary degrees of freedom. From this perspective we have not achieved a derivation of coarse scale physics, since the dynamics are still expressed in terms of fine scale degrees of freedom. Secondly, in order to define a renormalization group flow it is crucial to compare amplitudes after each coarse graining step. And thirdly, deriving amplitudes with more and more complex boundary data quickly becomes unfeasible, independent whether one is using analytical or numerical techniques, as one can already see for the 2D Ising model.

Hence, the next vital ingredient for a coarse graining scheme is the introduction of variable transformations, that map a collection of fine boundary degrees of freedom to a collection of coarse effective degrees of freedom. More precisely, we want to map states on a fine boundary Hilbert space $\mathcal{H}_{b'}$ on b' to a coarse boundary Hilbert space $\mathcal{H}_b$ on b . In the next section, we will explain the idea behind this concept and its physical interpretation. To do so, we work in the opposite direction and explain how to add degrees of freedom using embedding maps.

4.1. Embedding Maps and the Notion of Vacuum

As outlined above, a key idea of any renormalization procedure is to compare and relate theories defined on different scales. Given two spin foam amplitudes $\mathcal{A}_b$ and $\mathcal{A}_{b'}$, which are functionals for the Hilbert spaces $\mathcal{H}_b$ and $\mathcal{H}_{b'}$, respectively, these amplitudes can only be compared for the same physical processes. That is, given a state ψ_b in the coarse Hilbert space $\mathcal{H}_b$ one must represent ψ_b in the Hilbert space $\mathcal{H}_{b'}$. Then, each states can be evaluated with their respective amplitude and the results compared. For this purpose one defines so-called embedding maps:

$$\iota_{b'b} : \mathcal{H}_b \hookrightarrow \mathcal{H}_{b'} . \quad (6)$$

For this to work, the boundary b must be embeddable into the boundary b' , denoted as $b < b'$. Thus, the boundaries b form a partially ordered set. In case that two boundaries b and b' cannot be directly related, i.e., b cannot be embedded into b' , one embeds both into a common refinement b'' , written as $b < b''$ and $b' < b''$.

Hence, the goal is the following: given a state ψ_b in a coarse Hilbert space $\mathcal{H}_b$, we want to define an equivalence class of states in all finer, more complex Hilbert spaces $\mathcal{H}_{b'}$ in order to readily compare the associated amplitude functionals. This equivalence class of states is defined as follows: given two states $\psi_b \in \mathcal{H}_b$ and $\phi_{b'} \in \mathcal{H}_{b'}$,

$$\psi_b \sim \phi_{b'} \iff \iota_{b''b}(\psi_b) = \iota_{b''b'}(\phi_{b'}) \quad \forall b'' \text{ s.t. } b < b'' \text{ and } b' < b'' . \quad (7)$$

For this condition to be well-defined, the embedding maps need to satisfy a consistency condition, referred to as cylindrical consistency:

$$\iota_{b''b'} \circ \iota_{b'b} = \iota_{b''b} \text{ with } b < b' < b'' . \quad (8)$$

Essentially, it should not matter whether a state is directly embedded into a fine boundary b'' or via an (or any other)

intermediate boundary b' . Given these conditions and relations, one can (at least formally) define a continuum Hilbert space via an inductive limit: $\mathcal{H} := \bigcup_b \mathcal{H}_b / \sim$.

Beyond this formal definition, the action of embedding maps is best understood in the following way. As illustrated before, they serve the purpose of representing a coarse state in a finer Hilbert space, which can encode more complex configurations. Hence, embedding maps specify how and in which *state* degrees of freedom are added. Moreover, they thus define an inner product allowing us to compare states across Hilbert spaces. Since the information of the coarse state ought to be unchanged, these new degrees of freedom are added in a *vacuum state* prescribed by the embedding map. These concepts are familiar in the kinematical Hilbert space of loop quantum gravity [3] expressed in terms of spin network functions, where new degrees of freedom are added in the Ashtekar-Lewandowski vacuum [60, 61], which describes no space. In contrast, a dual BF representation [62–64] constructed in the last few years adds degrees of freedom that are peaked on flat connections. However, this notion of vacuum does not imply that this is a physical vacuum. Both examples given above are kinematical vacua in 4D gravity, i.e., they do not satisfy diffeomorphism and Hamiltonian constraints.

4.2. Renormalization Group Flow of Amplitudes

Once given such a choice of embedding maps, these can be readily used to compare spin foam amplitudes. Again, given two amplitudes $\mathcal{A}_b$ and $\mathcal{A}_{b'}$, a state $\psi_b \in \mathcal{H}_b$ and an embedding map $\iota_{b'b}$, we compare both amplitudes:

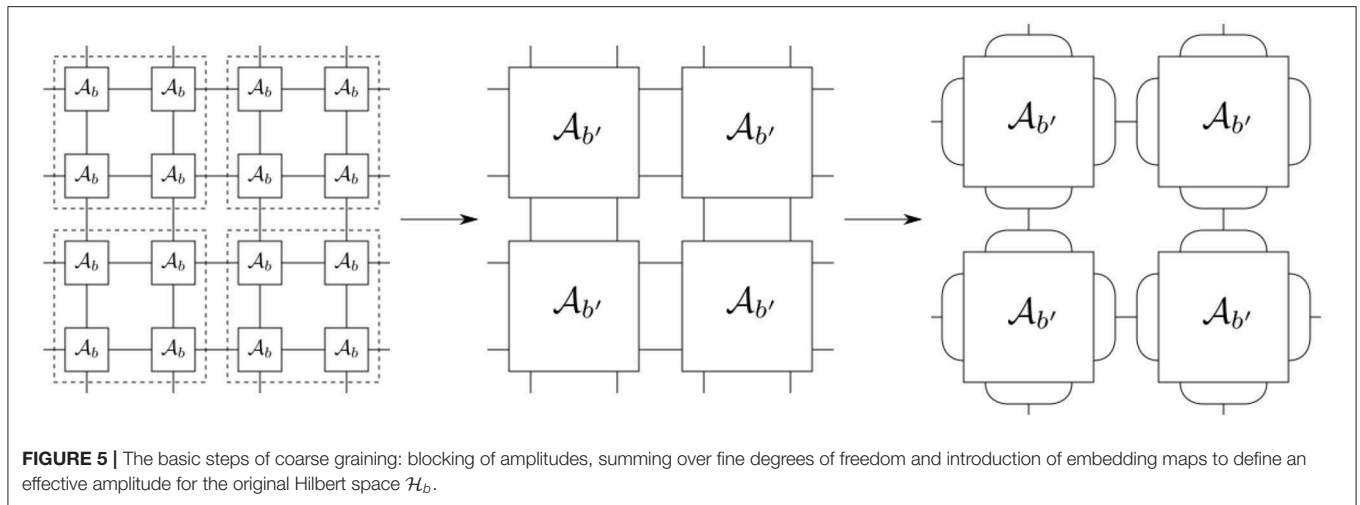
$$\mathcal{A}_b(\psi_b) \stackrel{?}{=} \mathcal{A}_{b'}(\iota_{b'b}(\psi_b)) =: \mathcal{A}'_b(\psi_b) . \quad (9)$$

Due to the embedding map, we define an effective amplitude $\mathcal{A}'_b$ for the coarse Hilbert space $\mathcal{H}_b$ from the fine one $\mathcal{A}_{b'}$ for $\mathcal{H}_{b'}$. If performed for all possible states in $\mathcal{H}_{b'}$, we obtain the coarse grained amplitude. Thus, embedding maps, which specify how to add degrees of freedom under refinement of states, serve as coarse graining maps for amplitudes. There, they specify how to define effective degrees of freedom. Consequently, since b' can capture more information than b , embedding maps also encode how to truncate degrees of freedom.

To summarize, a class of embedding maps defines a coarse graining/renormalization group flow of amplitudes, formulated with respect to their boundary. This flow is given by the following equation:

$$\mathcal{A}'_b = \mathcal{A}_{b'} \circ \iota_{b'b} . \quad (10)$$

To showcase the implications for the system as a whole, it is instructive to consider the partition function of the system. For simplicity, we assume that we can write it as a collection of



amplitudes $\mathcal{A}_b$ ⁶:

$$\begin{aligned} Z &= \sum_{j_b} \prod_b \mathcal{A}_b(j_b) = \sum_{j_{b'}} \prod_{b'} \left[\sum_{j_b \in \text{bulk of } b'} \prod_{b \supset b'} \mathcal{A}_b(j_{b'}, j_b) \right] \\ &=: \sum_{j_{b'}} \prod_{b'} \mathcal{A}_{b'}(j_{b'}) \end{aligned} \quad (11)$$

The original partition function is given as a product of amplitudes $\mathcal{A}_b$ assigned to building blocks with boundary Hilbert space $\mathcal{H}_b$. This Hilbert space is spanned by an orthonormal basis with labels j_b . In the second equality, we perform a blocking of amplitudes, e.g., of 16 vertex amplitudes for hypercubic combinatorics. The degrees of freedom j_b are split into two groups: one group makes up the boundary degrees of freedom $j_{b'}$ of the blocked amplitudes, while the other are block “bulk” degrees of freedom j_b and are summed over. The latter part is then summarized as the fine amplitude $\mathcal{A}_{b'}$. See **Figure 5**.

As the final step, we implement the embedding maps (or rather coarse graining maps) to derive the effective amplitude $\mathcal{A}'_b$ for the original Hilbert space.

$$\begin{aligned} Z &= \sum_{j_{b'}} \prod_{b'} \mathcal{A}_{b'}(j_{b'}) \approx \sum_{j_b} \prod_b \left[\sum_{j_{b'}} \iota_{b'b}(j_{b'}, j_b) \mathcal{A}_{b'}(j_{b'}) \right] \\ &=: \sum_{j_b} \prod_b \mathcal{A}'_b(j_b) \end{aligned} \quad (12)$$

In this context, the embedding maps serve as variable transformations and truncations, see again **Figures 5, 6** for more details. Indeed, this inclusion of embedding maps necessarily alters the partition function of the system, and we must ensure that we can still draw reliable conclusions about the original system. Thus, the combined embedding maps from neighboring

amplitudes should be close to the identity on the respective Hilbert space, in the sense that we only truncate irrelevant degrees of freedom. For example this is realized in tensor network renormalization, where these embedding maps are unitary as we explain in section 5.1.

Essentially, with this coarse graining procedure, we achieve two goals. On the one hand, we evaluate the partition function in parts, purely from local considerations of a subset of amplitudes. This is computationally efficient and makes calculations more accessible. On the other hand, we derive an effective theory on a coarser lattice, with less degrees of freedom, from a theory defined on a finer lattice. That way, we relate two theories on two different regulators by assigning different amplitudes to different lattices. Thus, the renormalization group flow of spin foam quantum gravity is defined as a family of amplitudes assigned to a family of 2-complexes/discretizations:

$$\mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}'' \rightarrow \dots \quad (13)$$

Before we discuss the consequences of this renormalization group flow in detail, it is important to discuss the role of the embedding/coarse graining and how they ought to be chosen.

4.3. Dynamical Embedding Maps and the Physical Vacuum

The embedding maps play a pivotal role in the renormalization group flow. Generically, different embedding maps result in different flows, since they determine how the effective coarse degrees of freedom depend on the fine ones. Thus, if one fixes the embedding maps *a priori*, this choice must be carefully checked. Instead, it is vital that these maps are directly determined from the dynamics encoded in the amplitudes themselves [30, 31, 40].

This reasoning is intuitive to follow, e.g., consider the Ising model. The effective degrees of freedom most suitably describing the coarse dynamics are sensitive to the temperature and significantly differ between low and high temperature. Hence, one size does not fit all: fixing embedding maps *a priori* can seriously distort the RG flow and give wrong results. Thus, we

⁶ E.g., this is the case for spin foam models defined on 2-complexes with regular, i.e., hypercubic, combinatorics. Due to the regularity, the system can be written purely as a collection of vertex amplitudes.

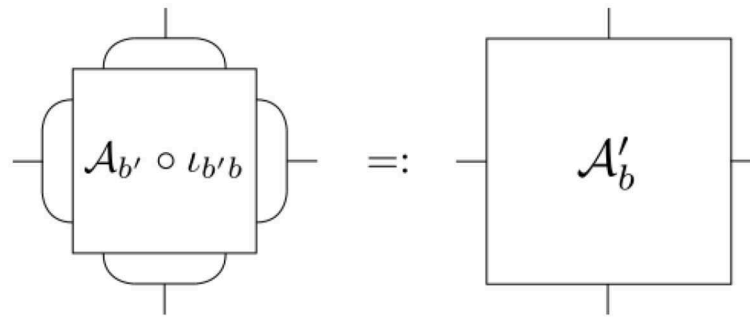


FIGURE 6 | Definition of the effective amplitude $\mathcal{A}'_b$ defined as the concatenation $\mathcal{A}_{b'} \circ \iota_{b'b}$.

are convinced that dynamical embedding maps are vital for a successful coarse graining scheme. We explain how to implement this in practice in section 5.1 on tensor network renormalization. For now, we discuss its implications and physical interpretation.

As discussed above, embedding maps are prescriptions how and in which state degrees of freedom are added under refinement. In particular they allow to relate and identify states across Hilbert spaces, defining an inner product and notion of vacuum. When these embedding maps are chosen dynamically, i.e., with respect to the dynamics encoded in the amplitude, the new degrees of freedom are added in a *dynamical* vacuum state.

General relativity is a totally constrained theory, i.e., in a canonical formulation its evolution is not governed by a Hamiltonian but rather by a sum of constraints, namely the diffeomorphism and Hamiltonian constraints. These constraints are generators of gauge transformations, which implies that evolution in gravity amounts to gauge transformations. This is known as the infamous problem of time [65]. For a quantum theory, the goal is to find the physical Hilbert space, i.e., the space of all states that are annihilated by all the constraints. Therefore, given an initial physical state, evolution by the constraint operators leave the physical state unchanged. Following this insight, dynamical embedding maps in a quantum gravity theory should be *physical* embedding maps that add degrees of freedom in the *physical* vacuum under refinement. Thus, such embedding maps do not add new information to the state and represent the same physical state on a finer boundary Hilbert space.

In the context of path integral formalisms of gravity, this insight is particularly intriguing: since evolution in the canonical formulation is governed by constraint operators, the path integral merely imposes the constraints, projecting out kinematical degrees of freedom and leaving physical states unchanged [30]. In short, the path integral serves as a projector onto the physical Hilbert space. Indeed, this insight is one of the original motivation behind constructing spin foam models in the attempt to define a rigging map/physical inner product for kinematical states of loop quantum gravity [3].

Following this line of thought, spin foams themselves are dynamical embedding maps. Consider a spin foam evolving an

initial state to a final state, where these states are defined on different boundaries⁷. When interpreting the spin foam as a map from one Hilbert space to another, instead of as an amplitude functional, it is by definition an embedding map. Moreover, if the spin foam acts as a projector onto the physical Hilbert space, concatenated spin foams still act as a projector, implying path independence of evolution. Conversely, this is interpreted as first evolving to an intermediate state, thus cylindrical consistency conditions of embedding maps are satisfied. Additionally, the projector property implies that this evolution is independent of the choice 2-complex/discretization, and it would mark a fixed point of the renormalization group flow:

$$\mathcal{A}_b = \mathcal{A}_{b'} \circ \iota_{b'b} . \quad (14)$$

This implies that assigning the same amplitude to all discretizations gives the same results.

All of the conditions mentioned above are highly non-trivial and rely on a perfect implementation of diffeomorphism symmetry in the discrete. Indeed, this assumption is hidden in the projector property of the path integral/spin foam, which implies an implementation of diffeomorphism and Hamiltonian constraints. Path/discretization independence follow immediately and underline the strong connection of diffeomorphism symmetry and discretization independence. In fact, this construction would be a realization of the perfect action program [38] for quantum gravity and would imply that the dynamics of quantum gravity are solved non-perturbatively and pulled back onto the discrete.

Unsurprisingly, these conditions are not met by spin foam models: spin foam amplitudes do not act like projectors [42, 69] and explicitly break diffeomorphism symmetry [34, 36]. Furthermore, it is unlikely that these conditions can be perfectly realized without approximations in full generality. Thus, the goal of the coarse graining scheme is to iteratively improve spin foam amplitudes in order to well approximate the ideal solution. We discuss this in the next section.

⁷ For an in-depth derivation of a canonical formalism for phase spaces and Hilbert spaces of varying dimension/complexity and the appearance of pre-/post-constraints, see [66–68].

4.4. Lessons From the RG Flow

The idea behind the coarse graining method outlined above is that it allows us to iteratively improve the amplitudes, such that the conditions mentioned above are approximately implemented. Furthermore, we can tackle discretization dependence of the theory. The expectation is that on a second order phase transition the regulator can be removed and diffeomorphism symmetry is restored. Let us explain this step by step.

Firstly, it is straightforward to recognize that the renormalization group flow addresses the question of discretization dependence and choice of 2-complex. By deriving an effective amplitude $\mathcal{A}_b$ from coarse graining $\mathcal{A}_{b'}$, we directly relate theories on 2-complexes defined from building blocks with boundaries b and b' , respectively. This information is vital for any discrete theory: it states that we perform (approximately) the same calculation on b when using $\mathcal{A}_b$ as on b' when using $\mathcal{A}_{b'}$. In particular, following this prescription, it does not matter whether we calculate a coarse observable on the coarse or the fine lattice. Thus, we account for the discretization dependence of the theory and ensure at the same time that the results are reliable. Indeed, understanding this behavior is indispensable when trying to make contact with experiments.

Understanding the lattice dependence of the theory is an important step toward determining when and how the regulator can be removed entirely. To this end, one has to study the whole coarse graining flow, that is choose an initial amplitude (e.g., given by a choice of parameters) and follow the flow until it reaches a fixed point. Within a certain approximation, e.g., restricting to finite dimensional boundary Hilbert spaces, this generically happens. These attractive fixed points frequently describe topological theories⁸, where a continuum limit can be trivially taken. However, these theories do not describe gravity in four dimensions, since they lack propagating degrees of freedom. Furthermore, these attractive fixed points denote the phases of the theory.

All initial amplitudes, e.g., all amplitudes from a certain region in parameter space, that flow under coarse graining to the same attractive fixed point lie in the same phase. These theories have the same dynamics on sufficiently coarse grained discretizations and thus lie in an universality class and share qualitative features, e.g., in expectation values of observables. An example would be the strong coupling phase in lattice gauge theory, in which one expects the Wilson loop operator to satisfy an area law. Frequently models possess multiple phases with phase transitions separating them. We are particularly interested in phase transitions of second order.

In standard lore, second order phase transitions are characterized by a diverging correlation length. This implies that degrees of freedom infinitely far away are correlated and, thus, infinitely many degrees of freedom are relevant for the dynamics. Moreover, right on the phase transition, the system is scale-invariant, i.e., physics do not change with scale. Therefore, on a second order phase transition one can take the continuum

limit to arrive at a continuum theory with propagating degrees of freedom.

We expect the same to hold for second order phase transitions in spin foam models, with a slightly different interpretation: Background independent theories lack an absolute length scale. Still, the regulator allows us to define a combinatorial distance. Essentially, the idea is to define a distance between vertices of the 2-complex, by counting the number of vertices one has to pass in order to reach the other one. If they are directly connected by an edge, this distance would be one⁹. Then, on a second order phase transition, degrees of freedom that are infinitely “far” away with regard to the lattice are correlated and the combinatorial correlation “length” diverges. Furthermore, the notion of scale invariance is consequently replaced by a discretization independence/invariance, fixed point equations for the amplitudes are satisfied and a continuum/refinement limit can be taken. The implications of constructing the theory on this fixed point must be stressed: Due to the discretization independence, calculations can be performed in the continuum or on any discretization, giving the same results. This exactly corresponds to the idea of perfect action, and thus solving the coarse graining flow corresponds to solving the theory on all lattices.

Nevertheless, two caveats must be observed: firstly, finding such a second order phase transition (if it exists) does not guarantee that the corresponding theory is a correct theory of quantum gravity. Secondly, if infinitely many degrees of freedom become relevant, truncated coarse graining schemes can only approximate the desired theory to a certain order, as it is the case in other renormalization schemes.

In the next section we discuss the notion of scale in more detail.

4.5. Background Independence and the Interpretation of Scale

Before we continue with reviewing how to coarse grain in practice, it is crucial to discuss the notion of “scale”—or the lack thereof. As a background independent approach, one cannot assign a scale to a spin foam since one sums/superimposes all possible geometries (allowed by a certain 2-complex). Thus, we use the 2-complex itself, here formulated via the boundary of the amplitudes, to order degrees of freedom according to a relative scale. Consequently, we replace the familiar notions of ultraviolet (UV) and infrared (IR) by “fine” and “coarse,” respectively. Following this perspective, we are not integrating out short scale degrees of freedom under coarse graining. Instead, we sum over finer discrete degrees of freedom and define effective coarse degrees of freedom, which encode (superpositions of) geometries of different scales.

Alternatively, one can introduce a specific scale in this coarse graining procedure via boundary states. That is, we do not

⁸This is expected for discretization independent theories with finitely many degrees of freedom. Examples are the Ponzano-Regge model in 3D [6], or more generically BF theory in any dimension [5].

⁹This idea is inspired by a similar concept in Causal Dynamical Triangulations [70], where the distance between two vertices of the triangulation is given by the minimal number of links between them, albeit with the notable difference that each length has a specified length assigned to it. This concept is used, e.g., to measure the geodesic distance between two vertices [71].

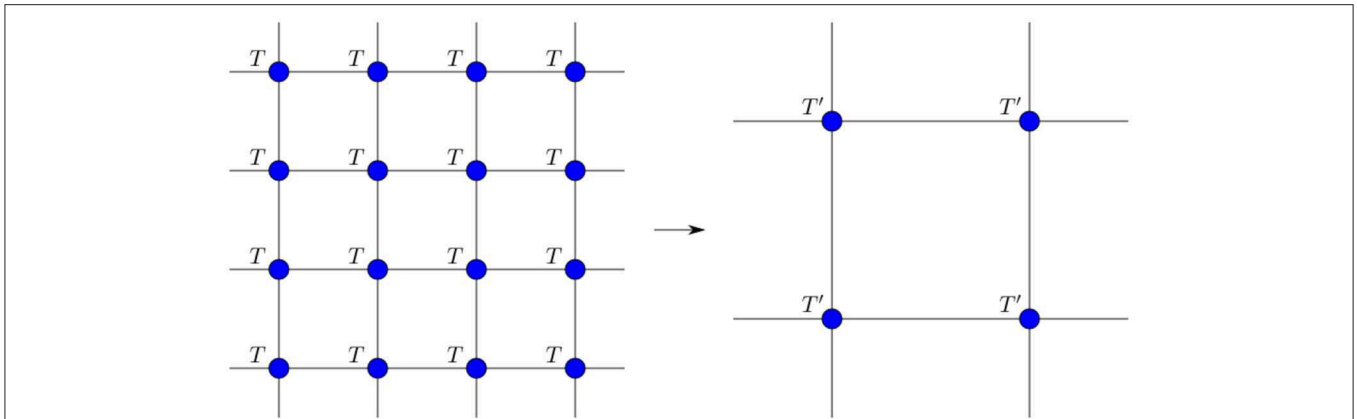


FIGURE 7 | The basic idea of tensor network renormalization: write the partition function as a contraction of a tensor network of tensors T and then locally manipulate the tensors, such that the same partition is approximated by a coarser network of effective tensors T' . This defines a flow in “tensor space”.

consider the entire Hilbert space, but only a specific state because we are interested in studying a transition of geometries. Then, this fixed boundary states introduces a physical scale via the encoded 3D geometry, e.g., implemented in the restricted path integral formalism, see section 5.2.

A further comment on the coarse graining scheme is in order: here we purely formulate it in terms of the boundary discretization and its associated boundary Hilbert space, not in terms of the bulk. From a practical perspective these questions are less important, e.g., if one assumes regular combinatorics such that the coarse graining procedure can be straightforwardly iterated; then both the boundary and bulk form totally ordered sets. Nevertheless, there is a proposal by Bahr [72] for formulating the coarse graining scheme in the bulk, essentially by defining embedding maps for 2-complexes.

5. COARSE GRAINING METHODS

For the rest of this article, let us focus on numerical methods that allow us to realize this coarse graining method in practice and review the results.

5.1. Tensor Network Renormalization Methods

Tensor network methods originate in the fields of quantum information and condensed matter and aim at efficiently studying quantum many body systems. In this review we focus on tensor network renormalization methods¹⁰ [46–48], a numerical algorithm for coarse graining discrete systems. To this end the partition function of the system is represented as a contraction of a tensor network. In the context of this article, a tensor $T_{abc\dots}$ is best understood as the amplitude assigned to a region. The boundary data of these amplitudes are represented as indices

¹⁰ There also exist tensor network methods that aim at constructing specific states of many body quantum systems, e.g., matrix product states (MPS), projected entangled pair states (PEPS) [73] or multi-scale entanglement renormalization ansatz (MERA) [74]. Their goal is to efficiently represent a small subspace of an exponentially large Hilbert space, containing the ground state.

of the tensor, which is graphically represented as a vertex with as many legs as it has indices. The partition function is then rewritten as a contraction of tensors:

$$Z = \sum_{a_1 b_1 c_1 \dots} T_{a_1 b_1 c_1 d_1} T_{c_1 b_2 c_2 d_2} \dots = \text{Tr}(T T \dots T). \quad (15)$$

Graphically, each identified and contracted index is represented by connecting the respective tensor indices. Thus, the partition function is represented by a collection of tensors connected to one another in a local fashion, a tensor network, see Figure 7.

So far, this is merely a rewriting of the original system. The goal is to locally manipulate the tensors in order to rewrite the partition function as a coarser tensor network, see again Figure 7. This may require truncations/approximations for which the error can be estimated. There exist several tensor network schemes, yet they all have a series of steps in common that we illustrate for a concrete example.

For simplicity, take a 2D quadratic tensor network. One step present in all tensor networks is an explicit summing of degrees of freedom, referred to contraction of indices. In our network we group together four tensors T and sum over their shared indices and obtain a new tensor $\tilde{T}$, which has twice as many indices, yet the network remains local, see Figure 8.

$$\tilde{T}_{a_1 a_2 b_1 b_2 \dots d_1 d_2} := \sum_{ijkl} T_{b_1 i l a_1} T_{l k d_1 a_2} T_{b_2 c_1 j i} T_{j c_2 d_2 k}. \quad (16)$$

We observe an immediate issue: if the original tensor had an index range of χ , called the bond dimension, the new tensor has a range of χ^2 . Thus, while we evaluate the partition function in steps, we cannot continue indefinitely without truncations/approximations. To implement those, dynamical variable transformations are derived from the $\tilde{T}$ via a singular value decomposition. That way, we define effective coarse degrees of freedom as functions of the fine ones. Crucially the effective degrees of freedom are derived from the dynamics encoded in the tensors. This works as follows:

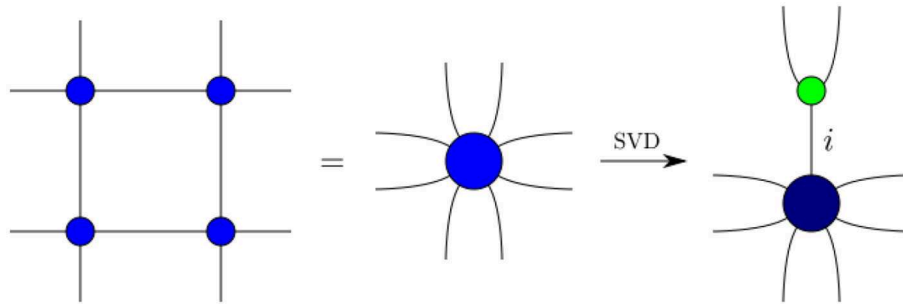


FIGURE 8 | The embedding maps are computed from the contracted tensor with multiple indices. It is split apart into two tensors, connected by a new effective edge labeled by the singular values indicating the relevance of the degree of freedom. The green, three-valent tensor then serves as an embedding/coarse graining map for the fine degrees of freedom.

Given the new tensor $\tilde{T}$, we intend to map two indices into an effective one. To do so, we split $\tilde{T}$ in two, separating the strands a_1, a_2 from all other variables. This is generically not possible unless the tensor factorizes. To split the tensor, we first rewrite it in terms of a matrix M_{AB} , where index $A = \{a_1, a_2\}$ and B contains all remaining indices. On this matrix we perform a singular value decomposition:

$$\tilde{T}_{(a_1, a_2), (b_1, b_2, \dots, d_1, d_2)} = : M_{AB} = \sum_{i=1}^{\chi^2} U_{A,i} \lambda_i (V)_{i,B}^\dagger. \quad (17)$$

The matrices U and V are unitary and contain the left and right singular vectors of M . λ is a diagonal matrix of singular values, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\chi^2} \geq 0$. See right side of **Figure 8**.

If we translate the matrix indices A and B back into the original tensor indices, we see that the singular value decomposition allows to write the tensor $\tilde{T}$ as the contraction of a three-valent and a seven-valent tensor, where the summed index i labels the singular values. The three-valent tensor U encodes the desired variable transformation, translating the degrees of freedom a_1, a_2 into an effective coarse degree of freedom/index i . This transformation is exact, since i (generically) has a range of χ^2 . Since U is a unitary matrix, we can introduce resolutions of identity $UU^\dagger$ in the partition function (see left of **Figure 9**) without changing it and sum over the indices a_1, a_2 as well as the indices c_1, c_2 on the opposite site. Then we repeat this procedure for the remaining indices to obtain a new effective tensor T' , see right side of **Figure 9**.

Hence, we define a new effective tensor, yet its index range is still χ^2 , and we cannot continue to iterate this procedure without truncations. The singular value decomposition allows us to implement this truncation in an optimal way. Since all singular values are positive semi-definite and ordered in size, $\frac{\lambda_i}{\lambda_1}$ indicates how significant i is with respect to the most significant one, $i = 1$. Indeed, we can approximate the rank χ^2 matrix M by a rank d matrix by ignoring all λ_i with $i > d$. Crucially, in terms of the least squared error, this matrix is the best rank d approximation of the matrix M_{AB} . Whether this is a good approximation can be readily inferred from the size of the singular values. Truncating the degrees of freedom i directly translates into truncations on the

variable transformations U and the new tensor T' , respectively. The accuracy of the simulations are then determined by the bond dimension, i.e., the number of degrees of freedom kept in each iteration.

The algorithm briefly sketched above is deliberately chosen to showcase that tensor network renormalization provides a concrete realization of the spin foam coarse graining scheme. Firstly, it blocks together tensors and exactly sums over their internal degrees of freedom. Secondly, the singular value decomposition provides dynamical variable transformations that fulfill the role of dynamical embedding maps, and, moreover, allow for efficient and controllable truncations. One way to check whether these approximations are justified is to gradually increase the bond dimension, i.e., the number of kept singular values, to see whether the properties of the system depend on this choice, e.g., the position of a phase transition in parameter space. In case the results converge, the approximation is sufficiently good and one can extrapolate to infinite bond dimension. However, there exist situations in which no truncation should be implemented, in particular on second order phase transitions [48]. There, one observes that more and more singular values are relevant the closer one tunes toward the phase transition, such that one would require an infinite bond dimension or equivalently infinitely fine boundary data. This is expected, since one models a highly non-local system by locally gluing amplitudes.

We would like to highlight some general advantages and disadvantages of tensor network methods, as well as modifications to the method that so far are not applied to models of quantum gravity. Compared to other numerical methods, like Monte Carlo methods, tensor network algorithms do not suffer from the sign problem; the algorithms are perfectly applicable to quantum (oscillating) amplitudes, like in spin foams. The reason is that tensor networks do not rely on (random) sampling methods for a large system, but usually focus on all possible configurations of an amplitude for one building block (tensor). However, this leads to one of their disadvantages: in order to save all configurations of a tensor, it must have a finite dimensional boundary Hilbert space. Moreover, the numerical costs, both in terms of computational time and memory usage, scale with the dimension of the boundary Hilbert space. In particular for

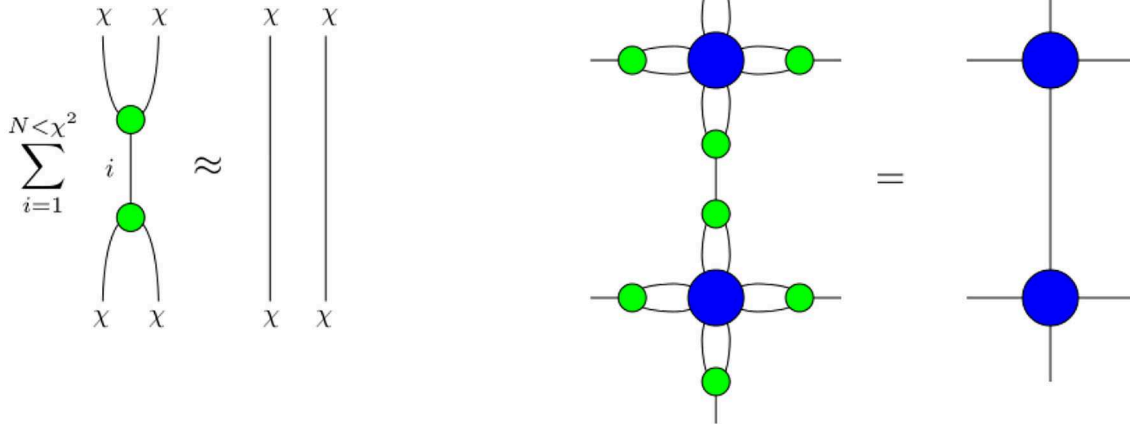


FIGURE 9 | (Left) In case the singular values for $i > N$ are negligible, the truncated maps $U^\dagger U$ approximate the resolution of identity well, such that inserting them in-between the pair of indices barely changes the partition function. **(Right)** Starting from the tensors in the middle of **Figure 8**, we insert the truncated resolution of identity for each pair of edge. Then we obtain the new effective tensor by contracting the previous tensor with its respective embedding maps.

lattice gauge theories and spin foams in higher dimensions, this requires extensive optimization to make the numerical simulations feasible.

Before reviewing results in quantum gravity (related) models, we would like to highlight a few methods from the tensor network community. One key modification is called entanglement filtering [47, 48, 75]. It removes entanglement between short-scale degrees of freedom, which would otherwise get promoted to larger scales and lead to unphysical fixed points in the renormalization group flow. Other modifications aim at including Monte Carlo methods into tensor network algorithms, e.g., for contracting tensor indices [76] or to sample over the probability distribution of coarse degrees given by the singular values [77].

In the following, we first review works on tensor network renormalization applied to 2D analog models. There we focus on the introduction of symmetry preserving methods that use the symmetry of the system to label the effective degrees of freedom with the original variables. In the second part, we discuss how to apply these methods to 3D lattice gauge theories and spin foam models, which require an efficient description of the model given by so-called decorated tensor networks.

5.1.1. Analog Spin Foam Models in 2D

By 2D analog spin foam models, sometimes also called spin net models, we mean spin systems with a global symmetry. The typical example is the Ising model (with vanishing external magnetic field) that has a global $\mathbb{Z}_2$ symmetry. Typically these models are written in terms of group variables colored $g_v \in G$ assigned to the vertices of the lattice, which only interact with their nearest neighbors expressed in “edge weights” $\omega_e(g_{s(e)}g_{t(e)}^{-1})$. In order to work with finite dimensional Hilbert spaces we restrict G to be a finite group (or quantum group later on) [78].

The partition function of the system is given by:

$$Z = \sum_{g_v} \prod_e \omega_e(g_{s(e)}g_{t(e)}^{-1}). \quad (18)$$

To be invariant under the global symmetry, i.e., an element $h \in G$ acting on all vertices at once, these edge weights must satisfy

$$\omega_e(h g h^{-1}) = \omega_e(g) \quad \forall h \in G. \quad (19)$$

Thus ω_e are class functions and Since the function ω_e are invariant under conjugation, each one can be expanded via Peter Weyl’s theorem [79] into a sum over irreducible representations ρ of the character χ_ρ of G :

$$\omega_e(g) = \sum_\rho \tilde{\omega}_\rho \chi_\rho(g). \quad (20)$$

$\tilde{\omega}_\rho$ stands for the group Fourier transform of the edge weight ω_e . Performing this for all edges and expanding the characters as a trace of representation matrices, the expression factorizes over all group elements g_v , such that the group integrations/summations can be performed analytically:

$$\mathbf{P}_v(\{\rho_e\}_{e \supset v}) := \sum_{g_v} \bigotimes_e \rho_e(g_v)_{n_e}^{m_e}. \quad (21)$$

$\mathbf{P}_v$ denotes the Haar projector of the group G , i.e., the projector onto the invariant subspace. We suppress its many indices for clarity of the notation. After performing all group integrations/summations, the partition functions reads:

$$Z = \sum_{\rho_e} \prod_e \tilde{\omega}_\rho \prod_v \mathbf{P}_v(\{\rho_e\}_{e \supset v}) \quad (22)$$

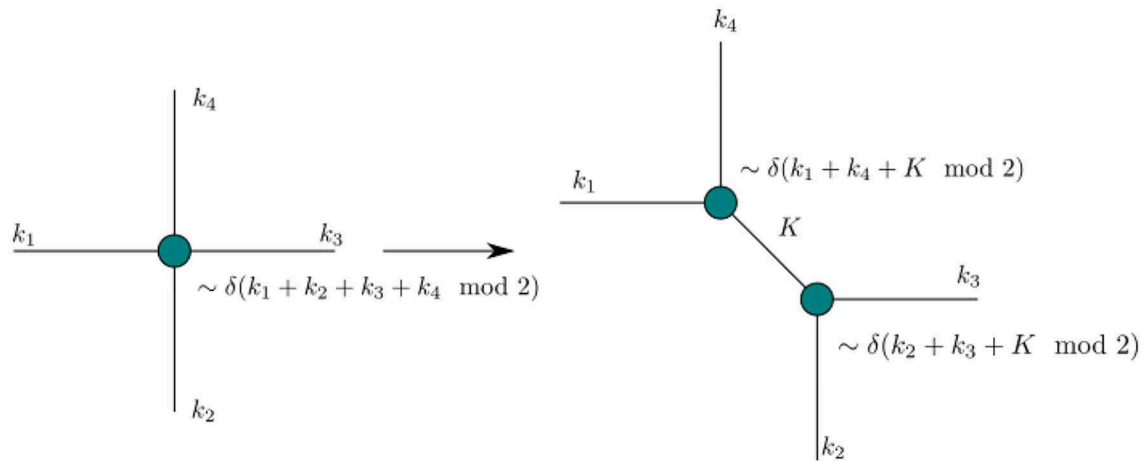


FIGURE 10 | By explicitly preserving the symmetry of the tensor, we assign an irreducible representation to the new effective edge, thus preserving also the original theory space. Here shown for the $\mathbb{Z}_2$ /Ising model case.

Note that the indices of the Haar projectors $\mathbf{P}_v$ are contracted with projectors on neighboring vertices. For more details on these models and their relation to spin foam models (with finite groups), see [78].

The expansion sketched here is completely analogous to the derivation of the spin foam representation familiar from spin foam literature. Thus, while the dimensionality is lower and the dynamics simpler, the dynamical ingredients—irreducible representations ρ and projectors onto the invariant subspace $\mathbf{P}$ /intertwiners ι —are the same as for spin foam models. Moreover, it is expected that these 2D spin systems share statistical properties with the 4D gauge theories of the same group [80]. As a final point, these models can be related to peculiar spin foams that only possess two vertices and many edges [81].

Hence these models represent ideal test cases for applying tensor network renormalization to spin foam models and derive first hints for the RG flow of the full theory. Fortunately, the translation of the partition function into tensor network language is straightforward: the projectors $\mathbf{P}$ are essentially tensors, whose variables are the irreducible representations ρ_e on the edges. Just the weights $\tilde{\omega}_{\rho_e}$ need be split per edge via a squareroot. While tensor network algorithms can be readily applied, it is vital to consider the symmetries encoded in $\mathbf{P}$: the irreducible representations ρ_e meeting at the vertex v must satisfy the coupling rules, i.e., they must couple to the trivial representation to satisfy gauge invariance. These restrictions can be used to optimize tensor network renormalization methods in two ways: firstly, by only storing and summing over configurations allowed by the coupling rules, the memory cost and numerical cost for the singular value decomposition and index contractions can be drastically reduced. For Abelian models this is straightforward, since all representations are one-dimensional and the projector $\mathbf{P} = \delta^{(G)}(\sum_{e \supset v} k_e)$ (modulo orientation of the edges). Thus, under splitting of a tensor, e.g., to define the variable transformations/embedding map, one defines an intermediate representation for the new effective edge satisfying the coupling

rules for both tensors. This new representation will be the label of the effective degrees of freedom and thus allows us to explicitly preserve the symmetry, see **Figure 10**. Moreover, ordering the entries of the matrix according to the intermediate representation turns the matrix into a block diagonal form. Thus, the algorithm can be further optimized by performing a singular value decomposition individually for each block¹¹.

For non-Abelian models, a further comment is necessary. As sketched in equation (21), the tensor possesses “magnetic” indices m, n per edge in addition to the irreducible representation ρ . More precisely, for fixed ρ , each edge carries the vector space $V_\rho \otimes V_{\rho^*}$, where ρ^* denotes the dual representation to ρ . Since V_{ρ^*} can be identified with the dual vector space $(V_\rho)^*$, we label each edge with a single representation. For tensor networks these magnetic indices pose a significant challenge: if we were to include them, the size of the tensor would render the simulations unfeasible. Fortunately, the dependence on these indices is entirely encoded in representation theory of the group G and does not change under coarse graining. Essentially these indices get “pre-contracted” [81, 82], which is accounted for by G recoupling symbols in the coarse graining equations. From these equations one can read off another feature of non-Abelian models: the renormalization group equations for the representations ρ and ρ^* are decoupled. Thus, it is possible that under coarse graining the effective edges will carry representations (ρ, ρ') with $\rho' \neq \rho^*$.

Mentioning the channels (ρ, ρ') is a good keyword to explain the flow as well as the approximation scheme. Due to the explicit symmetry preservation, the renormalized tensors are expressed in terms of the same variables as the original theory (with a slightly more general theory space). Thus, instead of directly comparing all entries of the tensors, we study the coarse graining flow by considering the singular values per channel (ρ, ρ') . This

¹¹ A SVD of a $p \times q$ matrix with $q > p$ scales with $p^2 q$ in terms of computational time. Thus, it is beneficial to perform multiple decompositions of smaller matrices.

is completely sufficient to characterize the flow and read off different fixed points. Then, in order to determine which degrees of freedom are more relevant, one needs to compare the singular values from all channels and truncate accordingly. This can then result in a higher multiplicity of the same representation labels and thus more general boundary data, which improves the accuracy of the simulation. However, in most cases [81–83] a simple scheme is used, where only the largest singular value per block is kept. While this is a strong simplification, it is sufficient to identify interesting phases, labeled by attractive fixed points of the flow, while keeping the simulations feasible, in particular for studies on quantum groups $SU(2)_k$ [81] and $SU(2)_k \times SU(2)_k$ [83].

The articles [84, 85] investigate spin net models for the Abelian finite groups $\mathbb{Z}_q$, for so-called cut-off models. The starting point are the edge weights in the zero-temperature limit with $\bar{\omega}_k = 1 \ \forall k \in \{0, \dots, q\}$. These weights are then truncated at different levels k , which breaks the topological symmetry and it is investigated whether this symmetry is restored under coarse graining. While for low- k and high- k cut-off the high and low temperature fixed points are found, respectively, there exist intermediate phases showing oscillating behavior. In [82] tensor network methods are generalized to non-Abelian groups applied to spin nets for S_3 , the permutation group of three elements. To keep these models feasible, the coupling rules are heavily used to optimize the algorithm. The models investigated build upon a holonomy representation of spin foam models [86] and their implementation of simplicity constraints. In general they find a non-trivial phase diagram of three phases, a low temperature S_3 ordered phase, a high temperature S_3 disordered phase as well as a $\mathbb{Z}_2$ ordered phase.

As a next step [81], finite non-Abelian groups are replaced by the quantum group $SU(2)_k$ with the deformation parameter $q = \exp(\frac{2\pi i}{k+2})$ a root of unity¹² [87, 88]. The advantage is that the integer level k defines a gauge-invariant cut-off $j_{\max} = \frac{k}{2}$. That way it is possible to study systems with more degrees of freedom by increasing the level k , while the representation theory remains similar. Moreover, one eventually approaches full $SU(2)$ as $k \rightarrow \infty$. Moreover, quantum groups are physically motivated from 3D spin foam models, where they describe gravity with a non-vanishing cosmological constant [53]. The models studied in [81] are constructed from so-called intertwiner model fixed points [89], which represent topological field theories. In a nutshell, intertwiner models are “half” of a spin net model, with an edge Hilbert space of V_ρ instead of $V_\rho \otimes V_{\rho^*}$. These models are interesting since one can directly investigate whether the two copies remain coupled or decouple under coarse graining. Indeed one finds a rich phase structure with potential second order phase transitions.

Eventually, the work [83] investigates spin net models for $SU(2)_k \times SU(2)_k$ that mimic the construction of 4D Riemannian spin foam models, namely the Barrett-Crane [10] and EPRL/FK model [13, 14]. For the BC model several attractive fixed points

are found, none of which correspond to topological BF theory. While this indicates that simplicity constraints are strongly implemented, no indications for a 2nd order phase transition are observed. This hints toward the fact that the constraints might be a too strongly implemented [15]. In contrast, the EPRL/FK model shows a highly intricate flow and partially oscillating behavior, most likely due to exciting only a few representations initially. This is a particularity of the implementation of the simplicity constraints in the Riemannian EPRL/FK model, which relate $Spin(4)$ representations (j^+, j^-) to an $SU(2)$ representation k via $j^\pm = \frac{1}{2} k |1 \pm \gamma|$. Note that $j^\pm$ as well as k must be half integer, such that γ must be rational.

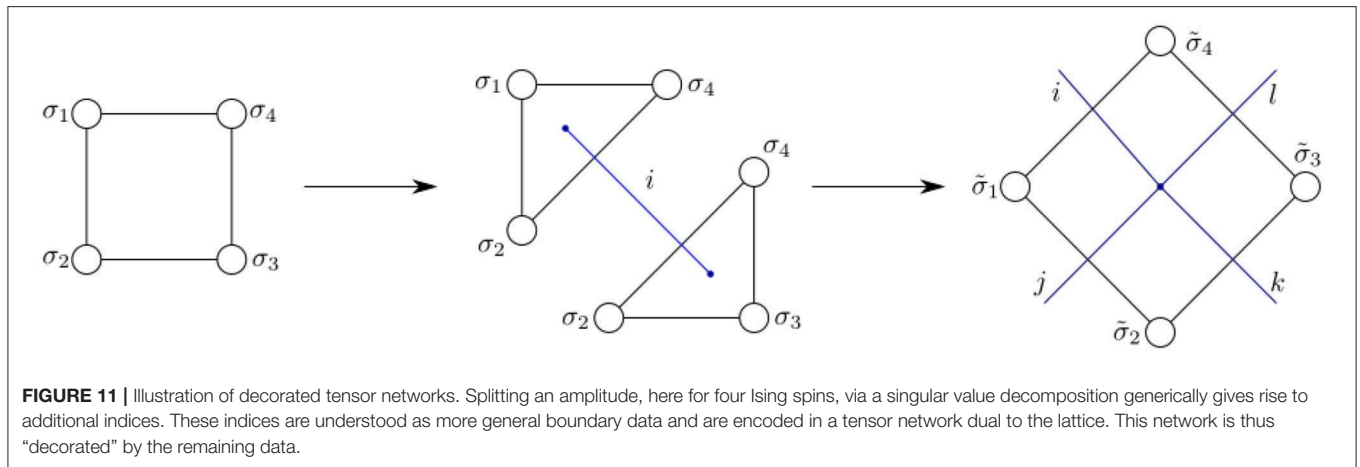
These results impressively show the potential of tensor network techniques for studying the renormalization group of spin foam quantum gravity. Moreover, they lead to the development of key optimizations and insights that are crucial for going to higher dimensional gauge systems. This is the subject of the next section.

5.1.2. Decorated Tensor Networks for Lattice Gauge Theories and Spin Foams

Dimensions larger than $d = 2$ and lattice gauge theories pose challenges for tensor network renormalization methods. Since higher dimensional tensors carry more boundary data, the algorithm generically is more costly than its lower dimensional counterpart. For lattice gauge theories, where due to the local gauge symmetry many degrees of freedom are redundant, it is thus imperative to develop an optimal representation if one intends to cast them into a tensor network form. Moreover, these networks are generically more complex than spin systems, since several data are shared among more than two building blocks. One example are spin foams in four dimensions, where a face and the representation it carries are shared among multiple 4-simplices. A possible tensor network representation is to assign a tensor dual to each 4-simplex, yet one must introduce auxiliary tensors [90] to ensure the correct identification of shared variables. See Dittrich et al. [91] for a more extensive discussion of possible representations.

To improve on these representations, decorated tensor network algorithms are developed and introduced in Dittrich et al. [91]. The idea is to shift the perspective away from a pure tensor network representation of the system toward amplitudes with more intricate boundary data. Yet the key ideas are retained to explicitly contract bulk degrees of freedom and to dynamically define effective degrees of freedom using a singular value decomposition. Instead of tensors, represented by vertices and legs, one works with a spin foam inspired representation where amplitudes are assigned to regions, which carry boundary data, e.g., spin network data. While the assignment of amplitudes remains local, the non-local nature of gauge theories requires more complex boundary data and intricate gluing rules that cannot be cast in a simple tensor network form without introducing additional structures. The rest of the algorithm remains essentially the same: amplitudes are glued together by suitably identifying variables among them. Depending on the considered situation, some of the identified variables are not summed over and remain part of the boundary Hilbert space. On

¹² Quantum groups do not allow for a holonomy representation. However, its representation theory is close to the one of $SU(2)$, such that one defines these models directly from the high temperature expansion of spin net models.



this fine amplitude one performs a singular value decomposition to derive an embedding map leading to coarse effective degrees of freedom.

The original algorithm in Dittrich et al. [91] works slightly differently by “splitting” the amplitudes explicitly. Let us briefly demonstrate the idea for the usual 2D Ising model, for which a decorated tensor network algorithm exists as well. Consider an amplitude assigned to a square given by four Ising spins σ_i , $i \in \{1, \dots, 4\}$, $\mathcal{A}(\sigma_1, \dots, \sigma_4)$, see **Figure 11**. The idea of the algorithm is alternately split the squares into regular triangles, such that four of these form a coarse, rotated square with a single Ising spin to sum over in their center. To do so we split the amplitude in two, separating the dependency on the spins opposite to the cut. For the singular value decomposition, we need to distinguish two sets of variables: the variables that we want to separate are encoded in the two indices of the matrix to decompose, while the shared variables will remain fixed similar to the symmetry preserving algorithm before. Hence, we perform a singular value decomposition for each configuration of shared variables, which is more efficient than a decomposing a big matrix. In a sense, this leads to a doubling of the shared variables, which is necessary for gluing them again in consecutive iterations. In our example, we get:

$$M_{\sigma_1, \sigma_3}^{(\sigma_2, \sigma_4)} := \sum_{i=1}^2 U_{\sigma_1, i}^{(\sigma_2, \sigma_4)} \lambda_i (V^\dagger)_{i, \sigma_3}^{(\sigma_2, \sigma_4)}. \quad (23)$$

By assigning a square root of the singular values λ_i to U and V , we derive the desired amplitudes assigned to the triangles. Note that each amplitude is not just given by the configuration of three Ising spins, but also by an additional index i , assigned to the coarse edge. When combining four triangular amplitudes, the resulting amplitude for the square is given by more general boundary data, four Ising spins and four new indices $\mathcal{A}(\sigma_1, \dots, \sigma_4, i_1, \dots, i_4)$, again see **Figure 11**. Since these indices are shared with neighboring amplitudes, we represent them by a tensor network on the lattice dual to the squares. Thus, we have a

tensor network encoding higher order corrections/more general boundary data “decorated” by the original boundary data of the system.

An algorithm is developed for 3D lattice gauge theories and first applied to Abelian $\mathbb{Z}_2$ lattice gauge theory in Dittrich et al. [91]. Instead of working with the original lattice, one works with the dual lattice in the strong coupling expansion. Thus, the variables are irreducible representations k_e of $\mathbb{Z}_2$ one edge of the dual lattice and Gauss constraints ($\mathbb{Z}_2$ δ -functions) on each face. The Gauss constraints can be explicitly solved to reduce the amount of data saved, and there is a freedom to choose which variables k_e to gauge fix. This choice is adapted to the intended splitting.

Let us briefly sketch the coarse graining algorithm: the idea is to cut cubes in half by cutting along the diagonal of one of its faces. Four of these amplitudes are glued together to form a new (distorted) cube. This coarse graining is continued in the other directions by “rotating” the amplitude and iterating the procedure. Compared to the 2D algorithms, the splitting is slightly more complicated. In order to cut the cube along the face, it is necessary to introduce another representation along the intended “cut.” This variable serves as the variable assigned to the coarse edge, and is introduced by splitting the Gauss constraint on the square face into two assigned to the two triangles. Then, the variables are gauge fixed such that equally distributed the remaining degrees of freedom are equally distributed among the split amplitudes, and some shared by both. As explained above, the shared variables will be kept fixed during the singular value decomposition and label the new amplitudes. Also, again more general boundary data arise in the form of a decorated tensor network due to this splitting, which will be assigned to edges and faces, see [91] for more details.

In its lowest order approximation, i.e., when truncating all tensor indices, the algorithm reproduces the phase diagram of $\mathbb{Z}_2$ lattice gauge theory with a strong and weak coupling phase, whereas the critical coupling is found within an error of a few percent compared to Monte Carlo simulations [92]. These results are improved by keeping more degrees of freedom after the singular value decomposition, yet the computational costs

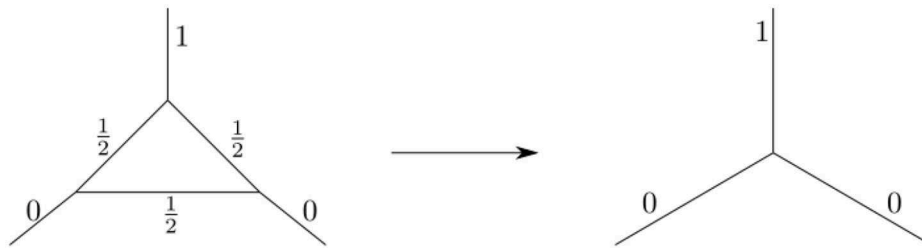


FIGURE 12 | Gauss constraint violations under coarse graining [for $SU(2)$]: we can define an effective vertex by summing over the $SU(2)$ representations associated to the inner edges. The configuration on the left is allowed by the coupling rules, thus the effective vertex on the right is allowed. However, this configuration is forbidden by $SU(2)$ coupling rules.

grow quickly¹³. As one of the first tensor network algorithms applied to 3D lattice gauge theories it already shows promising qualitative results.

In Delcamp and Dittrich [94] this algorithm is generalized to non-Abelian symmetry groups and applied to S_3 lattice gauge theory. While the basic idea and principle of the algorithm remains similar, it is significantly more complicated due to the non-Abelian group. The basic steps, splitting, gluing and choice of variables, are still in place, however in order to define them transformations between the holonomy and spin network representation are necessary. For the details, we refer the reader to the extensive and thorough explanation of the technical details in Delcamp and Dittrich [94]. Let us focus instead on the results.

As for the similar work on S_3 spin nets [82], analogous simplicity constraints for S_3 are implemented in the holonomy representation [86]. The coarse graining flow is studied and three different phases found, as in Dittrich et al. [82], that correspond to a strong coupling S_3 phase, a weak coupling S_3 phase as well as a weak coupling $S_3/\mathbb{Z}_3 \simeq \mathbb{Z}_2$ phase. The successful generalization to non-Abelian groups as well as the derivation of the phase diagram of the theory demonstrate the potential of decorated tensor network techniques. However, this work also revealed a short-coming of using the spin network basis for labeling the boundary Hilbert space.

As we discuss in great detail in this review, at some step of the coarse graining process one sums over fine degrees of freedom. In the spin network basis, where one assigns irreducible representations to the links and intertwiners to the nodes, this implies defining an effective vertex/intertwiner by summing over representations, see **Figure 12**. However, usually the coupling rules at the effective vertex are violated such that the vector associated to the vertex is not an intertwiner any more and gauge invariance is broken. This is a well-known shortcoming of the spin network basis under coarse graining [95, 96] and it is also expected for lattice gauge theories¹⁴. This can be overcome by

a different representation of the boundary Hilbert space that can accommodate Gauss constraint violations (electric charges) as well as curvature excitations (magnetic fluxes). In 3D this is accomplished by the so-called fusion basis.

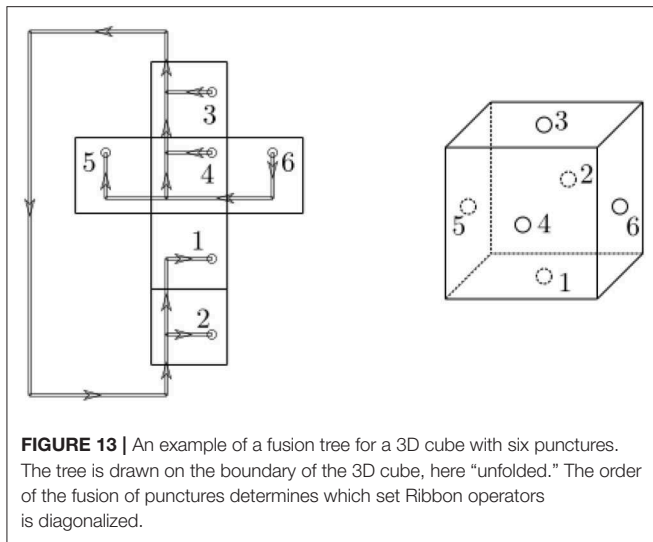
Here we will only briefly sketch the main features of the fusion basis, which arises in anyon systems [97, 98], (2+1)D lattice gauge theories [99] and 3D quantum gravity [100, 101]. The algebraic structures are called Drinfeld Doubles, see [99, 100, 102–105] for more details. Its main feature is that it diagonalizes a set of commuting operators, so-called Ribbon operators. These Ribbon operators, which contain both a Wilson loop operator as well as a t'Hooft operator, measure both the magnetic (curvature) as well as electric (torsion) excitations. Such excitations are localized on punctures carrying the magnetic and electric excitation. In lattice gauge theory, one can imagine one puncture per plaquette of the lattice. Ribbon operators surrounding a single or a collection of plaquettes then measure excitations associated with the puncture or collection of punctures.

These Ribbon operators commute among each other as long as they do not intersect. Hence, two Ribbon operators that surround a single puncture each commute with each other trivially, and they also commute with the operator surrounding both punctures. A choice of such a set of commuting Ribbon operators is encoded into the fusion basis by the choice of a fusion tree. The plaquettes are the leaves of the tree, and the connectivity of the tree determines which operators/observables are diagonalized by this choice of basis, see **Figure 13**. Moreover, the basis states can be transformed into one another, such that one can translate states to diagonalize the observables one intends to measure. This is a crucial concept that has a notion of coarse graining built into it. Imagine two cubes glued together: in order to derive an effective building block with effective degrees of freedom, one would like fuse the punctures of subdivided faces into one. To do so, the fusion basis must be chosen such that it diagonalizes the Ribbon operator around punctures. This ensures that the expectation values agree in both original and coarse grained case. Therefore, the fusion tree can be used to encode a choice of coarse grained observables, which is crucial for the decorated tensor network algorithm based on the fusion basis.

Such an algorithm is defined for quantum deformed 3D lattice gauge theories on a cubic lattice for the quantum group $SU(2)_k$ in Cunningham et al. [49]. More details about the fusion

¹³Dittrich et al. [91] also proposed an algorithm based on smaller building blocks that generically are more efficient, see also the “triangular” algorithm in 2D [83, 93].

¹⁴In order to define coarse grained fluxes in the discrete, the fine fluxes must be parallel transported to the same point. If the connection has curvature, the fluxes do not necessarily close any more, which is often referred to as curvature induced torsion.



basis for $SU(2)_k$ can be found in Dittrich and Geiller [100]. As in previous decorated tensor network algorithms, the basic ingredient is the amplitude associated to a cube. Its boundary Hilbert space is spanned by six-puncture states, for which a fusion tree needs to be specified. Then two cubes are glued together, which results in a cuboid with four coarse faces, each carrying two punctures. The goal is to compute an embedding map that fuses the two punctures on a subdivided face into one effective puncture. Generically, after gluing the fusion basis of the cuboid is not suited to do so, since it does not diagonalize the Ribbon operator surrounding both punctures. Thus, the basis must be transformed by a series of tree transformations involving $SU(2)_k$ recoupling theory, see [49] for details and consider **Figure 14** for an illustration.

Once one has arrived at a fusion basis that directly fuses the punctures as desired, one performs a singular value decomposition that splits the data of two punctures from all other variables (similar to section 5.1). In order to label the new puncture with the usual data, one keeps fixed the data of fusion tree directly after fusing the punctures. These data label the effective puncture, while the singular value decomposition gives the weight of these data in the effective amplitude. Also from the perspective of (coarse) observables this choice is viable, since the eigenvalue of a Ribbon operator surrounding two punctures is solely determined by these data. This allows us to use tensor network renormalization to approximately compute expectation values of Ribbon operators that we explain below.

After computing the embedding maps, they are used to coarse grain the punctures in such a way that the partition function is altered as little as possible (see again section 5.1). Again, only one non-vanishing singular value is kept per puncture label, such that the same boundary Hilbert space is pertained. Once all pairs of punctures are coarse grained, one obtains again an amplitude of six punctures that is associated with a cuboid. To complete one coarse graining iteration, the same procedure is performed in the other spatial directions. The cube is “rotated” by reordering the

punctures and the same procedure repeated in all directions to arrive at a coarse cube with six punctures.

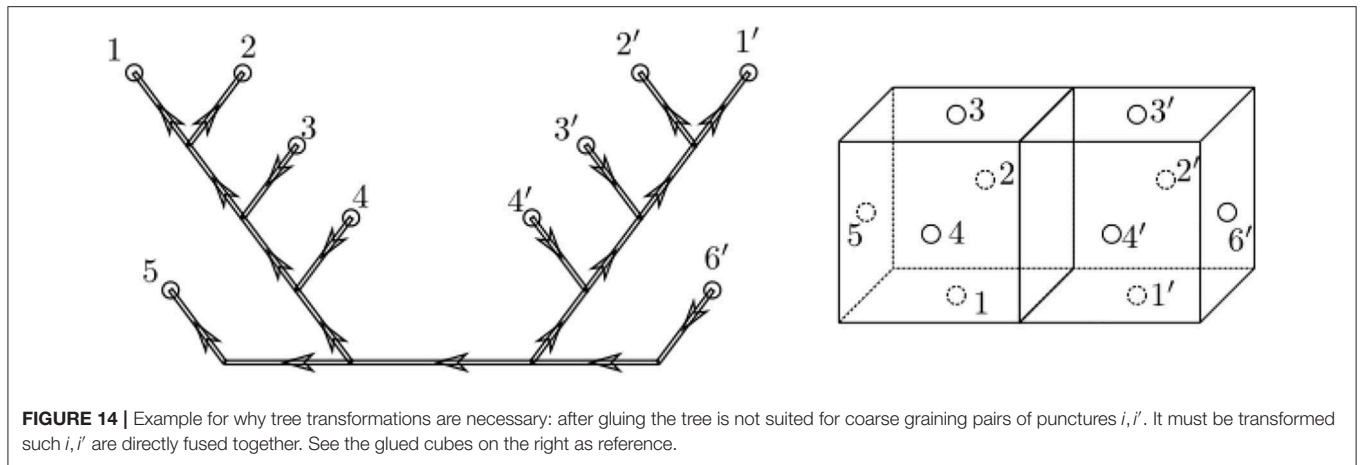
In Cunningham et al. [49] this algorithm is applied to 3D lattice gauge theories defined for the quantum group $SU(2)_k$. As for quantum group analog spin foam models [81, 83], the quantum group introduces a gauge-invariant cut-off on the irreducible representations $j_{\max} = \frac{k}{2}$. Thus, the boundary Hilbert spaces are finite dimensional and it is possible to study larger “groups” by increasing the level k [and approach $SU(2)$ in the limit $k \rightarrow \infty$]. The lattice gauge theory is modeled via a Heat kernel action for $SU(2)_k$ parameterized by a gauge coupling parameter g . Lastly, in the initial state each puncture only carries magnetic excitations as it usually is the case in lattice gauge theory.

Let us summarize a few of the main results: at each level k there are two phases separated by a phase transition given by a critical coupling g_c . For $g < g_c$, the system flows to the weak coupling fixed point $g = 0$ and is thus characterized as the deconfined phase. Conversely for $g > g_c$ it flows to strong coupling $g \rightarrow \infty$, which describes the confined phase. The position of the critical coupling g_c depends on the level k and decreases apparently linearly for small k . This tentatively suggests that for $SU(2)$, i.e., the limit $k \rightarrow \infty$, $g_c \rightarrow 0$ such that only the confining phase exists. Additionally, the fusion basis permits to track the appearance of electric excitations that get excited under coarse graining, even though the initial state had no electric charges. While they do not appear to be vital for the dynamics, e.g., the position of the phase transition is barely affected if electric charges are completely truncated, including electric charges is important for the behavior of the coarse graining flow as they serve as (non-dynamical) disentangling maps. See Cunningham et al. [49] for more details.

The final result we would like to mention is the expectation value of observables, here of Ribbon/Wilson loop operators. Since the fusion basis diagonalizes Ribbon operators, it is straightforward to approximately compute the expectation value of coarse Ribbon operators, i.e., Ribbon operators surrounding a larger number of plaquettes. In lattice simulations one usually has to simulate the entire system in order to measure coarse observables. Here, we first coarse grain the amplitude to arrive at an effective amplitude for the coarse cube, for which we measure the coarse Ribbon operator around the coarse plaquette. Thus, we first coarse grain/integrate out the fine degrees of freedom and account for them (with some truncations) in the effective amplitude, for which we then calculate the expectation values of the operators. Using this method, we derive different scaling behaviors of the expectation value with the enclosed area of the plaquette, in particular we recover the area law of the Wilson loop in the confined phase.

5.2. Restricted, Semi-Classical Path Integrals

Despite the tremendous progress in developing tensor network methods for spin foam models and lattice gauge theories, applying them directly to spin foam models of 4D quantum gravity (either Riemannian or Lorentzian) is



still out of reach, in particular for a continuous symmetry group. An attempt to make the 4D Riemannian spin foam models accessible is to study simpler models that represent a subset of the full gravitational path integral. These simplifications include restricting the degrees of freedom to specific intertwiners and representations as well as using asymptotic expansions of spin foam amplitudes valid only for large representations. Let us explain these assumptions in more detail.

Intertwiners, which determine the shape of dual 3D building blocks, can be expressed in terms of Perelomov coherent states/Livine-Speziale intertwiners [18, 106]: to each face of the intertwiner one assigns an $SU(2)$ coherent state $|j, \vec{n}\rangle$, where j labels the irreducible representation and $\vec{n}$ is a vector on S^2 . This vector is a maximum weight state diagonalizing the angular momentum operator $J_{\vec{n}} = \vec{n} \cdot \vec{J}$ in $\vec{n}$ direction. Given these states, the coherent intertwiner is given by:

$$|\iota\rangle := \int_{SU(2)} dg g \triangleright \bigotimes_{i=1}^N |j_i, \vec{n}_i\rangle. \quad (24)$$

Each coherent state $\sim |j_i, \vec{n}_i\rangle$ represents a face with area $\sqrt{j_i(j_i + 1)}$ peaked on a normal vector pointing in direction $\vec{n}_i$. The tensor product of these coherent states represents a 3D quantum building block sharply peaked on a classical geometry (if it exists) with areas and outward pointing normal encoded in the labels j_i and $\vec{n}_i$. The group integration (with Haar measure dg) defines an intertwiner. Note that the spin foam partition function itself can also be expressed in terms of coherent states and an integral over the labels of coherent states, e.g., see the review [1] for more details.

These coherent states play an important role in deriving semi-classical expressions for spin foam (vertex) amplitudes. When computing the vertex amplitude as a contraction of coherent intertwiners, these can be rewritten as several group integrations of contracted $SU(2)$ coherent states. The latter part

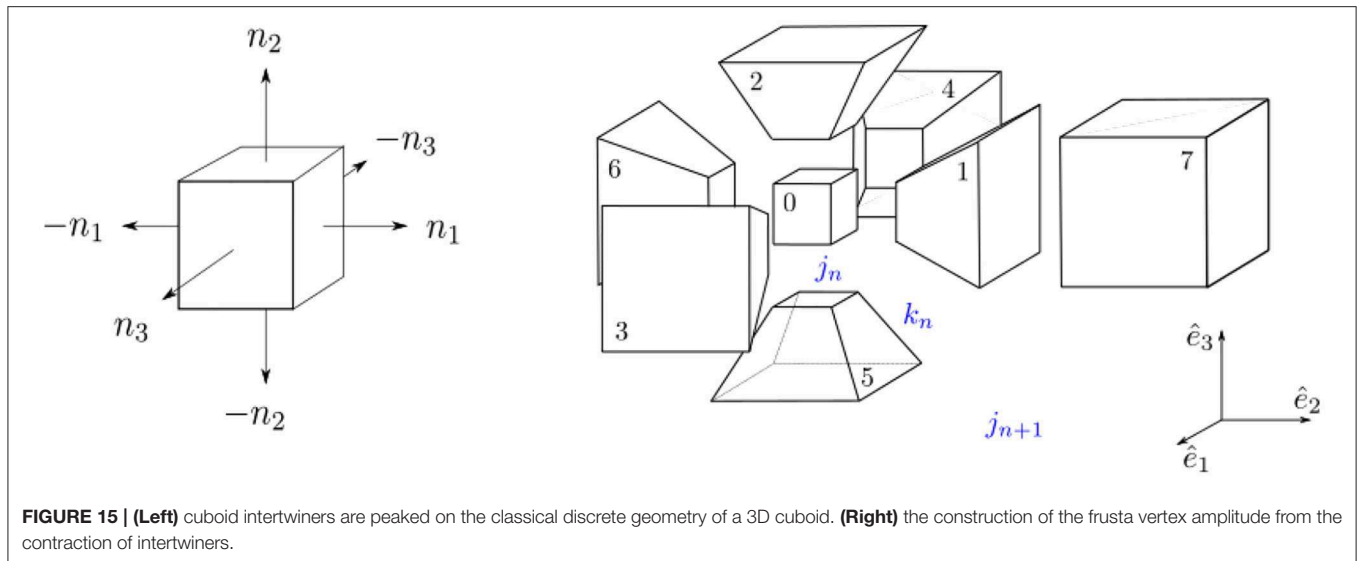
is then exponentiated and the group integration performed via a stationary phase approximation:

$$\begin{aligned} \mathcal{A}_v &= \int_{SU(2)^E} \prod_e dg_e \prod_{f \supset e} \langle j_{ab}, -\vec{n}_{ba} | g_b^{-1} g_a | j_{ab}, \vec{n}_{ab} \rangle \\ &= : \int_{SU(2)^E} \prod_e dg_e e^{\sum_{f \supset e} 2j_{ab} \ln \langle -\vec{n}_{ba} | g_b^{-1} g_a | \vec{n}_{ab} \rangle}. \end{aligned} \quad (25)$$

Since the stationary phase approximation is only valid when the argument in the exponential is highly oscillating, all representation j_f must be large. Hence this expansion is often called the large- j limit. For single vertex amplitudes it is shown that the “action” in the exponential evaluated on stationary and critical points is given by the Regge action of the building block dual to the vertex [22, 23].

Given these familiar results from spin foam literature, the idea is to restrict the spin foam partition function of the EPRL/FK model to specific coherent intertwiners (and representations) and use only the amplitudes derived in the large- j limit. That way the system depends on significantly fewer variables and the spin foam amplitudes, in particular the vertex amplitude, can be expressed in terms of closed formulas of the representations. Additionally, in the large- j limit the sum over representations can be approximated by an integral. The motivation is to employ numerical integration techniques, e.g., the Cuba package [107]. From now on, we only consider and discuss models defined on a 2-complex with hypercubic combinatorics which makes iterating the coarse graining steps straightforward.

So far, two models of restricted spin foams are defined that are also studied under coarse graining. The first one are so-called quantum cuboids [36], where the intertwiners are sharply peaked on a classical cuboid geometry. Opposite faces of the intertwiner carry the same representation and their normals are anti-parallel. Moreover, the outward pointing normal of a face is orthogonal to all normals of the adjacent faces, see the left part of **Figure 15**. Indeed these are severe restrictions, in particular the requirement that opposite faces in each intertwiner carry the same representation translate through the entire lattice.



The asymptotic expansion of the vertex amplitudes depends again on the Regge action, which generically vanishes for cuboid configurations. For larger complexes this implies that the flat cuboid building blocks are glued in a flat way. Thus, this model describes a superposition of flat discrete space-times of different distribution of sizes across its building blocks. While this is by no means a realistic model of quantum gravity, it captures an Abelian subgroup of diffeomorphisms corresponding to shifts of entire hypersurfaces in the lattice¹⁵. Notably, this spin foam model is not invariant under these transformations [36]. Due to its simplicity it does not have any free parameters, thus an additional parameter α is introduced in the face amplitude of the model, $(2j_f)^\alpha$, which can be understood as a modification of the path integral measure. This exponent simply emphasizes small or large representations/face areas in the partition function, and is motivated by a discussion in the community on the right choice of face amplitude [108].

A physically more interesting model is based on so-called frusta [109]. A frustum is a higher-dimensional analog of a trapezoid. In 4D, it consists of two cubes at its top and bottom, potentially of different size, which are connected by six 3D frusta, see the right of **Figure 15**. That way, a hyperfrustum can describe the evolution from one spatial cube to a larger/smaller spatial cube. Thus, the idea is to restrict the intertwiners to be cube/frusta shaped in order to study the expanding/contracting cubulated 3D spatial slices. The frustum intertwiner is then given by three representations, j_i and j_f correspond to the initial and final area of the initial and final cubes, respectively and k gives the side-face area, which also determines the “opening angle” ϕ of the frustum¹⁶. Crucially, in contrast to the cuboid model, the Regge action associated to a hyperfrustum in the asymptotic expansion no longer vanishes and one obtains the familiar cosine

formula [109]:

$$\mathcal{A}_v \sim \cos\left(\frac{S_R}{G} + \varphi\right) + \cos\left(\frac{\gamma S_R}{G} + \frac{\Lambda}{G} \mathcal{V}\right), \quad (26)$$

where S_R denotes the Regge action of the hyperfrustum, $\mathcal{V}$ its volume, G Newton’s constant, Λ the cosmological constant and γ the Barbero-Immirzi parameter.

Thus, this model captures several important generalizations compared to the cuboid model. As signified by the non-vanishing Regge action, frusta configurations allow for curvature to appear. Moreover, more parameters play a role in the dynamics: Newton’s constant G enters here as providing an explicit scale to the representations/areas on the boundary that serve as initial and final states. The cosmological constant Λ is added in Bahr and Rabuffo [110] (analogous to Han [111]) by deforming the vertex amplitude. The parameter α remains as in the cuboid model¹⁷.

After this basic introduction of these models, let us discuss the simplified coarse graining scheme and results.

5.2.1. Coarse Graining Setup and Results

While in spirit the coarse graining setup is similar to the general method outlined in section 4, there are several noteworthy difference and assumptions being made. Firstly, the embedding maps are chosen on geometric grounds instead of determining from the dynamics/the amplitudes. The intuitive idea is, e.g., in case of the hypercuboids, that a coarse hypercuboid arises as a superposition of fine hypercuboids consistent with the coarse hypercuboid geometry. Secondly, the coarse graining flow is computed for one fixed coarse boundary state, not the whole coarse boundary Hilbert space. Thus, in the path integral context, this coarse graining flow is performed for a fixed transition. The third and final assumption includes a projection back onto the

¹⁵ Given a flat space-time decomposed into flat hypercuboid lattices, it should not matter how the 4-volume is split among the building blocks.

¹⁶ The representation j_i, j_f and k must satisfy a relation that is spelled out in Bahr et al. [109].

¹⁷ The Immirzi parameter γ is explicitly kept fixed. Due to the particularities of the Riemannian EPRL/FK model, γ is necessarily a rational number, which significantly impacts the amount of allowed representations. Hence, the models for slightly different γ are substantially different.

original form of the amplitude, such that the flow is formulated as a flow in parameter space of the theory. This projection is defined by comparing expectation values of observables in the coarse and fine calculation. In a sense, the logic of the consistent boundary formulation is inverted: the RG flow assigns a family of amplitudes to different lattices such that expectation values of observables agree on all lattices. Instead, we derive the RG flow by identifying theories/parameters across lattices for which the expectation values of some (sensitive) observables agree. In case of the cuboids, which are just given by the parameter α , this would read:

$$\langle \mathcal{O} \rangle_{b'}^{\alpha'} \approx \langle \mathcal{O} \rangle_b^{\alpha}, \quad (27)$$

which defines the flow $\alpha' \rightarrow \alpha$ from fine b' to coarse b . Note that all these three assumptions are strong simplifications that need to be lifted to verify their validity.

Let us first discuss the setup for coarse graining in the quantum cuboid model reported in Bah and Steinhaus [112]. Consider two hypercuboids glued together along a common 3D cuboid. The total geometry of both hypercuboids is fixed in the coarse boundary state, i.e., the total area of each coarse face is fixed, yet the distribution of 4-volume among the two cuboids fluctuates. Obviously, the expectation value of the volume of a single hypercuboid is always exactly half of the total volume, yet its variance depends sensitively on the parameter α . Thus, for fixed coarse boundary state, one studies the variance of a single coarse in the coarse and in the fine case, where each of the hypercuboids is subdivided into 16. The geometric embedding map is prescribed such that the fine areas sum up to the coarse area. This setup and observable is particularly interesting since it is closely connected to the Abelian subgroup of diffeomorphisms that can be represented in the quantum cuboid model.

With this setup, the variance of the 4-volume is computed in both the coarse and fine case for various α/α' , respectively. In both cases the observable is monotonously decreasing and both curves intersect once in the value α^* . This particular value of α defines a fixed point of the renormalization group flow $\alpha' \rightarrow \alpha$, which is repulsive, i.e., $\alpha > \alpha'$ for $\alpha' > \alpha^*$ and $\alpha < \alpha'$ for $\alpha' < \alpha^*$. Thus, the fixed point also separates two phases, which are dominated by different configurations. For $\alpha < \alpha^*$, small representations j are preferred, such that subdivided faces contain one large area and several small ones. In contrast for $\alpha > \alpha^*$, the configuration dominates in which a face is equally subdivided since then all spins are as large as possible. Remarkably, the value α^* is close to the one at which diffeomorphism symmetry is almost restored [36]. This result together with the repulsive behavior at the phase transition indicate that this transition might be of second order, and that on it the subgroup of diffeomorphisms might be restored.

The same calculation is repeated for different coarse boundary states in Bahr and Steinhaus [113] and results in the same qualitative behavior, yet the position of the fixed point changes. Thus, we do not include the exact value. This result sheds a light on the possible interpretation of these results. Since the coarse boundary state is kept fixed, this coarse graining derives a family of amplitudes on a family of lattices for this specific

transition. Therefore, it contains the information whether and for which parameters the regulator/the lattice can be removed and the results are consistent (within the given approximations and truncations). Note however that this is a weaker condition than the coarse graining flow defined in section 4, which refers to all transitions/boundary states. In a sense, the fixed state becomes part of the observable for which the coarse graining flow is defined. That way it provides first insights of coarse graining flow in a truncated theory space.

A similar analysis of the coarse graining flow is performed for frusta spin foams in Bahr et al. [114] with a slightly different setup. Here the boundary is made up of two parts, an initial and a final 3D spatial cubulations each prescribed by a single representation j_i and j_f , which are chosen to be equal. Again, the goal is to compute expectation values of observables in a fine and a coarse setting and define a renormalization group flow in parameter space (α, G, Λ) such that these observables agree. In Bahr et al. [114] a few different setups are examined, here we just discuss the main result of the RG flow in three dimensional parameter space.

Due to the high symmetry of frusta configurations, the lattice is prescribed by spatial and temporal subdivisions. The former fixes the fineness of the spatial cubulation while the latter determines the number of time steps. The coarse lattice then has two spatial subdivisions, i.e., 4^3 spatial cubes, and one intermediate time step. The fine lattice has one more spatial and temporal subdivision, i.e., 8^3 cubes and two intermediate time steps. The boundary states in both settings are straightforwardly related by requiring that the total 3D spatial volume encoded in initial/final state agrees in both settings. Moreover, the total “height”/“time” is fixed in both settings as well as each time step is chosen to be equal. That way, only the intermediate spatial volume is integrated over, while the side panels are fixed, which greatly reduces the numerical cost.

In order to derive a renormalization group flow in a parameter space with three parameters, one must consider at least three observables. In Bahr et al. [114], the 3D spatial volume of the intermediate slice, its variance as well as the total 4D volume are considered. The expectation values for all these observables are computed in a range of all parameters G , Λ , and α and compared for both settings. Then a coarse graining flow is derived by matching theories with the smallest relative error of observables¹⁸. Under this premise, indications for a fixed point around $\alpha^* \approx 0.677$, $G^* \approx 0.037$, and $\Lambda^* \approx 0.08$ are found. While the exact numerical values are less relevant and most likely subject to change for different boundary states, qualitatively the numerics indicate that this fixed point has one repulsive and two attractive directions. As for the cuboid case, the repulsive direction appears to be (mostly) related to the parameter α , while G and Λ seem to be the attractive directions. In standard lore, this would imply that both G and Λ are irrelevant couplings and fixed by the RG trajectory.

¹⁸Moreover, one is only comparing theories that are “close” in parameter space. This is justified since one is mainly interested in fixed points of the coarse graining flow, given the truncations introduced in the model.

At first sight, this result appears to be at odds with results in asymptotic safety [115], where both G and Λ correspond to free parameters/relevant directions. However, note that this setting here is significantly different: due to considering only a specific transition, a scale is introduced into the system, which is not changed by the coarse graining flow. Thus, in contrast to asymptotic safety where one derives theories at different scales, this coarse graining flow teaches us whether and for which parameters the regulator/lattice might be removed. In this sense, the fixed point gives the correct discretization independent amplitude (given the introduced approximations) for this specific transition. So G^* and Λ^* mark the correct parameters for one specific transition and are thus irrelevant in this flow, yet they might correspond to relevant directions when different scales are related.

5.2.2. Numerical Methods

A short comment on the numerical methods is in order. In the semi-classical, restricted spin foam models, Monte-Carlo and numerical integration techniques are used [107]. This works particularly well for the quantum cuboid model, where the action vanishes and the amplitudes do not show an oscillating behavior. Nevertheless, also for the frusta models, which feature oscillating amplitudes, can be explored with these methods with slower convergence. In general, convergence slows down for higher dimensional integrals and larger discretizations, such that this method appears to be feasible for systems with a few building blocks and symmetries that reduce the amount of degrees of freedom.

5.3. Semi-Classical Continuum Limit

Before concluding this review, we would like to briefly discuss the semi-classical continuum limit approach [116, 117], since it aims at defining a flow across a family of triangulations and might at first sight be similar to the restricted path integral method.

This approach discussed in the papers [116, 117] aims at defining a semi-classical continuum limit for spin foam models in the following sense. As mentioned before, it is a well-established result that one obtains (area) Regge calculus in the asymptotic expansion of spin foam vertex amplitudes [21–25]. Often this is called a semi-classical limit, by scaling all representations $j \rightarrow \lambda j$ by a parameter λ . In Han et al. [117] a Gaussian weight is introduced into this semi-classical limit that suppresses non-length-Regge like geometries, i.e., geometries prescribed by areas that do not correspond to triangulation given by edge lengths. The parameter for this Gaussian weight is δ , where $\delta \rightarrow 0$ removes this weight. These systems are studied in the regime $\lambda \gg \delta^{-1} \gg 1$, such that the semi-classical formula is valid and higher curvature corrections (in the deficit angle) are suppressed.

Essentially the idea is to define a continuum limit as in Regge calculus: for a sequence of triangulations $\mathcal{K}_N$ of the same manifold continuum general relativity is restored if all lengths and deficit angles converge to zero as $N \rightarrow \infty$. In Han et al. [117] it is argued that this is achieved for particular scaling relations for λ , δ , and μ , where μ rescales the Planck length. Thus, one defines a flow across triangulations in this parameter space, where in the limit $N \rightarrow \infty$, both areas and deficit angles converge to zero.

Invoking the continuum limit of Regge calculus, it is argued that general relativity is obtained in this limit.

The existence of such a regime, where one obtains general relativity as the continuum limit of Regge calculus, would be intriguing and it is suggestive to think it should exist, given the close relation of spin foam models and Regge calculus. However, there are several points that must be considered before this can be confirmed: firstly, the assumptions and modifications made that must be carefully cross-checked. Most strikingly, a term that suppresses non-Regge like geometries is not present in spin foam models and one might argue that such a role ought to be already implemented in the simplicity constraints. Secondly, the conditions under which the formulas are valid are highly specific and it must be validated whether these are satisfied in generic situations. Finally, the defined flow of parameters is not dynamical, in the sense that it is not derived by relating dynamics across different triangulations. Thus, it is not clear whether this continuum limit gives well-defined continuum dynamics.

6. OUTLOOK: TOWARD RENORMALIZATION IN 4D

In this article we provide a detailed review of coarse graining in spin foams at the conceptual and practical level. Attentive readers notice that these methods have not been applied yet to the full 4D theory, e.g., the EPRL/FK in the Riemannian or Lorentzian setting, and it is currently out of reach. In this outlook, we would like to discuss the open issues and questions that need to be addressed.

A first point, which is relevant for all calculations performed in spin foam quantum gravity, is the computability of spin foam amplitudes, more precisely the vertex amplitude. As the amplitude associated to a 4D building block, it is the centerpiece of the theory and the most intricate to compute. Analytical formulas are known for the asymptotic expansion of the amplitude [21–25], where the boundary data is given by coherent states peaked on classical discrete geometries. However, these results are not valid for small representations, the quantum regime of the theory. To compute the amplitude in this regime requires numerical techniques, e.g., by explicitly contracting intertwiners to obtain the vertex amplitude. Significant progress was made in recent years for the Lorentzian EPRL model in Donà et al. [28, 29, 45] using the results from [118]. Nevertheless, these calculations require significant numerical resources, which makes it difficult to explore systems with multiple vertex amplitudes. Two ideas might be helpful in exploring larger 2-complexes: Firstly, storing computed vertex amplitudes, e.g., for an orthonormal basis of intertwiners, in an open-data database, such as the “Encyclopedia of Quantum Geometries”¹⁹ would make them accessible to interested researchers and avoid computing the same amplitudes multiple times. The second idea relies on the fact that the asymptotic formula well approximates the vertex amplitude for fairly small representations, $j \sim 10$ for a 4-simplex in the Riemannian EPRL model [119]. Exploiting this

¹⁹<https://zenodo.org/communities/enquegeo/?page=1&size=20>.

fact could lead to an efficient hybrid algorithm, similar to the idea used in loop quantum cosmology [120], that only uses the costly to compute quantum amplitude in case the asymptotic formula is not accurate.

An alternative route toward studying spin foams with multiple simplices lies in defining simplified models. In section 5.2 we review one example for such models, namely restricted spin foam models. Instead of exploring the full spin foam path integral, only a subset of configurations is explored using the asymptotic formula. Thus, the number of degrees of freedom is drastically reduced and the issue of exactly computing the vertex amplitude is circumvented, which makes it possible to renormalize these models. Clearly, as next steps these restrictions need to be lifted in order to explore more of the dynamics of the theory. This could either be by allowing more configurations in the path integral, see e.g., [121], or by going beyond the asymptotic formula and including the full vertex amplitude. Recently, another simplified model has been constructed in a similar direction [122]. Again, the asymptotic formula of the vertex amplitude is invoked to define a simplified vertex amplitude. Special emphasis is given to an implementation of simplicity constraints akin to spin foam models as weak conditions on 3D dihedral angles, which might give new insights into spin foam dynamics for large 2-complexes. Note that this model does not restrict the allowed configurations in contrast to the restricted models discussed in this review.

The most holistic approach to coarse graining spin foam models, tensor network renormalization discussed in detail in section 5.1, faces two main challenges when going to 4D. One is the increased complexity of the amplitude which results both in larger memory cost as well as computational time. Related to this is the second issue, how to define a tensor network algorithm for systems with infinitely many configurations or continuous variables like the 4D spin foam models defined for Lie groups. A solution to the former challenge might lie in defining a representation of the model suited for renormalization, similar to the fusion basis in 3D [49, 100]. Alternative formulations of 4D models are investigated, e.g., in [123, 124]. Using observables might again serve as a guiding principle to find such representations. The second issue might be tackled in a similar direction, where a rewriting of the model might lead to an efficient tensor network description. One such example is [125], where the renormalization group flow of ϕ^4 scalar field theory

is accessible for tensor network methods by performing a simple transformation.

Another important research direction, on which renormalization and coarse graining can shed a new light, is matter coupling in spin foam quantum gravity. Since spin foams are a purely gravitational theory, matter degrees of freedom must be added in order to adequately describe the universe. Different ways to couple matter to spin foams exist in the literature [126–131], yet the intriguing dynamics of the coupled matter gravity system are hardly explored. Applying a coarse graining scheme to the combined system allows us to renormalize matter and gravitational degrees of freedom at the same time, uncovering the phase diagram of the system. This idea is realized for a simplified toy model in Steinhaus [93]. Without a question, a system consisting of both spin foam and matter degrees of freedom is more difficult to study than the former alone. Nevertheless, adding matter to simplified models might be accessible and lead toward intriguing new features and insights, e.g., it would be interesting to see how the matter sector influences the quantum gravitational theory as in Donà et al. [132].

Beyond the methods discussed in the review, ideas from other fields and approaches to quantum gravity might help us advance coarse graining in spin foam models to 4D. These might be novel numerical techniques, like deep learning, or well-established ones like Monte Carlo methods, which might be efficiently applicable in certain settings.

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Why the Cosmological Constant Seems to Hardly Care About Quantum Vacuum Fluctuations: Surprises From Background Independent Coarse Graining

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Background Independence is a *sine qua non* for every satisfactory theory of Quantum Gravity. If one tries to establish a corresponding notion of Wilsonian renormalization, or coarse graining, it presents a major conceptual and technical difficulty usually. In this paper, we adopt the approach of the gravitational Effective Average Action and demonstrate that, generically, coarse graining in Quantum Gravity and in standard field theories on a non-dynamical spacetime are profoundly different. By means of a concrete example, which, in connection with the cosmological constant problem, is also interesting in its own right, we show that the surprising and sometimes counterintuitive implications of Background Independent coarse graining are neither restricted to high energies nor to strongly non-perturbative regimes. In fact, while our approach has been employed in most studies of Asymptotic Safety, this particular ultraviolet behavior plays no essential role in the present context.

Keywords: asymptotic safety, background independent quantum gravity, renormalization group, cosmological constant, functional renormalization group

1. INTRODUCTION

The perhaps most remarkable feature of classical General Relativity is its ability to select and to describe the stage upon which all physics, both gravitational and non-gravitational, takes place. This stage, the set of all events, is what we call spacetime and try to model by means of (topological, differentiable, causal, pseudo-Riemannian, ...) manifolds.

(1) The theory of General Relativity complies with the principle of Background Independence, which proclaims that no particular such “stage” should enjoy a privileged status a priori but rather should be a computable result of the dynamics. While this seems to be a natural and almost self-evident requirement for any classical or quantum theory of the physical world, all of our present non-gravitational physical theories violate Background Independence quite explicitly. In particular, the standard model of elementary particle physics is formulated on an externally prescribed and, thus, unexplained Minkowski spacetime.

In the realm of classical physics, it is well-known how to overcome this deficiency and to set up a matter-coupled gravity theory that determines the spacetime metric dynamically along with the matter fields. Historically, corresponding progress at the quantum level has been hampered by the

non-renormalizability of perturbatively quantized General Relativity and the problems in finding a satisfactory microscopic theory of Quantum Gravity of any other sort.

As a result, two issues have gotten mixed up that, however, are quite independent logically, at least as long as no additional information is available. These are

- (a) the difficulty of setting up a non-perturbative fundamental quantum theory of the gravitational (“spin-2”) interaction and
- (b) the problem of repairing, in one way or another, the background dependence of the standard model and similar local quantum field theories on Minkowski space.

The respective classical variants of both problems are solved by General Relativity. A theory of “Quantum Gravity” in the modern sense of the word [1] likewise must address, and ultimately solve, the dynamics-related problem (a) as well as the Background Independence issue (b).

In the past, many discussions failed to appreciate (b) as an additional and independent point on the agenda, which has often led to severe misconceptions. One of them is the widespread prejudice that quantum gravity effects are numerically small and can be neglected for all practical purposes. It was nurtured by the observation that graviton corrections to standard model physics on a rigid Minkowski space tend to be numerically small, classically as well as quantum mechanically (to the extent they are under control). If one believes, however, that the ultimate Quantum Gravity theory is a Background Independent one, this kind of reasoning is flagrantly wrong.

Its deficiency is not so much that it leaves unexplained the Minkowskian spacetime, which we observe on the scale of our laboratories; the real flaw is that it closes its eyes toward the possibility that a state, which looks Minkowskian on laboratory scales, may well possess a completely different (metric, causal, ...) structure on the shorter length scales from which we have no experimental information about gravity and the structure of spacetime. Instead, in a Background Independent theory, this structure is a computable prediction rather than a phenomenological input based upon incomplete experimental data.

(2) In this paper, we advocate the point of view that Quantum Gravity, regarded as a non-perturbative and Background Independent theory, can have substantial implications well beyond the areas envisaged in the past, questions of ultraviolet renormalizability, or tiny loop corrections due to gravitons.

To support this view, we analyzed the gravitational impact of vacuum fluctuations as an exemplary problem. It has the advantage that it allows for an almost perfect separation of the points (a) and (b) on our to do list. In fact, the dynamics-related part (a) is essentially trivial, while a number of unexpected results contradicting traditional beliefs follow from (b), that is, the imposition of Background Independence on an otherwise unspectacular dynamic.

(3) We employ a continuum approach to Quantum Gravity which quantizes pure or matter-coupled metric gravity in terms of the gravitational Average Action, a concept that is both covariant under diffeomorphisms and Background Independent

[2]. This scale-dependent action functional satisfies a functional renormalization group equation. From very early on, it provided strong evidence for the theory’s Asymptotic Safety, i.e., the non-perturbative renormalizability at a non-Gaussian ultraviolet fixed point [2–8], see [9, 10] for a general exposition and [10–25] for later extensions.

As a matter of principle, the *Quantum Einstein Gravity* constructed in this manner shares Background Independence and the choice of the metric as the field variables with General Relativity, though the microscopic action may well turn out different from the Einstein-Hilbert action. (See [9] for a detailed description of the various steps involved in the overall “Asymptotic Safety Program”).

While the gravitational Average Action does depend on a background metric besides the dynamical one, the setting complies with the requirements of Background Independence since the background metric is determined self-consistently by the so called “tadpole equation,” a generalization of Einstein’s equation.

Self-consistent background metrics depend on the RG scale at which the Effective Average Action is evaluated. It follows that “going on-shell” at a given point of the renormalization group (RG) flow requires understanding two types of scale dependencies. First is the (familiar) direct dependence of the EAA on the RG scale, and second is an equally important indirect one associated to the re-adjustment of the self-consistent background metric as a solution to the scale dependent-tadpole condition.

The scale dependence of the Effective Average Action is due to a coarse graining or averaging process on the spacetime manifold. The occurrence of the *indirect* scale dependence is a concrete manifestation of the abstract principle of Background Independence.

(4) In this paper, we give a detailed account of the corresponding notion of *Background Independent coarse graining*, and we illustrate the discussion by studying explicitly the case where the self-consistent background metric is determined essentially by the cosmological constant and its RG flow.

To this end, we begin, in section 2, by reviewing the relevant aspects of the gravitational Effective Average Action (EAA). Then, in section 3, we turn to an object of central mathematical importance, namely the spectral flow induced by the Laplacian of the self-consistent background metric. As we shall see, such spectral flow acquires an explicit scale dependence.

Most importantly, this spectral flow tells us which field modes constitute the degree of freedom of the respective effective field theory at a given RG scale. We discover a surprising, seemingly paradoxical, behavior for a broad class of RG trajectories. A Background Independent theory of (matter coupled) quantum gravity *loses* rather than gains degrees of freedom at increasing energies in contrast with the expectation based on the off-shell formalism.

In section 4, we study to what extent the Background Independent theory can be reformulated as a theory of matter and gravity fluctuations on a *rigid flat space*. We will show that,

in vacuo, such reformulation is possible only for a very short RG time since a “scale horizon” prevents one to go further.

Finally, in section 5 we use the insights gained to critically revisit, and refute, the argument leading to the naturalness problem of the energy density obtained by summing up the zero-point energies in quantum field theory.

Our presentation partly follows [26] to which the reader is referred for further information.

2. THE BACKGROUND INDEPENDENT EFFECTIVE AVERAGE ACTION

In this section we review some relevant properties of the gravitational EAA [2]. We focus mostly on properties that go beyond the standard EAA for matter fields on flat space, see [27–31].

(1) The EAA for Einstein gravity is defined by a functional integral over the metric $\hat{g}_{\mu\nu}$. The integral is then expressed in terms of a fluctuation field $\hat{h}_{\mu\nu}$ and a background field $\bar{g}_{\mu\nu}$. In the case of a linear split, the following relation holds: $\hat{h}_{\mu\nu} = \hat{g}_{\mu\nu} - \bar{g}_{\mu\nu}$. The functional integral is characterized by a diffeomorphism invariant bare action $S[\hat{g}, \dots]$ and a suitable gauge-fixing term together with the associated Faddeev-Popov ghosts C^μ and $\bar{C}_\mu$ [2]:

$$W_k[J; \bar{g}] = \log \int \mathcal{D}\hat{\varphi} \exp \{-S_{\text{tot}}[\hat{\varphi}, \bar{g}] + \int d^4x \sqrt{\bar{g}} J_i \hat{\varphi}^i - \Delta S_k[\hat{\varphi}, \bar{g}]\}, \quad (1.1)$$

where $\hat{\varphi} \equiv (\hat{\varphi}^i) \equiv (\hat{h}_{\mu\nu}, C^\mu, \bar{C}_\mu, \dots)$ is a multiplet of fields, with the dots denoting possible matter fields, and $J \equiv (J_i)$ is a set of sources conjugate to them. On top of the bare action, the gauge-fixing, and the ghost terms, the total action includes a term ΔS_k , the so-called cutoff action, which gives a mass of order k to the modes of $\hat{\varphi}$ which have a (covariant momentum)² smaller than k^2 .

(2) The background metric plays a crucial role in the present approach. By means of $\bar{g}_{\mu\nu}$ one constructs the associated Laplacian $\square_{\bar{g}} \equiv \bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$, with $\bar{D}_\mu$ being the associated Levi-Civita connection. The spectrum of the Laplacian is determined by the following eigenvalue problem:

$$-\square_{\bar{g}} \chi_n(x) = \mathcal{E}_n \chi_n(x). \quad (1.2)$$

By expanding the fields on the associated eigen-modes $\{\chi_n\}$, i.e., $\hat{\varphi}(x) = \sum_n a_n \chi_n(x)$, we can view the functional integral as an integral over the coefficients a_n , $\int \mathcal{D}\hat{\varphi} \equiv \prod_n \int da_n$. The cutoff action can then be expressed as a sum over the eigen-modes:

$$\Delta S_k \propto k^2 \sum_n \int d^4x \sqrt{\bar{g}} R^{(0)} \left(\frac{\mathcal{E}_n}{k^2} \right) \chi_n(x)^2, \quad (1.3)$$

where $R^{(0)}(z)$ satisfies $R^{(0)}(0) = 1$, and $R^{(0)}(\infty) = 0$, and is a monotonically decreasing function which smoothly “crosses

over” near $z = 1$. As a consequence, a field eigen-mode $\chi_n(x)$ associated with an eigenvalue $\mathcal{E}_n$ smaller than k^2 is equipped with an effective mass term $\propto k^2 \chi_n(x)^2$. The other modes remain essentially unaffected. This mechanism provides the IR cutoff that will cause the scale dependence of the EAA. In practice, it is convenient to rewrite (1.3) as $\Delta S_k = \frac{1}{2} \int d^4x \sqrt{\bar{g}} \hat{\varphi}(x) \mathcal{R}_k \hat{\varphi}(x)$, without resorting to an explicit mode decomposition, with the pseudo-differential operator

$$\mathcal{R}_k[\bar{g}] = Z_k k^2 R^{(0)} \left(\frac{-\square_{\bar{g}}}{k^2} \right), \quad (1.4)$$

where Z_k is a matrix acting in field space taking into account the different normalization of the fields.

(3) It is important to emphasize that the eigenvalue problem (1.2), the spectrum $\{\mathcal{E}_n[\bar{g}]\}$, and the set of eigenmodes, $\{\chi_n[\bar{g}](x)\}$, depend on the background metric. This fact will play a crucial role later on.

(4) The gravitational EAA $\Gamma_k[\varphi; \bar{g}]$ is defined as the Legendre-Fenchel transform of $W_k[J; \bar{g}]$ with respect to the sources J_i , holding $\bar{g}_{\mu\nu}$ fixed and subtracting $\Delta S_k[\varphi; \bar{g}]$ from it. The EAA is a functional of the variables “dual” to J , viz. $\varphi \equiv \langle \hat{\varphi} \rangle \equiv (h_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu, \dots)$. The expectation value of the metric fluctuation is given by $h_{\mu\nu} \equiv \langle \hat{h}_{\mu\nu} \rangle = \langle \hat{g}_{\mu\nu} \rangle - \bar{g}_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, with $g_{\mu\nu} \equiv \langle \hat{g}_{\mu\nu} \rangle$.

(5) The path integral representation of W_k allows one to derive a number of properties satisfied by Γ_k , such as BRST- and split-symmetry Ward identities. In particular, one can derive the following exact functional RG equation (FRGE),

$$\partial_t \Gamma_k[\varphi; \bar{g}] = \frac{1}{2} \text{STr} \left[\left(\Gamma_k^{(2)}[\varphi; \bar{g}] + \mathcal{R}_k[\bar{g}] \right)^{-1} \partial_t \mathcal{R}_k[\bar{g}] \right]. \quad (1.5)$$

At least superficially, it has the same appearance as for matter theories [29–33]. Moreover, the following source-field relation (“effective Einstein equation”) holds

$$\frac{1}{\sqrt{\bar{g}}(x)} \frac{\delta \Gamma_k[\varphi; \bar{g}]}{\delta \varphi^i(x)} + \mathcal{R}_k[\bar{g}]^i_j \varphi^j(x) = J^i(x). \quad (1.6)$$

Instead of the pair $(h_{\mu\nu}, \bar{g}_{\mu\nu})$ one may employ $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ as two independent variables and define

$$\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu] \equiv \Gamma_k[h_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu; \bar{g}_{\mu\nu}]. \quad (1.7)$$

When setting the ghost fields to zero, $\xi = \bar{\xi} = 0$, we write $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}] \equiv \Gamma_k[h_{\mu\nu}; \bar{g}_{\mu\nu}]$, which is the proper vertex generating functional for the 1PI correlators of $\hat{g}_{\mu\nu}$ ¹. In this work, we limited ourselves to consider the EAA defined in this section. It could be interesting to extend our study to related functionals, see in particular [46], and pinpoint possible advantages and disadvantages of each choice.

For further details on the EAA, we refer to [2] and the comprehensive account in [9].

¹More general composite operators $O(\hat{g}_{\mu\nu})$ can be included in the gravitational EAA [34–38] by coupling them to independent sources [39–45].

3. SCALE-DEPENDENT SPECTRA AS A DIAGNOSTIC TOOL

(1) In our argument, a central role will be played by the eigenbasis of $\square_{\bar{g}}$, henceforth denoted by $\Upsilon \equiv \{\chi_n\}$. The eigenmodes satisfy the $\bar{g}$ -dependent eigenvalue equation

$$-\square_{\bar{g}} \chi_n[\bar{g}](x) = \mathcal{E}_n[\bar{g}] \chi_n[\bar{g}](x). \quad (2.1)$$

We refer to modes with $\mathcal{E}_n[\bar{g}] < k^2$ as *IR modes*, while all others are *UV modes*. The lowest lying UV mode is the so called *cutoff mode*, χ_{COM} . Its eigenvalue is either precisely equal to k^2 , or slightly larger if the spectrum is discrete. In the former case:

$$\begin{aligned} \text{UV modes: } \mathcal{E}_n[\bar{g}] &\geq k^2 \\ \text{cutoff mode: } \mathcal{E}_n[\bar{g}] &= k^2 \\ \text{IR modes: } \mathcal{E}_n[\bar{g}] &< k^2. \end{aligned} \quad (2.2)$$

According to this division of the functions $\chi_n(x)$, the eigenbasis $\Upsilon \equiv \Upsilon[\bar{g}]$ decomposes as

$$\Upsilon[\bar{g}] = \Upsilon_{\text{UV}}[\bar{g}](k) \cup \Upsilon_{\text{IR}}[\bar{g}](k) \quad (2.3)$$

with Υ_{IR} and Υ_{UV} containing the IR and UV modes, respectively.

It needs to be emphasized that the decomposition (2.3) depends not only on the scale, k , but also on the background metric. Hence, dealing with a fixed functional

$$(\varphi, \bar{g}) \mapsto \Gamma_k[\varphi; \bar{g}], \quad (2.4)$$

the attribute of being “UV” or “IR” depends on the $\bar{g}_{\mu\nu}$ -argument the functional is evaluated at. In particular the set of quantum numbers n_{COM} that characterizes the cutoff mode $\chi_{\text{COM}} \equiv \chi_n|_{n=n_{\text{COM}}}$ depends on the background metric:

$$n_{\text{COM}} \equiv n_{\text{COM}}[\bar{g}](k). \quad (2.5)$$

The spectrum of $-\square_{\bar{g}}$ is schematically represented in **Figure 1** together with the cutoff mode at $k = k_1$.

(1) $\Upsilon_{\text{IR}}[\bar{g}](k)$ and effective field theory. In the EAA-based quantization on a k -independent background metric the classification of the eigen-modes according to (2.3) can be interpreted physically:

- At a scale $k = k_1$ the effect of the modes belonging to $\Upsilon_{\text{UV}}[\bar{g}](k_1)$ is encoded in the running (i.e., scale dependent) couplings that parametrize $\Gamma_{k_1}[\varphi; \bar{g}]$. Essentially, the modes in $\Upsilon_{\text{UV}}[\bar{g}](k_1)$ have been “integrated out.”
- At the scale k_1 , the running couplings do not take into account the fluctuations of the modes in $\Upsilon_{\text{IR}}[\bar{g}](k_1)$, i.e., these modes have not been integrated out yet. It follows that Γ_{k_1} can be interpreted as an *effective field theory* that governs these modes at scales close to k_1 .

The term “effective field theory” has the following meaning for us. When employing the action functional Γ_{k_1} to compute some observable, only the modes belonging to $\Upsilon_{\text{IR}}[\bar{g}](k_1)$ remain to

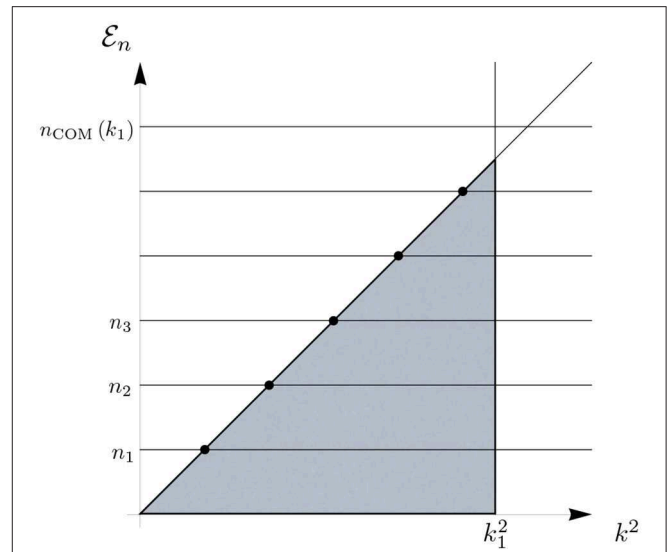


FIGURE 1 | Representation of the spectral flow of $-\square_{\bar{g}}$ constructed via a scale independent background metric. This schematic representation shows a trivial spectral flow with the horizontal lines representing k -independent eigenvalues and the diagonal representing the identity $k^2 \mapsto k^2$. The black dots denote the intersection points at which a mode is “integrated out.” At the scale k_1 , the IR degrees of freedom $\Upsilon_{\text{IR}}[\bar{g}](k_1)$ are associated to the eigenvalue lines passing through the shaded triangle.

be quantized. Therefore, *the scale k_1 plays the role of an UV cutoff from an effective field theory point of view*. All modes the effective field theory governs have eigenvalues $\mathcal{E}_n[\bar{g}] < k^2$.

Let us note that one may “integrate out” the IR-modes in $\Upsilon_{\text{IR}}[\bar{g}](k_1)$ by employing the FRGE and running the RG flow down to a lower scale (eventually $k_1 \rightarrow 0$). However, one may also “integrate out” these modes by any other suitable technique in principle.

(2) **Self-consistent background geometries.** Assume we solved the flow equation and obtained a certain RG trajectory $k \mapsto \Gamma_k$, a curve on theory space. By differentiating the corresponding running action $\Gamma_k[\varphi, \bar{g}]$ with respect to φ , we may compute arbitrary proper vertices from which one may eventually calculate any arbitrary correlation function,

$$\langle \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) \rangle. \quad (2.6)$$

These correlators describe the dynamics of $\varphi \equiv (h_{\mu\nu}, \dots)$ -fluctuations on a classical spacetime whose metric, $\bar{g}_{\mu\nu}$, is enforced by unspecified external means. By appropriately changing the second argument of $\Gamma_k[\varphi, \bar{g}]$ and of its φ -derivatives, the RG trajectory informs us about the fluctuation dynamics on any given background geometry and at all scales.

A quantity of special interest is the one-point function $\langle \hat{h}_{\mu\nu} \rangle_{\bar{g}}$ since it determines the expectation value of the metric operator $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$, i.e.,

$$\langle \hat{g}_{\mu\nu} \rangle_{\bar{g}} = \bar{g}_{\mu\nu} + \langle \hat{h}_{\mu\nu} \rangle_{\bar{g}}. \quad (2.7)$$

In general, the expectation value (2.7) will be quite different from the background metric when the fields $\varphi \equiv (h_{\mu\nu}, \dots)$ are quantized on a randomly chosen geometry.

Now, we go one step further and ask which metric expectation value the quantum system will realize when it is free from all external interferences. More concretely, if we quantize the set of fluctuation fields on a geometry with a given $\bar{g}_{\mu\nu}$, we can ask which background(s) they would “like” most, in the sense that they dynamically produce a $\hat{g}_{\mu\nu}$ -expectation value that agrees precisely with the background metric prescribed. Such geometries are the called *self-consistent geometries*, and their metric is denoted $(\bar{g}_k^{\text{sc}})_{\mu\nu}$. Hence,

$$(\hat{g}_{\mu\nu})_{\bar{g}} = \bar{g}_{\mu\nu} \Leftrightarrow \langle \hat{h}_{\mu\nu} \rangle_{\bar{g}} = 0 \quad \text{if } \bar{g} = \bar{g}_k^{\text{sc}}. \quad (2.8)$$

Self-consistent background metrics can be found from the condition of a vanishing fluctuation one-point function, for historic reasons termed the *tadpole condition*. It comprises the equation

$$\frac{\delta}{\delta h_{\mu\nu}} \Gamma_k [h, \dots; \bar{g}] \Big|_{h=0, \bar{g}=\bar{g}_k^{\text{sc}}} = 0 \quad (2.9)$$

coupled to similar conditions where the differentiation is with respect to the other fluctuation fields.

In the following, it is of central importance that, generally, self-consistent backgrounds depend on the RG scale. By (2.9), it is clear that solutions $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ inherit a certain k -dependence from Γ_k .

(3) Generalized RG trajectory. Henceforth, we assume that we solved the RG flow equation and have a certain trajectory $k \mapsto \Gamma_k$ in our hands. Furthermore, we assume that, using this running action as an input, we solved the tadpole equation and found a family of metrics $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ labeled by k , or, stated differently, a curve in the space of metrics. It is natural therefore to refer to the map

$$k \mapsto \left(\Gamma_k, (\bar{g}_k^{\text{sc}})_{\mu\nu} \right) \quad (2.10)$$

as a *generalized RG trajectory* and to visualize it as a parametrized curve in the product of theory space with the space of metrics.

(4) Spectral flow. At every point of the generalized RG trajectory, we use the metric $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ in order to construct the associated Laplacian $\square_{\bar{g}_k^{\text{sc}}}$. This results in a family of Laplacians whose members are distinguished by their respective value of k . Each family member gives rise to its own eigenvalue equation. It reads, for every value of k ,

$$-\square_{\bar{g}_k^{\text{sc}}} \chi_n(x, k) = \mathcal{F}_n(k) \chi_n(x, k). \quad (2.11)$$

Solving the family of eigenvalue problems (2.11), we obtain a “curve of spectra,” i.e., a *spectral flow*,

$$k \mapsto \{ \mathcal{F}_n(k) \}, \quad (2.12)$$

and the corresponding eigenbasis, $\{ \chi_n(\cdot; k) \}$.

If $\bar{g} = \bar{g}_k^{\text{sc}}$, the effective field equations implied by Γ_k admit the simple solution $h = 0$. The correlation functions (2.6) are thus taken “on-shell” when we evaluate them, separately for every k , at the self-consistent metric and simultaneously set $h = 0^2$. In this sense, the graviton n -point functions, for instance,

$$\langle \hat{h}(x_1) \cdots \hat{h}(x_n) \rangle_{\text{on-shell}} \quad (2.13)$$

enjoy the property of being on-shell for each scale separately.

(5) Direct vs. indirect k -dependence. While on-shell at all points along the generalized RG trajectory, the n -point functions (2.13) possess a rather complicated scale dependence in general, which has two independent sources: the (naively expected) *direct scale dependence*, stemming from the k -dependence of the Γ_k , and the *indirect scale dependence*, caused by the continually changing, dynamically selected background metric.

The indirect scale dependence makes the physical interpretation of the coarse graining procedure rather non-trivial in general and striking surprises can occur, as we shall see.

(6) At the heart of Background Independent coarse graining. Recall that, when still “off-shell,” the Effective Average Action maps k -independent arguments onto a k -dependent number,

$$\Gamma_k : (h, \bar{g}) \mapsto \Gamma_k [h; \bar{g}], \quad (2.14)$$

such that k^2 is a cutoff in the spectrum of an operator, namely, $-\square_{\bar{g}}$, which is determined by the functional’s second argument, $\bar{g}$.

By taking Γ_k and its h -derivatives on-shell, this operator gets concretely specified as $-\square_{\bar{g}_k^{\text{sc}}}$, which possesses an explicit parametric dependence on k . k^2 thus appears to be a *cutoff in the spectrum of an operator that is k -dependent in itself*.

With this somewhat confusing and paradoxical-looking situation, we have reached the very core of the Background Independent coarse graining: since physics (i.e., on-shell data) may not depend on any distinguished metric that was chosen *ad hoc*, spectral information of physical relevance is bound to come from *operators which are determined dynamically* [9].

The following steps are aiming at a first physical understanding of what it means to “coarse grain” under such conditions in a fully Background Independent fashion.

(7) Local IR-UV separation along the trajectory. We may assume that the eigenvalue problems (2.11) have been solved all the way along the generalized RG trajectory so that the spectral flow

$$k \mapsto \{ \mathcal{F}_n(k), \chi_n(\cdot; k) \} \quad (2.15)$$

can be analyzed explicitly. At first, we determine the cutoff modes of all the spectra occurring on the trajectory. At a given scale, say $k = k_1$, we require that

$$\mathcal{F}_{n_{\text{COM}}}(k) \Big|_{k=k_1} \stackrel{!}{=} k_1^2, \quad (2.16)$$

²Alongside with $h = 0$ we also fix other fluctuation fields in the multiplet $\varphi = (h, \dots)$, e.g., the ghosts and matter fields, according to their solution from the coupled field or tadpole equations.

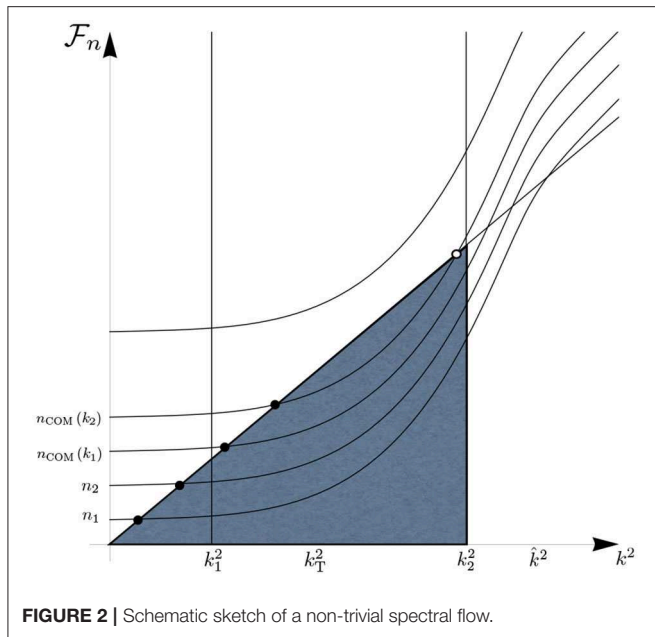


FIGURE 2 | Schematic sketch of a non-trivial spectral flow.

and solve this condition for $n_{\text{COM}} \equiv n_{\text{COM}}(k_1)$. Equation (2.16) determines the label associated to the cutoff mode in the spectrum of the (on-shell!) background Laplacian at a given point of the theory space, which is visited by the RG flow when $k = k_1$. When the spectrum is discrete, the cutoff mode corresponds to the smallest eigenvalue $\mathcal{F}_{n_{\text{COM}}}(k_1)$ equal to or above k_1^2 .

Furthermore, we distribute the modes of the eigenbasis $\{\chi_n(\cdot; k_1)\}$ over two sets, putting those with eigenvalues $\mathcal{F}_n(k_1) \geq \mathcal{F}_{n_{\text{COM}}}(k_1)$ and $\mathcal{F}_n(k_1) < \mathcal{F}_{n_{\text{COM}}}(k_1)$ into the sets $\Upsilon_{\text{UV}}(k_1)$ and $\Upsilon_{\text{IR}}(k_1)$, respectively.

By performing the outlined algorithm for all k_1 , one can construct the map $k \mapsto n_{\text{COM}}(k)$ or, more explicitly, $k \mapsto \chi_{n_{\text{COM}}}(\cdot; k)$. In the same manner, we can construct the “curves” $k \mapsto \Upsilon_{\text{UV/IR}}(k)$, and the decomposition of the eigenbasis,

$$\{\chi_n(\cdot; k)\} = \Upsilon_{\text{UV}}(k) \cup \Upsilon_{\text{IR}}(k) \quad (2.17)$$

which replaces (2.3) when going on-shell.

(8) Spectral flow and mode reshuffling. In Figure 2, we sketch a typical spectral flow of the kind that will arise later in our example. The trajectory’s curve parameter k is on the horizontal axis, while two specific values, $k = k_1$ and $k = k_2$, are represented by two vertical lines. Figure 2 is analogous to Figure 1, the difference being that the eigenvalues $\mathcal{E}_n$ are replaced by $\mathcal{F}_n(k)$. We refer to the k -dependence of the spectrum $\{\mathcal{F}_n(k)\}$ as the spectral flow induced by the (scale dependent) self-consistent background metric.

(8.1) The cutoff mode at the scale $k = k_1$ can be determined as follows. First one identifies all the intersection points between the graphs $\mathcal{F}_n(k)$ and the vertical line $k = k_1$. The modes are then separated into two sets, i.e., $\Upsilon_{\text{UV}}(k_1)$ and $\Upsilon_{\text{IR}}(k_1)$, according to whether the intersection point lies above or exactly on the diagonal or below the diagonal, respectively.

The cutoff mode is defined as the mode associated to the smallest eigenvalue in $\Upsilon_{\text{UV}}(k_1)$. At the scale $k = k_1$, the cutoff mode is labeled by $n_{\text{COM}}(k_1)$ as illustrated in Figure 2.

The mode carrying the label $n = n_{\text{COM}}(k_1)$, k_1 fixed, is associated to a scale dependent eigenvalue $\mathcal{F}_{n_{\text{COM}}(k_1)}(k)$. For values $k \neq k_1$, this mode plays no special role in general.

(8.2) As we explained, the effective action $\Gamma_k|_{k=k_1}$ governs the degrees of freedom associated to the modes in $\Upsilon_{\text{IR}}(k_1)$. These latter modes correspond to the eigenvalues passing within the shaded area to the left of the vertical $k = k_1$ -line in Figure 2. At scales lower than k_1 these eigenvalues intersect the diagonal only once. The intersection is marked by a black dot similarly to the case of constant $\mathcal{E}_n$ displayed in Figure 1.

This behavior can be interpreted as follows. By lowering k_1 , the vertical $k = k_1$ -line sweeps over one of the black dots on the diagonal. This implies that the associated mode is moved from $\Upsilon_{\text{IR}}(k_1)$ to $\Upsilon_{\text{UV}}(k_1)$. At first sight, one may suspect that this is what has to be expected in general since by lowering the cutoff one “integrates out” more and more modes.

(8.3) However, this picture changes dramatically at higher scales. Let us consider the scale $k = k_2$, in Figure 2. As we shall see explicitly later on, the crucial point is that, if the cosmological constant increases with k rapidly enough, then the graph of an eigenvalue $\mathcal{F}_n(k)$ may intersect the diagonal more than once below k_2 . In Figure 2, we observe eigenvalues both entering and exiting from the shaded area to the left of the vertical k_2 -line, and the intersection points are marked by black dots or open circles accordingly. By lowering k_2 , it is possible for the k_2 -line to sweep over an open circle. This implies that a certain mode has changed its UV/IR status. However, this time, the mode moved from $\Upsilon_{\text{UV}}(k_2)$ to $\Upsilon_{\text{IR}}(k_2)$!

At first glance, such behavior appears paradoxical and may seem “unphysical.” Indeed, we normally expect that, by integrating out further field modes, we are actually relocating them from the set Υ_{IR} to the set Υ_{UV} . In the present case, however, the opposite happens and a UV-mode in Υ_{UV} becomes an IR-mode in Υ_{IR} by lowering the RG scale.

(8.4) This conundrum is solved by recalling that the standard expectation, i.e., $(k \text{ lowered}) \Leftrightarrow (\text{mode transfer } \Upsilon_{\text{IR}} \rightarrow \Upsilon_{\text{UV}})$, is valid for k -independent (off-shell) field arguments of the functional $\Gamma_k[\varphi; \bar{g}]$. It must be emphasized that, during the computation of the EAA, this expectation holds true also in the present case. Such unexpected spectral behavior is due to the fact that, when the fields are taken on-shell and one employs the self-consistent background, they acquire a further scale dependence, which causes this non-trivial spectral flow.

It follows that there is nothing “unphysical” if, by lowering the value of k_2 , one observes a transition $\Upsilon_{\text{UV}} \rightarrow \Upsilon_{\text{IR}}$. Actually, such transition encodes the physically important fact that the effective field theory described by Γ_k has gained a degree of freedom, whose fluctuations have not been taken into account in the values of the renormalized couplings in Γ_k .

As displayed in Figure 2, by lowering k further, the new IR-mode crosses the diagonal again and eventually becomes a UV-mode.

(9) Illustrative example: Einstein-Hilbert truncation. Let us pause for a moment and introduce an approximation that will be invoked for illustrative purposes in the following.

We truncate the gravity action to the Einstein-Hilbert form, and we either consider pure gravity, or matter coupled gravity in situations where the matter stress tensor in the effective field equation plays no significant role (at least at the level of the qualitative discussion we present here).

As a result, the tadpole condition (2.9) happens to assume the form of the classical Einstein equation in vacuo with a scale dependent cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_k g_{\mu\nu} = 0. \quad (2.18)$$

Herein, $k \mapsto \Lambda_k$ is one of the functions that constitute the RG trajectory on theory space.

- (i) As for the solution $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ to Equation (2.18), we focus on the instructive, yet technically simple, class of *solutions of the scaling type*:

$$(\bar{g}_k^{\text{sc}})_{\mu\nu} = \frac{\Lambda_0}{\Lambda_k} (\bar{g}_0^{\text{sc}})_{\mu\nu}. \quad (2.19)$$

Here $(\bar{g}_0^{\text{sc}})_{\mu\nu}$ is an arbitrary solution to (2.18) for the cosmological constant Λ_0 . (Instead of the reference point $k = 0$, any other would do as well).

- (ii) It is easy to determine the spectral flow caused by the k -dependence of the self-consistent background metric (2.19):

$$\mathcal{F}_n(k) = \frac{\Lambda_k}{\Lambda_0} \mathcal{F}_n(0), \quad (2.20)$$

$$\chi_n(x; k) = \chi_n(x; 0). \quad (2.21)$$

Moving along the generalized RG trajectory, the eigenvalues $\mathcal{F}_n(k)$ get rescaled, while the eigenfunctions remain unaltered.

- (iii) For every fixed spectrum occurring along the trajectory we must determine the cutoff mode, i.e., the label $n_{\text{COM}} \equiv n_{\text{COM}}(k)$. It is easy to show that this can be done by the following two-step algorithm:

- Determine the cutoff mode in the reference spectrum obtained with Λ_0 . In this case, denote the cutoff by q^2 rather than, as usual, k^2 . Set up the equation

$$\mathcal{F}_{n_{\text{COM}}}(0) \stackrel{!}{=} q^2 \quad (2.22)$$

and solve it for n_{COM} . Denote the result by

$$n_{\text{COM}} \equiv n_{\text{COM}}^0(q), \quad (2.23)$$

thus defining the function $n_{\text{COM}}^0(\cdot)$.

- We would like to solve

$$\mathcal{F}_{n_{\text{COM}}}(k) \stackrel{!}{=} k^2 \Rightarrow n_{\text{COM}} = n_{\text{COM}}(k). \quad (2.24)$$

Upon defining the function $q(k)$ by

$$q(k)^2 = k^2 \frac{\Lambda_0}{\Lambda_k}, \quad (2.25)$$

the solution to the problem (2.24) can be found in terms of the above $n_{\text{COM}}^0(\cdot)$ as follows:

$$n_{\text{COM}}(k) = n_{\text{COM}}^0(q(k)) \quad (2.26)$$

In the discussion of **Figure 2**, we have determined this k -dependence of n_{COM} by graphical means.

(10) Illustrative example: S^4 spacetimes. Assuming a positive cosmological constant, the maximally symmetric solution to the (Euclidean) Einstein equation (2.18) is a sphere S^4 . Its radius $\bar{r}_k$ follows from $R = 12/\bar{r}_k^2 = 4\Lambda_k$, i.e., $\bar{r}_k = \sqrt{3/\Lambda_k}$, implying

$$\bar{r}_k = r_0 \left(\frac{\Lambda_0}{\Lambda_k} \right)^{1/2}. \quad (2.27)$$

The radius $\bar{r}_k$ can be thought of as the Euclidean analog of the Hubble length.

- (i) On S^4 , the eigenmodes of the tensor Laplacian are labeled by a positive integer, n , and a set of further quantum numbers the associated eigenvalue is independent of. The latter generalize the familiar magnetic quantum number m that appears as a label of the spherical harmonics $Y_{l,m}$, i.e., the scalar eigenfunctions on S^2 , while the former is analogous to l which determines the eigenvalue, $l(l+1)$. For not too small values of n , the eigenvalues of the S^4 harmonics, for tensors of any rank, are given by n^2/r^2 , whence for the radius $r = \bar{r}_k$,

$$\mathcal{F}_n(k) \approx \left(\frac{n}{\bar{r}_k} \right)^2 \text{ for } n \gg 1. \quad (2.28)$$

The approximation behind Equation (2.28) is analogous to replacing $l(l+1)$ with l^2 in the S^2 case. Its advantage is that it is valid for tensors of any rank, contrary to the exact formula [47–50]. For the purposes of the present discussion we do not lose any relevant information by specializing for $n \gg 1$. In fact, treating n as a large, quasi-continuous number also helps avoiding a number of inessential technicalities.

- (ii) Using (2.27) and (2.28), the above algorithm yields the following answer for the quantum number of the cutoff mode:

$$n_{\text{COM}}(k) = \bar{r}_0 q(k). \quad (2.29)$$

Herein, $q(k)$ is given by Equation (2.25), which we rewrite in the suggestive form

$$q(k)^2 = \Lambda_0 \frac{k^2}{\Lambda_k} = \frac{k^2}{\lambda_k} \quad (2.30)$$

where $\lambda_k \equiv \Lambda_k/k^2$ is the dimensionless cosmological constant in cutoff units.

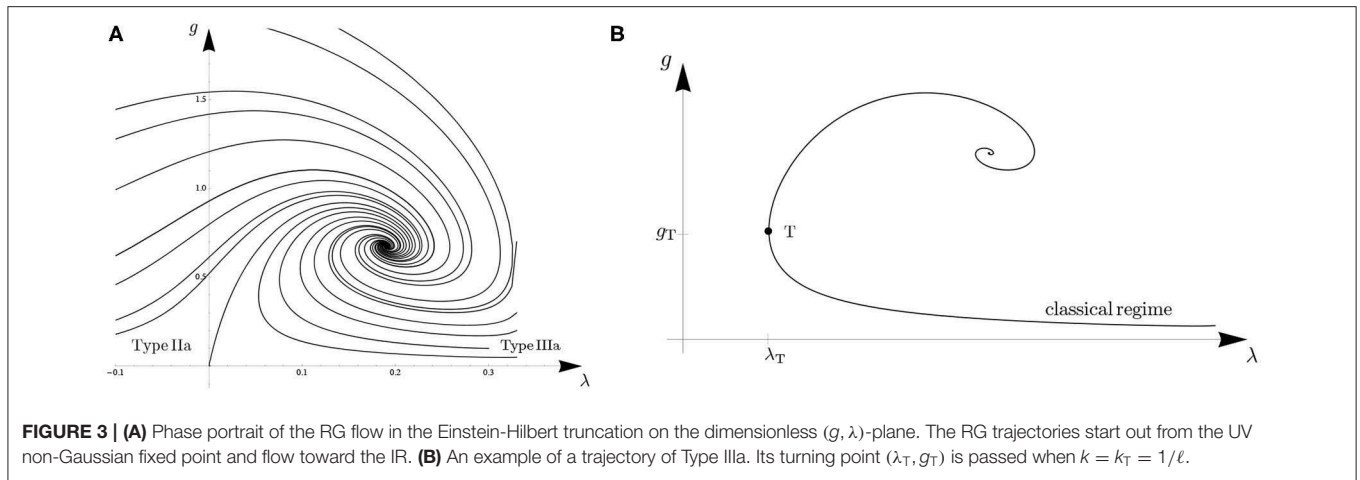


FIGURE 3 | (A) Phase portrait of the RG flow in the Einstein-Hilbert truncation on the dimensionless (g, λ) -plane. The RG trajectories start out from the UV non-Gaussian fixed point and flow toward the IR. **(B)** An example of a trajectory of Type IIIa. Its turning point (λ_T, g_T) is passed when $k = k_T = 1/\ell$.

(11) The generic semiclassical RG trajectory. To go on and study the contents of (2.29), (2.30) we must pick an RG trajectory which then supplies a concrete function $k \mapsto \Lambda_k$. In this paper, we focus on the semiclassical regime below the Planck scale ($k \lesssim m_{\text{Pl}}$), where qualitatively the k -dependence of Λ_k is essentially the same for a large class of trajectories in pure gravity and also in matter-coupled gravity with many different matter systems. It reads, with constants Λ_0 and ℓ ,

$$\Lambda_k = \Lambda_0 [1 + \ell^4 k^4] \text{ for } 0 \leq k \lesssim m_{\text{Pl}}. \quad (2.31)$$

The concomitant running of Newton's constant is trivial, $G_k = G_0 = \text{const.}$ This behavior applies in particular to the semiclassical regime of the Type IIIa trajectories in pure Quantum Einstein Gravity, see **Figure 3**.

There, thanks to Asymptotic Safety, the $k \rightarrow \infty$ behavior is determined by the non-Gaussian UV fixed point, being $\lambda_k \rightarrow \lambda_*$, and so $\Lambda_k \rightarrow \lambda_* k^2 \equiv \Lambda_0 L^2 k^2$. In what follows, use will not be made of this specific UV completion, and many others would do as well for what concerns our main argument. It will only rely on the semiclassical formula (2.31). Despite this, the choice of trajectory, having a positive cosmological constant in the IR, Λ_0 , is essential.

The simple formula (2.31) should be seen as a “caricature” of a generic semiclassical behavior that is precise enough to display the crucial feature of a *turning point* when the trajectory is plotted on the dimensionless $g - \lambda$ -plane. There,

$$\lambda_k = \Lambda_0 \left[\frac{1}{k^2} + \ell^4 k^2 \right], g_k = G_0 k^2, \quad (2.32)$$

and so λ_k is seen to switch from decreasing to increasing when k passes the turning point scale $k_T = 1/\ell$ from below, see **Figure 3B**.

We assume that, on the one hand, k_T is much smaller than the Planck scale, but, on the other, it is much larger than the Hubble parameter at $k = 0$:

$$\Lambda_0^{1/2} \ll k_T \ll m_{\text{Pl}}. \quad (2.33)$$

For illustration's sake, we may fit the formula (2.31) to the values of Λ_0 and G_0 measured in Nature. Up to factors of order unity, this yields

$$k_T \approx 10^{-30} m_{\text{Pl}} \approx 10^{30} H_0 \quad (2.34)$$

with the present Hubble parameter $H_0 \approx \Lambda_0^{1/2}$. Both inequalities in (2.33) are well-satisfied then.

(12) The S^4 family in the semiclassical regime. With Λ_k given by (2.31), the members of the S^4 family of maximally symmetric self-consistent backgrounds have radii

$$\bar{r}_k = \frac{\bar{r}_0}{\sqrt{1 + \ell^4 k^4}} = \frac{\bar{r}_0}{\sqrt{1 + \left(\frac{k}{k_T}\right)^4}} \quad (2.35)$$

where $\bar{r}_0 \equiv \sqrt{3/\Lambda_0}$. In the strictly classical regime ($k \ll k_T$) the radius is essentially constant, $\bar{r}_k \approx \bar{r}_0$, while it decreases rapidly ($\bar{r}_k \propto 1/k^2$) when $k \gg k_T$. As a consequence, the eigenvalues on the S^4 with radius $\bar{r}_k$ are

$$\mathcal{F}_n(k) = \left(\frac{n}{\bar{r}_k}\right)^2 = \left(\frac{n}{\bar{r}_0}\right)^2 [1 + \ell^4 k^4]. \quad (2.36)$$

This spectral flow has the qualitative features anticipated in **Figure 2**.

The k -dependent cutoff quantum number $n_{\text{COM}}(k) = \bar{r}_0 q(k)$ is explicitly known at this point, with (2.30) yielding

$$q(k) = \left(\frac{k^2}{1 + \ell^4 k^4}\right)^{1/2}. \quad (2.37)$$

The functions $q(k)$ and $n_{\text{COM}}(k)$ are plotted in **Figure 4**. They possess a maximum at the turning point scale $k = k_T \equiv 1/\ell$, where $q(k)$ assumes the value

$$q_{\text{max}} = q(k_T) = \frac{1}{\sqrt{2}} k_T. \quad (2.38)$$

Under the condition (2.33), this maximum is situated well within the semiclassical regime. The value of the quantum number

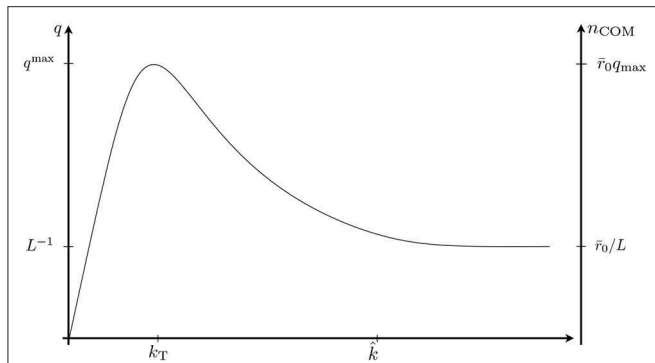


FIGURE 4 | Representations of the functions $q(k)$ and $n_{\text{COM}}(k) = \bar{r}_0 q(k)$ along a trajectory of the Type IIIa. The semiclassical regime extends from $k = 0$ to $k = O(m_{\text{PL}})$. Beyond this point, the Asymptotic Safety result is shown for concreteness, being $q(k) = L^{-1} = \text{const}$, with $L \equiv \sqrt{\lambda_*}/\Lambda_0$.

n_{COM} never can become really large. At the very least in the semiclassical domain, it is bounded above:

$$n_{\text{COM}}(k) \leq n_{\text{COM}}(k_T) = \bar{r}_0 q_{\text{max}} = \bar{r}_0 k_T / \sqrt{2} \text{ for all } k.$$

(2.39)

(13) Fuzzy appearance of spacetime. Recall that the integer n , much like the quantum number l of Y_{lm} , measures the degree of complexity (number of nodes etc.) of the corresponding eigenfunctions. Hence $n_{\text{COM}}(k)$ characterizes the maximum “resolving power” or “fineness” that can be achieved on the self-adjusting 4-sphere with an eigenfunction expansion that is truncated at $n = n_{\text{COM}}(k)$.

Since $n_{\text{COM}}(k)$ is bounded above by its value at the turning point, $n_{\text{COM}}(k_T)$, it follows that *on the family of self-consistent spacetimes, whatever is the value of k , the “resolving power” of the eigenmodes is never perfect.*

The best angular resolution that can be achieved on S^4 is of order $2\pi/n_{\text{COM}}(k_T)$, and this renders spacetime a kind of “fuzzy sphere.” (See also [51–53] for a discussion of a related dynamically generated minimum length, and [54] for the concomitant effect on the entanglement entropy).

(14) Anomalous mode reshuffling explained. Our usual intuition being trained on k -independent metrics, the behavior of $n_{\text{COM}}(k)$ shown in **Figure 4** comes as a surprise: while we expect that by increasing the characteristic momentum of the coarse graining, i.e., k , higher eigenmodes of the Laplacian with shorter wavelength get involved, the opposite happens according to **Figure 4** at scales above the turning point ($k > k_T$). Increasing k leads to a lower cutoff mode then, i.e., a function with less structure (having fewer nodes, etc.).

Another side of the same medal is that, above the turning point, *lowering* k converts UV-modes to IR-modes (rather than vice versa). Hence, at low scales, the effective field theory has *more* degrees of freedom to deal with than at high scales. This is again in conflict with the naive fixed k intuition, which would suggest

that lowering k means “integrating out,” hence a relocation of modes in the opposite direction, $\Upsilon_{\text{UV}} \rightarrow \Upsilon_{\text{IR}}$.

Thus, we observe that the spectral flow at hand, Equation (2.36), does indeed realize the possibility of eigenvalues which, in **Figure 2**, cross the diagonal twice.

Given the explicit form of the eigenvalues in (2.36), we can explain the above “paradoxes” in elementary physical terms:

when k is increased, the radius $\bar{r}_k$ of the self-consistent sphere *shrinks*, and this causes $\mathcal{F}_n(k)$, n fixed, to *grow*. There are two ways of making $\mathcal{F}_n(k)$ large: the familiar one of increasing n at fixed radius and the new one of keeping n fixed, or making it smaller, while decreasing the radius. Above the turning point, this second mechanisms turns out to be the dominant one.

The analysis carried out in this section assumes a positive curvature, the extension of our investigation to the cases of negative curvature or Lorentzian signature is interesting and requires further study (Ferrero R and Reuter M, work in progress).

4. RUNNING VS. RIGID PICTURE OF THE RG EVOLUTION

(1) The familiar “running picture.” Assume we are given a certain solution to the functional RG equation, $\Gamma_k[h, \psi; \bar{g}]$, describing gravity coupled to a set of matter fields, ψ . Then, as for the associated effective field theory, the couplings it encapsulates apply to the “particle physics” of $h_{\mu\nu}$ and the other quanta *when they propagate on the running $(\bar{g}_k^{\text{sc}})_{\mu\nu}$ geometries.*

The high- k matter physics predictions supplied by Γ_k are valid only in conjunction with a high- k gravitational background.

This is the standard way of interpreting the RG trajectories. It refers all “particle physics” to the k -dependent on-shell geometry and is therefore called the “running picture” of the generalized RG trajectory.

(2) The novel “rigid picture” . . . To describe the alternative “rigid picture” let us put ourselves in the place of collider physicists who are able to measure the matter couplings governed by $\Gamma_k[h, \psi; \bar{g}_k^{\text{sc}}]$, but are unable to explore the microscopic spacetime structure. They would find it natural to construct a new action functional, Γ_q , which makes no reference to a running metric and eliminates $\bar{g}_k^{\text{sc}}$ everywhere in favor of the (essentially flat) macroscopic metric $\bar{g}_0^{\text{sc}} = \bar{g}_{k=0}^{\text{sc}}$. In the sought for description only the particle physics runs, while the metric stays fixed, being always $\bar{g}_0^{\text{sc}}$.

Among other changes of an essentially kinematic character, the construction of Γ_q from Γ_k involves re-interpreting the cutoff scale as an eigenvalue of $(-\square_{\bar{g}})$ built from $\bar{g} = \bar{g}_0^{\text{sc}}$ rather than the usual $\bar{g}_k^{\text{sc}}$. It is easy to see that the two operators have the same eigenfunctions and that, if the eigenvalue in the latter case is k^2 , then it equals q^2 in the former. (Note that $\square_{\bar{g}_k^{\text{sc}}} = (\Lambda_k/\Lambda_0) \square_{\bar{g}_0}$ for metrics of the rescaling type).

In this *rigid picture*, therefore, the quantity q plays the same role the usual cutoff k plays in the running picture and, consequently, the notation Γ_q for the new running action. From the perspective of Γ_q , all momenta are proper with respect to the fixed metric $\bar{g}_0^{\text{sc}}$, as desired by the collider physicists.

In order to actually construct the new action functional, we would have to reparametrize the RG time axis,

$$\Gamma_q = \Gamma_{k(q)}, \quad (3.1)$$

which requires inverting the function $k \mapsto q(k)$ to obtain $k = k(q)$. This is impossible though.

In the semiclassical regime a given value $q < q_{\max}$ is associated to two k -values via Equation (2.37), see **Figure 4**. It follows that, globally speaking, the map $k \mapsto q(k)$ is not a valid reparametrization of the whole RG trajectory since it does not provide a diffeomorphism on the RG-time axis.

Locally, however, it is possible to invert Equation (2.37) for either $k < k_T$ or $k > k_T$. The inversion yields the following two maps $k = k(q)$ for $q \in [0, q_{\max}]$:

$$\begin{aligned} k_{\pm}(q) &= \frac{1}{\sqrt{2}\ell^2 q} \left[1 \pm \sqrt{1 - (2\ell^2 q^2)^2} \right]^{1/2} \\ &= \sqrt{2} \frac{q_{\max}^2}{q} \left[1 \pm \sqrt{1 - \left(\frac{q}{q_{\max}} \right)^4} \right]^{1/2}. \end{aligned} \quad (3.2)$$

The functions $k_+(q)$ and $k_-(q)$ joins at $k_{\pm}(q_{\max}) = \sqrt{2}q_{\max} = k_T$ while, for a generic q , the upper branch is given by $k_+(q) > k_T$ and the lower branch by $k_-(q) < k_T$.

(3) ... **and its breakdown.** The non-invertibility of $q(k)$ implies that the rigid picture is applicable from $k = 0$ up to $k = k_T$ only. It breaks down at the turning point, which acts as a sort of horizon in the one-dimensional space of scales [26].

Figure 5 illustrates the role played by this “scale horizon” in connection with the cosmological constant. The action Γ_q includes a term $\propto \Lambda_{k(q)} \int d^4x \sqrt{g_0^{SC}}$, with $\Lambda_{k(q)} \equiv \Lambda_{\text{rigid}}(q)$ the natural scale dependent cosmological constant in the running picture. From $\Lambda_{k(q)} = \Lambda_0 k^2(q)/q^2$, and with (3.2) we obtain the double-valued relation

$$\Lambda_{\text{rigid}}(q) = 2\Lambda_0 \left(\frac{q_{\max}}{q} \right)^4 \left[1 \pm \sqrt{1 - \left(\frac{q}{q_{\max}} \right)^4} \right]. \quad (3.3)$$

The behavior of Λ_{rigid} is displayed in **Figure 5**, with the minus (plus) sign corresponding to the lower (upper) branch of the function.

A hypothetical collider physicist that insists on using the scale q has no problems in interpreting the lower branch of $\Lambda_{\text{rigid}}(q)$ but is not able to go beyond the horizon located at $q = q_{\max}$.

By viewing q as a curve parameter for the RG trajectory, we note that it provides a “good” coordinate on the RG time axis only below the turning point. A different coordinate is needed to go beyond the horizon, an example is offered by k , which is valid globally. The situation is somewhat similar to the usual coordinate horizons in spacetime. In terms of q the rigid picture based on the (perturbative) k_- -branch is valid for momenta $q \leq q_{\max}$.

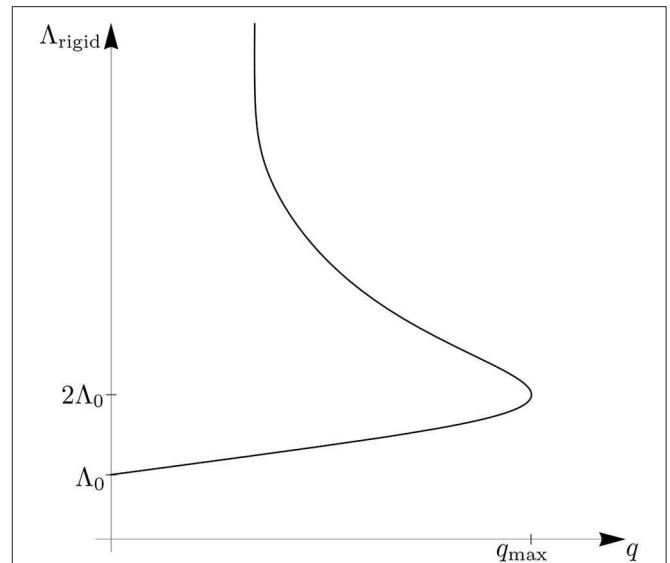


FIGURE 5 | The cosmological constant appearing in the rigid picture's Γ_q in dependence on its natural RG scale q . While we are able to consistently interpret the diagram's lower branch ($\Lambda_0 \leq \Lambda_{\text{rigid}} \leq 2\Lambda_0$), the “scale horizon” at $q_{\max}$ prevents us from passing to the upper branch straightforwardly.

5. IMPLICATIONS FOR GRAVITATING VACUUM FLUCTUATIONS AND THE COSMOLOGICAL CONSTANT

The cosmological constant has been puzzling physicists for a long time [55–58]. The problem involves classical and quantum aspects of both matter and gravitation. There is a general consensus that such a small cosmological constant poses an extraordinary naturalness problem. According to usual arguments the cosmological constant is unnaturally small in comparison to the vacuum energy density due to the quantum fluctuations of the quantum field theories describing particle physics. In another variant of the argument, the cosmological constant is small in relation to the Planck scale.

In this section we will focus on the former version of the “cosmological constant problem” and we shall revisit it from the perspective of Background Independent quantum field theory.

5.1. The Standard Argument

The best-known argument showing the claimed tension between quantum field theory and general relativity goes as follows. In Minkowski space, one assumes that each mode pertaining to a certain quantum field behaves like a harmonic oscillator, which contributes to the field's ground state energy by an amount $\frac{1}{2}\hbar\omega$. In flat space, the modes of a quantum fields are labeled by the 3-momentum $\mathbf{p}$. By summing over all momenta, one obtains the total vacuum energy as $\sum_{\mathbf{p}} \frac{1}{2}\hbar\omega(\mathbf{p})$. For instance, in the case of a massless free field the energy density is given by the integral

$$\rho_{\text{vac}} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}|, \quad (4.1)$$

which is ultraviolet divergent and requires regularization. For instance, one may regularize the integral (4.1) via a sharp cutoff $|\mathbf{p}| \leq \mathcal{P}$. Clearly, different regularization can also be employed. In any case the vacuum energy density is quartically divergent, i.e.,

$$\rho_{\text{vac}} = c\mathcal{P}^4, \quad (4.2)$$

where the particular value of c depends on the chosen regularization scheme and is of order unity. Next, the UV cutoff $\mathcal{P}$ is fixed to some high value (typically related to a new physics scale). The energy density ρ_{vac} is then taken into account in the Einstein's equation as a contribution to the cosmological constant in the amount of $\Delta\Lambda = 8\pi G\rho_{\text{vac}} = 8\pi Gc\mathcal{P}^4$.

Similar semiclassical arguments goes back to Pauli [57]. He had already realized that a cosmological constant of order $\Delta\Lambda$ would produce a curvature, which is unacceptable even if the UV cutoff $\mathcal{P}$ is taken to be the scale of atomic physics.

In the modern version of the argument, the UV cutoff often corresponds to the Planck scale ($\mathcal{P} = m_{\text{Pl}}$). In this latter case, the contribution to the cosmological constant $\Delta\Lambda$ is roughly 10^{120} times bigger than the observed cosmological constant, Λ_{obs} . Then, by expressing the observed cosmological constant as the sum of a bare cosmological constant Λ_{bare} and $\Delta\Lambda$, i.e., $\Lambda_{\text{obs}} = \Lambda_{\text{bare}} + \Delta\Lambda$, one observes that Λ_{bare} must be fine tuned at the level of 120 digits. This is thus considered a major naturalness problem.

Similar issues arise with essentially any plausible choice for the UV cutoff $\mathcal{P}$. This has triggered the suspicion that there may be something incorrect in the previous argument. In the following we argue that this is indeed the case. Let us note that, along different lines with respect to the ones invoked in the present work, quartic divergences on a fixed Minkowski background have been shown incompatible with Lorentz symmetry [59–61].

5.2. Lessons From the Rigid Picture

Comparing the above standard argumentation to our approach we observe that

(1) The standard calculation amounts to the quantization of a free matter field's modes in $\Upsilon_{\text{IR}}(\mathcal{P})$. They constitute a low energy effective field theory with UV cutoff at $\mathcal{P}$. The field quantization it amounts to is equivalent to staying within the EAA framework and lowering the cutoff from $\mathcal{P}$ down to zero.

(2) Since the calculation includes no gravitational back reaction on the Minkowski metric, it possesses a translation to the Background Independent EAA language at best if the “rigid picture” of the RG flow is invoked.

(3) The domain of applicability of the rigid picture restricts the cutoff $\mathcal{P}$ to the interval $0 \leq \mathcal{P} \leq q_{\text{max}}$ below the turning point. Within this interval, the vacuum fluctuations change the cosmological constant by *not more than a factor 2*; according to (3.3), $\Lambda_{\text{rigid}}(q)$ increases from $\Lambda_{\text{rigid}}(0) = \Lambda_0$ to $\Lambda_{\text{rigid}}(q_{\text{max}}) = 2\Lambda_0$ along the k_- -branch.

It therefore follows that the traditional argument on the gravitational impact of summed up zero-point energies overstretches its domain of validity quite considerably.

We just learned that the enormous cosmological constants that are often claimed to be induced by quantum vacuum fluctuations, like $10^{120}\Lambda_0$, can never result from such a calculation if one restricts it to the momentum scales it is valid for, i.e., those where the rigid picture is available ($q \leq q_{\text{max}}$). A calculation neglecting the backreaction on the metric becomes invalid already when the zero-point energies have changed the cosmological constant from Λ_0 to $2\Lambda_0$.

Because no large numbers are involved in the renormalization $\Lambda_0 \rightarrow 2\Lambda_0$, we can also say that *it is incorrect to claim on the basis of the traditional argument that a small value of the cosmological constant is necessarily afflicted by a naturalness problem* [26].

5.3. Lessons From the Running Picture

Let us now move to scales above the turning point and ask about the physical contents of the generalized RG trajectory there.

(1) Since the rigid picture is unavailable at scales $k > k_{\text{T}}$, we fall back upon the running picture which applies everywhere along the trajectory. Now, the complication is that the running of the consistent background metric cannot be “transformed away” any longer and must be taken into account in explicit form.

(2) The metric $\bar{g}_k^{\text{sc}}$ is determined by the k -dependent Einstein equation (2.18). To see the essential point, its contraction is sufficient:

$$R(\bar{g}_k^{\text{sc}}) = 4\Lambda_k. \quad (4.3)$$

We are interested in the question why the rapidly increasing cosmological constant $\Lambda_k \propto k^4$ for $k \gg k_{\text{T}}$ seems in no way mirrored by our cosmological observations. On the basis of the effective field theory description with Equation (4.3), the answer is as follows:

when Λ_k grows with increasing k beyond the experimental bounds of the observed cosmological constant $\Lambda_0 \equiv \Lambda_{k=0}$, the effective field theory with the Einstein equation (4.3) ascribes the associated growing curvature to *much smaller, non-cosmological distance scales*; the smaller they are, the larger is k . On those sub-cosmological length scales, however, we have no observational tools (yet) that could measure $R(\bar{g}_k^{\text{sc}})$.

This explains why to date we have seen no manifestation of the huge values that $\Lambda_k \propto k^4$ can reach and that play a central role in the traditional discussions of the cosmological constant. [see [26] for further details, and [62] for a purely classical discussion of “hiding” the cosmological constant at small distances].

(3) Paraphrasing a well-known concise summary of classical General Relativity, it can be said that *Matter at scale k tells space at scale k how to curve, and space at scale k tells matter at scale k how to move*.

Modeling the gravitational effect of vacuum fluctuations by simply declaring their summed zero-point energies to be a part of the cosmological constant in an otherwise classical Einstein equation violates this principle spectacularly.

Being the coefficient of the zero-derivative term in the classical gravity action, the cosmological constant should play a role for the universe on its largest scales only. However, the traditional approach, limited by the simple two-parameter form of the Einstein-Hilbert action, cannot but package the energy and

momentum of even a Planck scale fluctuation, say, into this IR-related parameter.

Clearly, this hints at the necessity of much more general actions to better describe the generation of spacetime curvature scale by scale [26].

(4) Assume we were able to measure the spacetime curvature on sub-cosmological scales, say in a terrestrial lab, and that k_T is indeed in the milli-electron Volt range, as suggested by (2.34). Can we observe the scale dependence of the vacuum curvature then?

The answer is that, even then, this would be extremely difficult since $\bar{g}_k^{sc}$ has a significant k -dependence only when the $\Lambda_k g_{\mu\nu}$ term in the Einstein's equation dominates over the matter field's stress tensor $T_{\mu\nu}$.

As long as $T_{\mu\nu}$ is k -independent, the effect we are after requires ordinary matter and its fields to be very "diluted." The late Universe, the present epoch of cosmology, is one instance where this condition is met. It remains to be seen if there are also others.

6. CONCLUSION

In this paper, we advocated the general expectation that the lessons from Quantum Gravity may reach far beyond its traditional realm of small Planck mass suppressed effects and questions of UV renormalizability. We emphasized that what defines modern Quantum Gravity and makes it radically different from all present theories of particle physics is the key desideratum of Background Independence. As such unrelated to any specific

scale, there is no reason a priori why it should have implications for the microscopic world only.

Indeed, we argued that it is relevant to one of the purported problems surrounding the cosmological constant, namely, the gravitational effect of quantum vacuum fluctuations. Exploiting Background Independence in an essential way we demonstrated that most of the vacuum fluctuations could not manifest themselves in the cosmological constant Λ measured at cosmological scales since such fluctuations affect the curvature of spacetime only at sub-cosmological scales.

In principle, a mechanism of this sort could resolve the conundrum regarding the *invisibility* of spacetime curvature due to quantum vacuum fluctuations and the associated energy density, which is possibly the most mysterious facet of the cosmological constant problem.

From the perspective adopted in this work, our analysis shows no tension or "clash" between theoretical expectations and actual observations.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Effective Asymptotic Safety and Its Predictive Power: Gauge-Yukawa Theories

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Effective field theory provides a new perspective on the predictive power of Renormalization Group fixed points. Critical trajectories between different fixed points confine the regions of UV-complete, IR-complete, as well as conformal theories. The associated boundary surfaces cannot be crossed by the Renormalization Group flow of any effective field theory. We delineate cases in which the boundary surface acts as an infrared attractor for generic effective field theories. Gauge-Yukawa theories serve as an example that is both perturbative and of direct phenomenological interest. We identify additional matter fields such that all the observed coupling values of the Standard Model, apart from the Abelian hypercharge, lie within the conformal region. We define a quantitative measure of the predictivity of effective asymptotic safety and demonstrate phenomenological constraints for the associated beyond Standard-Model Yukawa couplings.

Keywords: renormalization group, interacting fixed points, effective field theory, (beyond) standard model physics, asymptotic safety

1. MOTIVATION

Effective field theory (EFT) describes all of high-energy physics remarkably well—see [1] for a review of Standard Model (SM) EFT, and [2] for a well-defined EFT of gravity below the Planck scale. EFTs are solely governed by their field content and symmetries (both global and local). The theory space of all possible realizations of an EFT is spanned by the couplings associated with the (infinite) set of all independent symmetry invariants. A specific realization is characterized by its coupling values at some Renormalization Group (RG) scale. Despite the infinitely many couplings, perturbative and local EFTs are predictive toward the infrared (IR) since the infinite tower of higher-order interactions permitted by symmetries and field content is suppressed by powers of the ratio between experimentally accessible scales and the cutoff scale, i.e., by the RG structure around the free fixed point. A sufficiently high cutoff scale thus gives retrospective insight into the success of perturbatively renormalizable gauge-Yukawa theories such as the SM.

On the other hand, it has been of paramount interest to identify fundamental, i.e., ultraviolet (UV) complete, quantum field theories in which the cutoff can be removed—first asymptotically free [3–21], and more recently, asymptotically safe [22–43] gauge-Yukawa theories. See also [44–58] (with potential caveats discussed in [59–61]) for asymptotically safe gauge-Yukawa theories from resummation at a large number of matter fields and [62] for a recent review including lattice results.

The RG flow of asymptotically free theories emanates from a fixed point at which all interactions vanish and the theory exhibits classical scale invariance. Asymptotic safety [63] generalizes asymptotic freedom to include UV-complete theories that emanate from (partially) interacting fixed points at which some of the couplings remain finite and the fixed-point theory exhibits quantum not classical scale invariance, cf. [64].

Quantum scale invariance of asymptotically safe theories (including the special case of classical scale invariance of asymptotically free theories) can entail enhanced predictivity. Close to a fixed point, this predictivity can be quantified by the eigenvalues of the linearized RG flow, i.e., by the number of IR-attractive opposed to IR-repulsive directions in theory space, cf. e.g., [65] for an introduction. Toward the IR, the RG flow can emanate from the fixed point only along IR-repulsive directions. Hence, the subset of EFTs emanating from the fixed point, referred to as its UV-critical hypersurface, is spanned only by the subset of IR-repulsive directions. On the contrary, IR-attractive directions become predictions of such fundamental theories because their coupling values have to remain fixed to the UV-critical hypersurface. A fundamental theory is predictive whenever the UV-critical surface is finite-dimensional. All perturbative fixed points—both free and interacting—are automatically predictive because perturbative quantum fluctuations are (by definition) too weak to cause classically irrelevant couplings to become IR-repulsive.

The present work is limited to non-gravitational theories. Concerning gravity, a considerable body of evidence, pioneered by [66], suggests the existence of an interacting fixed point for Euclidean quantum gravity, cf. [67–69] for introductory texts. If present, such a fixed point could extend EFTs beyond the Planck scale Λ_{Planck} . Here, we will only be concerned with perturbative EFTs at energies below Λ_{Planck} . Nevertheless, the Planck scale plays a crucial role. Most conservatively, it is to be regarded as the unavoidable cutoff scale for any non-gravitational theory. Hence, phenomenological implications of (non-gravitational) asymptotic safety should be discussed in the framework of an EFT that is valid only between Λ_{Planck} and the scale Λ_{NP} at which the new physics decouples. Assuming that new physics below the electroweak scale Λ_{ew} is excluded by collider experiments¹, the EFTs of interest are therefore valid over at most 17 orders of magnitude in energy scales, i.e.,

$$10^2 \text{ GeV} \approx \Lambda_{\text{ew}} \lesssim \Lambda_{\text{NP}} \lesssim \Lambda_{\text{Planck}} \approx 10^{19} \text{ GeV}. \quad (1)$$

This motivates us to explore *effective asymptotic safety*, i.e., the predictivity of RG fixed points over a finite range of scales, cf. also [65, 70–72]. Moreover, we are interested in the global RG structure encompassing all fixed points available in perturbation theory. Effective—in comparison to fundamental—asymptotic safety can alter conclusions about phenomenology as well as about the exclusion of specific models. To put the results of this paper in a wider context, we make the following simple

observation about the RG flow in the theory space of perturbative gauge-Yukawa theories:

The respective boundaries of the set of all UV-complete, IR-complete, and both UV- and IR-complete theories constitute hypersurfaces in theory space that cannot be crossed by the RG flow of any EFT. With respect to other directions orthogonal to such a boundary hypersurface, the latter inherits the IR-attractive properties of the fixed points by which it is delimited. In these cases, the entire boundary surface, not just the fixed point, can constitute an IR-attractor and generic EFTs tend to cluster close to it².

Possible proof of this claim in more general settings is beyond the scope of this work and might be provided elsewhere in the future. Besides its potential importance for a structural understanding of the behavior of RG flows, it can have phenomenological implications which, in our opinion, deserve more attention. In the following, we will demonstrate this observation for the case of gauge-Yukawa theories. These make for a particularly suitable example because (i) their fixed-point structure is both rich enough and perturbatively well-controlled [34, 36, 38, 39, 42, 73] and (ii) they are of direct phenomenological significance as possible extensions of the SM [37, 40, 41, 43].

1.1. Synopsis of Results

- In section 2, we review the different phases, i.e., the possible perturbative fixed-point structures, of simple gauge-Yukawa theories identified in [22, 73]. This discussion allows us to delineate how the above observation is realized. Readers who are familiar with the fixed-point structure of gauge-Yukawa theories and are not interested in a respective discussion of effective asymptotic safety may want to skip this section.
- In section 3, we look at each simple SM subgroup by itself which leads to a transparent understanding of why within perturbation theory: (i) additional matter fields can induce fully IR-attractive interacting fixed points for the non-Abelian SM subgroups, while (ii) interacting fixed points with UV-attractive directions are not available, and (iii) Abelian subgroups will always remain trivial.
- Turning to phenomenological implications, we introduce a novel quantitative measure for the global predictivity of EFTs in section 4. This effective notion of predictivity applies to (finite-dimensional truncations of) perturbative as well as non-perturbative EFTs, more widely.
- In section 5, the SM serves as a first example to demonstrate the predictivity measure. Here, we also conclude that whenever the non-Abelian sectors remain perturbative, the Abelian Landau pole of the SM remains safely beyond the Planck scale.
- In section 6, we discuss phenomenological conclusions of effective asymptotic safety for extensions of the SM by

¹This assumption can be circumvented by very weakly coupled particles, in which case the new-physics scale may lie below the electroweak scale. We will not discuss these cases here.

²We caution that these observations require a truncation of the perturbative series or any other expansion which is sufficiently converged to have revealed all physical fixed points. Otherwise, the statement still applies to the truncated RG flow but might lose its phenomenological significance.

additional matter fields along the lines of [22]. We identify specific BSM matter for which all the SM coupling values (apart from the Abelian hypercharge coupling) lie within the conformal region.

We conclude in section 7. Throughout this analysis, we work with well-established perturbative $\overline{\text{MS}}$ beta functions. The respective collection of NLO and NNLO beta-functions required for this work, cf. [3–5, 74–103] for original references, is relegated into **Appendices**.

2. RG STRUCTURE OF GAUGE-YUKAWA THEORIES: AN EFT POINT OF VIEW

Before explicitly discussing the SM and its possible extensions, we briefly review the available fixed-point structures of simple gauge-Yukawa theories previously discussed in [22, 73]. This serves as a specific example to characterize the global RG structure, effective asymptotic safety and their significance for generic EFTs. For the purpose of this section, we focus on a simple gauge group for which we denote the squared gauge coupling by $\alpha_g = g^2/(4\pi)^2$, cf. [34] for a generalization to semi-simple gauge groups.

Weyl-consistency conditions suggest that the RG equations of gauge-Yukawa theories should be obtained in hierarchical schemes [38, 104–108]. In particular, Yukawa couplings contribute to gauge couplings only at 2nd loop order. Quartic couplings contribute to Yukawa and gauge couplings only at 2nd and 3rd loop order, respectively. Therefore, included loop orders of gauge, Yukawa, and quartic couplings should relate as $(n+2, n+1, n)$, respectively. Throughout this paper, we will neglect quartic couplings for simplicity and work in the $(2, 1, 0)$ -scheme (subsequently referred to as NLO). We check that fixed points remain perturbatively well-controlled by extending to the $(3, 2, 0)$ -scheme (subsequently referred to as NNLO)³. In the notation of [38], what we call NLO (NNLO) is referred to as NLO'' (2NLO''). The explicit RG equations of the latter are discussed in **Appendices** since they merely serve to ensure perturbative control. Following [73], the beta-function of general Yukawa couplings $\mathcal{L}_{\text{Yukawa}} = -\text{Y tr}[\bar{\psi}_L \chi \psi_R + \bar{\psi}_R \chi^\dagger \psi_L]$, suppressing indices, takes the form

$$\beta_Y = \mathbf{E}(Y) - \alpha_g \mathbf{F}(Y), \quad (2)$$

where $\mathbf{E}$ and $\mathbf{F}$ are matrices cubic and linear in the Yukawa-coupling matrices Y , respectively. Therefore, besides a trivial fixed point at $Y_* = 0$, additional non-trivial (partial) Yukawa fixed-points exist. The latter depend parametrically on α_g [73], i.e.,

$$Y_{(*)}(\alpha) = \sqrt{\alpha_g} C, \quad (3)$$

³Given this setup, we are only able to make statements about those non-vanishing fixed points that potentially arise from a balance between leading order (LO) and NLO contributions. In principle, there could be further fixed points for which NNLO (or even higher) loop orders are required. However, one should then always be careful to test their nature by confirming that (at least) they persist upon inclusion of the subsequent higher-loop order. Since 3NLO contributions are not available at present, such an analysis cannot reliably be made.

TABLE 1 | Different perturbative Renormalization Group phases for simple gauge-Yukawa theories.

	Conditions	d_{IR}	d_{UV}	$d_{\text{UV-IR}}$
Complete asymptotic freedom (CAF)	$B < 0, C < 0, C' < 0$	0	2	0
Banks-Zaks (BZ) conformal window	$B < 0, C > 0, C' < 0$	1	2	1
Gauge-Yukawa (GY) conformal window	$B < 0, C > 0, C' > 0$	2	2	2
Litim-Sannino (LS) conformal window	$B > 0, C > 0, C' < 0$	2	1	1
Complete triviality (CT)	$B > 0, C > 0, C' > 0$	2	0	0

Conditions on B , C , and C' can be translated into conditions on parameters like the size of the gauge group and number of fermionic/scalar representations. We also indicate the dimensionalities d_{IR} , d_{UV} , and $d_{\text{UV-IR}} = \text{Min}(d_{\text{UV}}, d_{\text{IR}})$ of the IR-complete, UV-complete and conformal, i.e., UV- and IR-complete, region in theory space, respectively.

where the C is independent of the gauge coupling, cf. [73]. These partial fixed points (also referred to as Yukawa-nullcline) always exist and occur at positive (but not necessarily perturbative) values of the Yukawa couplings. Under the RG flow, they focus the values of Yukawa couplings toward a small IR interval, as for instance in the SM. We will see in section 6 that they are of phenomenological importance, cf. also [109–111]. Evaluating the (2-loop) running of the gauge coupling α_g by use of the above partial fixed-point solution results in

$$\beta_{\alpha_g} = \alpha_g^2 [-B - C\alpha_g + 2D\alpha_g]. \quad (4)$$

The scalar coefficients B , C , and D are purely group-theoretic and can be found in [73]. B and C arise from gauge-coupling contributions, while D arises from Yukawa couplings at their partial fixed point. Since $\alpha_g = g^2/(4\pi)^2$, the fixed points for g_* are physical, i.e., real, only if $\alpha_{g*} \geq 0$. While $D \geq 0$ ($D = 0$ for the vanishing Yukawa fixed point), the signs of B and C depend on the matter content of the theory⁴. Defining $C' = C - 2D$ (note that $C' < C$, always), one can fully classify the general theory by two types of interacting fixed points, cf. [22, 73]: one with vanishing and one with non-vanishing Yukawa couplings, i.e.,

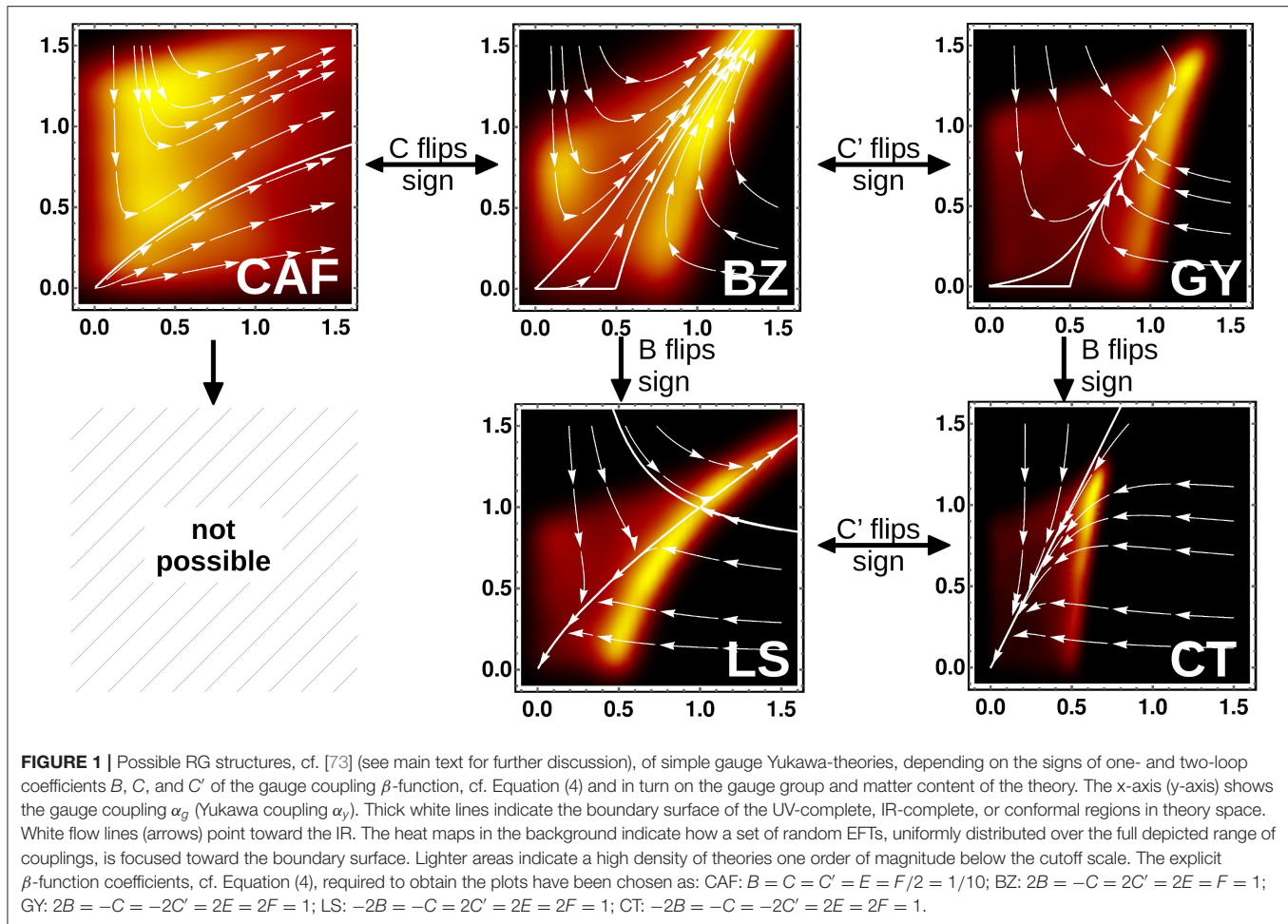
$$\alpha_{g*, \text{BZ}} = -\frac{B}{C}, \quad Y_{*, \text{BZ}} = 0, \quad (5)$$

$$\alpha_{g*, \text{GY}} = -\frac{B}{C'}, \quad Y_{*, \text{GY}} = \sqrt{\alpha_g} C, \quad (6)$$

respectively. Depending on which of these are physical, i.e., occur at $\alpha_{g*} \geq 0$, [73] have classified the five possible phases, i.e., perturbative fixed-point structures, of simple gauge-Yukawa theories. These are summarized in **Table 1** and depicted schematically in **Figure 1**.

Preceding the EFT discussion of these phases, it is important to distinguish the following terminology. A set of gauge-Yukawa theories is determined by its gauge group and matter content, parameterized, for instance, by the number of fermionic representations N_F . To agree with previous literature [112], we

⁴We have chosen the signs to reflect the antiscreening non-Abelian case without matter content. Note that this choice agrees with [112] but differs from [22].



refer to the possible values of N_F which realize certain gauge-Yukawa phases as “windows.” This is distinct from a particular realization within a set of gauge-Yukawa theories. The latter is further parameterized by coupling values, i.e., by a choice of RG trajectory. When referring to possible values of the couplings, we talk about “regions.” In particular, we say that the set of all UV-complete trajectories makes up the UV-complete region, the set of all IR-complete trajectories makes up the IR-complete region, and the set of all UV- and IR-complete trajectories makes up the “conformal” region. Crucially, the terminology “conformal window” and “conformal region” are to be distinguished.

2.1. Complete Asymptotic Freedom (CAF)

For antiscreening $B > 0$ as well as $C > 0$, the only physical fixed point is the Gaussian one, cf. left-hand upper panel in **Figure 1**. The former is completely asymptotically free when approached from below the Yukawa nullcline, i.e., whenever $Y < Y_*(\alpha_g)$. This case also encompasses asymptotic freedom of Yang-Mills theory without Yukawa couplings, cf. the RG flow along the x-axis in the left-hand upper panel of **Figure 1**. The UV-complete region is 2-dimensional and extends to infinite coupling values (or more accurately beyond perturbative control) although it is partially bound by the Yukawa-nullcline (white line). This

boundary surface inherits the IR-attractive property of the free fixed point along the Yukawa direction (y-axis) and is hence IR-attractive from above, i.e., for $Y > Y_*$. This entails that generic UV-incomplete EFTs will be attracted to the boundary. The IR-complete (and thus also the conformal) region is reduced to the trivial theory. All other theories eventually escape perturbative control toward the IR.

2.2. Banks-Zaks (BZ) Conformal Window

Scalar, as well as fermionic matter, adds screening fluctuations and modifies the running of non-Abelian gauge couplings. This can flip the signs of C , C' , and B . Independent of the specific matter representation, the sign of C is always flipped first and the theory (with vanishing Yukawa couplings) enters the so-called conformal window [112], cf. upper panel in the middle of **Figure 1**. As C flips sign (but before C' or B do so), the Banks-Zaks fixed point becomes physical. For vanishing Yukawa coupling and $0 < \alpha_g < \alpha_{g*,BZ}$, the theory is now both UV- and IR-complete. In fact, since the IR-complete and thus the conformal region is still only one-dimensional, the RG-scale can be mapped directly to a unique gauge-coupling value. Put differently, there only exists a single conformal theory. The gauge-Yukawa fixed point is still not physical and

thus every theory with non-vanishing Yukawa coupling will eventually diverge in the IR. This is a consequence of the Banks-Zaks fixed point being IR-attractive in the direction of the gauge coupling but IR-repulsive in the direction of the Yukawa coupling. To distinguish this situation, we refer to this as the “BZ conformal window.” However, the UV-complete region is still two-dimensional. Its boundary inherits the partial IR-attractive nature of the two fixed points. (The free fixed point is IR-attractive in the Yukawa-coupling direction and the Banks-Zaks fixed point is IR-attractive along the gauge-coupling direction). The corresponding sections of the boundary surface act as an IR-attractor, in particular for EFTs outside of the UV-complete region. We emphasize that it is the boundary and not a single fixed point which is IR-attractive.

Adding further matter representations can flip the sign of C' or B first. Therefore, there are now two distinct phases that can occur when further matter is added. Which of these is realized depends on the ratio of scalar and fermionic matter and on the set of possible Yukawa interactions.

2.3. Gauge-Yukawa (GY) Conformal Window

Whenever C' turns negative before B does, the theory develops a fully IR-attractive gauge-Yukawa fixed point, cf. right-hand upper panel in **Figure 1**. As C' is varied, the fixed point formally enters from infinity (or from outside the perturbative regime) along the direction in which the two nullclines of the BZ phase join. The gauge-Yukawa fixed point serves as an endpoint of the two nullclines and delimits the two-dimensional UV-complete region, which is now also IR-complete. As a consequence, there is now a two-dimensional region of distinct conformal theories. In correspondence to the “BZ conformal window,” we refer to this as the “gauge-Yukawa (GY) conformal window.” This case is particularly predictive. If realized only over a finite range of scales, e.g., due to the decoupling of massive modes, this realizes *effective asymptotic safety*. All EFTs are attracted first to the boundary of the “gauge-Yukawa conformal window” and eventually into the gauge-Yukawa fixed point.

2.4. Litim-Sannino (LS) Conformal Window

If, on the other hand, B turns negative before C' does, a Litim-Sannino fixed point [22] becomes available, while the Banks-Zaks fixed point [112] disappears (formally it escapes the perturbative regime in direction of increasing gauge coupling) and the free fixed point becomes fully IR-attractive, cf. lower panel in the middle of **Figure 1**. It is now the IR-complete region which is two-dimensional. However, the UV-complete region and hence the set of conformal theories, is just one-dimensional. The latter is delimited by the free and the Litim-Sannino fixed point, while the former also extends beyond the Litim-Sannino fixed point and corresponds to its UV-critical hypersurface. Concerning generic EFTs, the conformal theory which splits the IR-complete region, i.e., the separatrix between the Litim-Sannino and the free fixed point, acts as an IR-attractor because it inherits this property from the shared IR-attractive direction of both its delimiting fixed points. The boundary of the IR-complete region, however, is *not* IR attractive since it inherits the IR-repulsive direction

of the Litim-Sannino fixed point. Again, generic EFTs tend to cluster close to the UV-complete theories, i.e., exhibit *effective asymptotic safety*.

2.5. Complete Triviality (CT)

The final possibility occurs if all three signs are flipped, i.e., $B < 0$ and $C < C' < 0$. Since all contributions have now turned screening, the theory remains only with the free fixed point. The latter is now fully IR-attractive. This phase occurs for the perturbative range of any Abelian gauge group, cf. section 3.2. Formally, the UV-complete and conformal regions reduce to the trivial theory to which all EFTs are attracted. The IR-complete region now covers all of the theory space. The triviality problem can therefore be seen as a consequence of “effective asymptotic freedom.”

This concludes the review of all possible fixed-point structures [73] which can occur due to different cancelations at NLO in simple gauge-Yukawa theories. One can schematically think of semi-simple cases, such as the SM, as the higher-dimensional combinations of these phases, cf. [34] for an explicit discussion. In the following, we will always check whether potential fixed points persist at NNLO. We present our formal definition of perturbativity in section 4. Before doing so, we provide insight into the single gauge groups of the SM which is sufficient to qualitatively understand the available fixed points that we identify in the coupled system in section 6. In all phases, some form of IR-attractor dominates the RG flow.

3. AVAILABLE PHASES FOR THE SIMPLE STANDARD-MODEL SUBGROUPS

Following [22], we remain focused on a simple gauge group with N_F copies of a single type of fermionic representation R_F and uncharged scalars to allow for Yukawa couplings, cf. **Appendix A** or [22] for the explicit Lagrangian. In this case, the Yukawa-coupling matrices $E(Y)$ and $F(Y)$ in Equation (2) reduce to scalar coefficients E and F of a single Yukawa coupling y for which we introduce $\alpha_y = y^2/(4\pi)^2$. The NLO coefficients that determine the interacting fixed-points, cf. Equations (5) and (6) and the resulting RG-structure are given by, cf. [73],

$$B = \frac{2}{3d^{\text{adj}}} \left(11 d^{\text{adj}} C_2^{\text{adj}} - 2 N_F d^{R_F} C_2^{R_F} \right),$$

$$C = \frac{68 \left(C_2^{\text{adj}} \right)^2}{3} - \frac{4 d^{R_F} C_2^{R_F} N_F (5 C_2^{\text{adj}} + 3 C_2^{R_F})}{3 d^{\text{adj}}}, \quad (7)$$

$$C' = \frac{4}{3} \left[- \frac{5 d^{R_F} C_2^{\text{adj}} C_2^{R_F} N_F}{d^{\text{adj}}} + \frac{3 d^{R_F} \left(C_2^{R_F} \right)^2 N_F (10 N_F - d^{R_F})}{d^{\text{adj}} (2 N_F + d^{R_F})} \right. \\ \left. + 17 \left(C_2^{\text{adj}} \right)^2 \right]. \quad (8)$$

Here, C_2^{adj} and d^{adj} refer to the second Casimir and the dimension of the adjoint representation, respectively. Similarly, $C_2^{R_F}$ and d^{R_F} denote the same for the fermionic representation⁵.

Naively, there are two ways to achieve perturbativity of the possible fixed points $\alpha_{*,\text{BZ}} = -\frac{B}{C}$ and $\alpha_{*,\text{GY}} = -\frac{B}{C'}$, i.e., either by (i) making B small, or by (ii) making C or C' large. It is typically not possible to achieve the latter (as a function of N_C and N_F for instance) without invalidating perturbation theory at higher orders. The subsequent discussion of the $U(1)$ in section 3.2 will serve as an explicit example. On the contrary, non-Abelian gauge groups can allow for perturbatively small B without invalidating perturbation theory [22]. A dedicated 3-loop analysis of a simple $SU(N)$ gauge group with fermions in the fundamental representation [38] provides strong indications that perturbative yet interacting gauge-Yukawa fixed points are only possible for $N_C \geq 5$. However, this does not necessarily imply that the same conclusions hold for arbitrary representations. For extensions of the SM, this has been tested by an explicit grid search for a single type of BSM representation in [40]. Before extending such a grid search to multiple different types of BSM representation, we discuss each of the simple SM subgroups on its own. This provides a good intuition of why certain phases, cf. **Figure 1**, are possible and others are not.

3.1. The Non-Abelian Subgroups of the SM

Which of the gauge-Yukawa phases is accessible in perturbation theory depends on the sign of C' in the region close to a sign-change of B , cf. **Figure 1**. Note that the sign of C is always fixed close to a sign change of B , cf. [73]. The sign of C' , in turn, depends on the specific gauge group and matter representations. In particular, additional fermionic representations without (or with negligibly small) Yukawa couplings result in additional screening contributions to B and C , while they do not contribute to $(C' - C)$ since they do not participate in Yukawa interactions. Hence, charged fermions without Yukawa couplings will influence which phases are available.

The latter also occurs in the SM where there are 32 light Weyl degrees of freedom with negligibly small Yukawa couplings⁶. We explicitly visualize their significance for the existence of gauge-Yukawa fixed points in the case of $SU(2)$ and $SU(3)$ in **Figure 2**. Without the SM fermions, there exist BSM representations [such as the $d_2 = 3$ dimensional for $SU(2)$ and the $d_3 = 8$ dimensional for $SU(3)$] for which, with growing number N_F of BSM fermions, B changes sign before C' does. As a function of N_F one moves from complete asymptotic freedom to the Banks-Zaks phase and into the Litim-Sannino phase, i.e., through the chain $\text{CAF} \rightarrow \text{BZ} \rightarrow \text{LS}$, cf. first and third panel in **Figure 2**. In particular, one enters the LS phase via a sign change in B , i.e., in the

region in which the interacting fixed points can be perturbatively controlled. Inclusion of the SM fermions prohibits the realization of this chain, i.e., there is no possible BSM representation for which B changes sign before C' does. When adding additional BSM representations one therefore always follows a different chain with growing N_F : starting from complete asymptotic freedom and moving through the Banks-Zaks phase, one instead enters the gauge-Yukawa and ends up in the completely trivial phase, i.e., this realizes the chain $\text{CAF} \rightarrow \text{BZ} \rightarrow \text{GY} \rightarrow \text{CT}$. For most BSM representations which can be added to the SM case, the BZ and GY phase only occur at non-integer values of N_F such that this formal chain is effectively reduced to $\text{CAF} \rightarrow \text{GY} \rightarrow \text{CT}$ or $\text{CAF} \rightarrow \text{CT}$, cf. **Figure 2**. Formally, this chain can be prolonged and the LS phase can still be entered from the CT phase, cf. upper-right area of the second panel in **Figure 2**. The minimal (but quite large) number of BSM fermions identified in [37] realizes this formal window of the LS phase. While this can occur at small values of the couplings if $C' \gg B \gg 1$, the latter invalidates perturbation theory and such fixed points are lost at NNLO, cf. also [40].

Regarding extensions of the SM, we can conclude that the SM fermions with negligibly small Yukawa couplings prohibit from entering the LS phase, i.e., no perturbatively controlled, interacting fixed points with UV-attractive directions are possible. On the contrary, fully IR-attractive interacting gauge-Yukawa fixed points in the GY phase remain possible for special dimension and number of BSM representations, cf. red upward triangles in **Figure 2**. As we shall see in section 6, both conclusions persist for the full SM gauge group. The GY phase realizes *effective asymptotic safety* if the theory space is extended to include mass terms or scalar vacuum expectation values. In this case, the theory departs from (close to) the otherwise fully IR-attractive fixed point at RG scales below this mass threshold.

3.2. Persistence of Abelian Triviality

Despite the complete asymptotic freedom of both non-Abelian subgroups, the SM is not UV-complete, i.e., it eventually breaks down at a transplanckian but finite energy scale. Due to the lack of antiscreening self-interactions in the $U(1)$ gauge group, matter fluctuations dominate and screen the associated Abelian gauge coupling. At $\sim 10^{41}$ GeV, the latter grows beyond perturbative control and eventually results in a perturbative divergence—the Landau pole [113]. Beyond perturbation theory, the $U(1)$ triviality problem has been confirmed by different non-perturbative methods [114–116], but so far only in the absence of Yukawa couplings.

Indeed, the presence of a Yukawa coupling formally places an Abelian gauge group in the Litim-Sannino phase, cf. section 2. Unfortunately, the corresponding interacting pseudo-fixed-point cannot occur within the perturbatively controlled regime. Since we found no explicit discussion of the latter statement in the literature, we will provide it in the following.

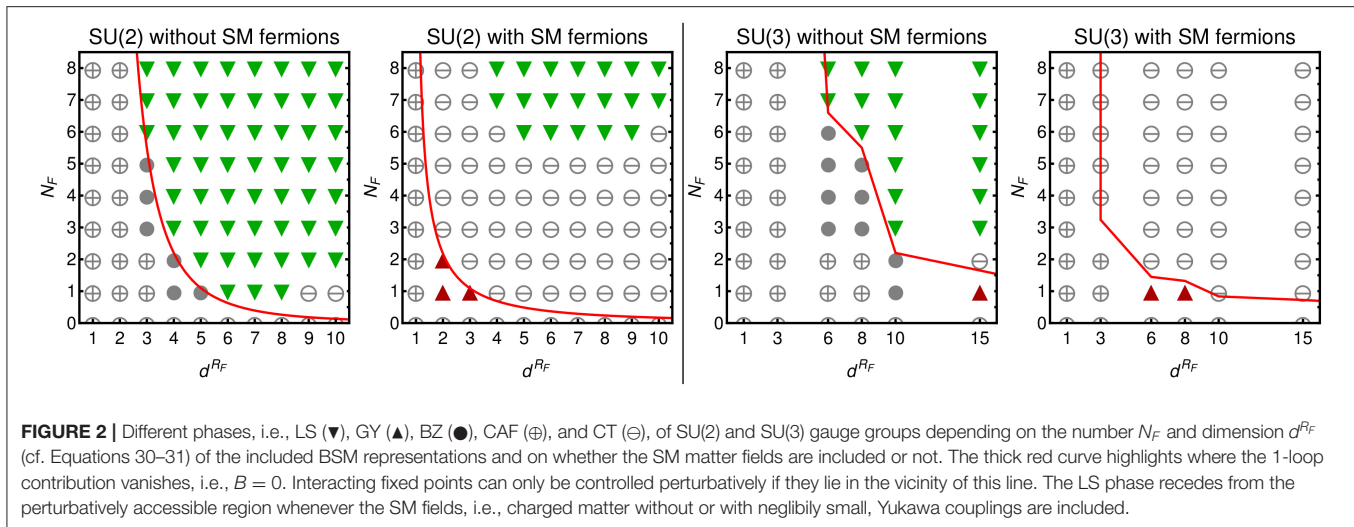
In principle, every $U(1)$ gauge group with N_F fermions of charge Y and associated scalars to facilitate Yukawa couplings is in the LS phase which would indicate the presence of an

⁵Either of the latter group-theoretic invariants can be traded for the Dynkin index

$$S^{R_F} \equiv \frac{d^{R_F}}{d^{\text{adj}}} C_2^{R_F} \quad (9)$$

but note that the latter is defined only up to a constant and varying conventions are used in the literature. Here, we use the dimension and the second Casimir.

⁶This counting excludes the top quark since its Yukawa coupling is not negligibly small. It also excludes potential right-handed neutrinos which are SM singlets anyway.



interacting UV fixed point for the gauge coupling at

$$\alpha_{U(1)*} = \frac{1}{15 Y^2}, \quad \alpha_{y*} = \frac{2}{5 N_F}. \quad (10)$$

The explicit NLO and NNLO β -functions are presented in **Appendix C**. It would seem as if this fixed point becomes more perturbative for large Y^2 and large N_F but this ignores the accompanying growth of higher-loop contributions and the resulting breakdown of perturbation theory. To properly analyze the above fixed point, one has to introduce a t'Hooft-like rescaling of the couplings. More specifically, one has to rescale the couplings α_g and α_y such that all higher-loop contributions either vanish or at least converge to finite values at large N_F and large Y^2 . In the present case, the minimal rescaling that suppresses all higher-loop contributions with growing N_F and Y^2 is given by

$$\alpha_{U(1)} = \frac{\tilde{\alpha}_{U(1)}}{N_F Y^2}, \quad \alpha_y = \frac{\tilde{\alpha}_y}{N_F}. \quad (11)$$

The correspondingly rescaled β -functions reveal that (in contrast to the non-Abelian case in [22]) only the trivial fixed point persists in the perturbative large-charge-large- N_F limit. We have explicitly confirmed that for *any* combination of integer $N_F \geq 0$ and arbitrary Y^2 , the absolute value of the NNLO contributions is larger than that of the NLO contributions when evaluated at the fiducial fixed point in Equation (10)—a clear sign that perturbation theory is not valid anymore.

The physical mechanism through which the Litim-Sannino fixed point arises, i.e., the balance of screening contributions from fermionic fluctuations against antiscreening contributions from Yukawa couplings, is present nevertheless. Thus, it might be worthwhile to conduct a non-perturbative analysis of this fixed-point mechanism in Abelian theories with Yukawa couplings in the future. However, for the present perturbative analysis, we conclude that the Abelian gauge group of the SM will always remain trivial.

4. A QUANTITATIVE MEASURE OF PREDICTIVITY

For a given gauge-Yukawa theory with fixed gauge group and matter content, we define the perturbative range of coupling values α_i by the condition that all NNLO contributions remain smaller than the respective NLO contributions, i.e.,

$$\text{perturbativity} \Leftrightarrow |\beta^{(NNLO)}| < \frac{1}{2} |\beta^{(NLO)}|. \quad (12)$$

The factor $\frac{1}{2}$ is included such as to avoid the regime of novel fiducial fixed points arising at NNLO. Another reason for the inclusion of this factor is the U(1) Landau pole, as will become clear below. This perturbativity condition is rather non-conservative, meaning that perturbation theory may break down earlier. The resulting set of perturbative EFTs encloses a finite volume $\mathcal{V}$ in the (truncated) theory space of all couplings⁷. More explicitly, we use the volume of the convex hull obtained from a Delauney-triangulation of a large enough random set of points in theory space which fulfill the perturbativity criterion⁸. We ensure convergence of this discrete volume measure by averaging over several individual random sets of perturbative EFTs and making sure that the statistical error is subleading.

The theory-space volume $\mathcal{V}$ depends on the definition of couplings: for instance, a simple rescaling of couplings will also rescale $\mathcal{V}$. It is thus certainly a scheme-dependent statement.

⁷Since we work in the perturbative regime, all higher-order couplings will necessarily remain irrelevant. Hence, the UV-complete region does not extend in any of these directions and its volume, if finite in truncated theory space, remains finite in full theory space. Technically, this is not necessarily true for the overall EFT volume in the theory-space volume which permits an extension in any higher-order direction of the full theory space. We restrict to truncated theory space in the following.

⁸One can easily see that the convex hull is not always a good approximation to the theory-space volume enclosed by the separatrices between fixed points, cf. upper right-hand panel in **Figure 1**. However, it is (to our knowledge) the only mathematically well-defined discrete notion of such a volume. It certainly suffices to quantify the statements of this study.

However, ratios of such volumes at different scales measure something like an overall critical exponent and should, therefore, capture scheme-independent effects⁹. Taking such ratios allows us to define a quantitative measure of predictivity. The theory-space volume $\mathcal{V}$ can be evolved by following the RG flow to the IR. Given the initial volume $\mathcal{V}_\Lambda$ at the cutoff scale Λ , and its evolution following the RG flow, i.e., $\mathcal{V}_k$, at RG scale k , we define predictivity $\mathcal{P}(k)$ by

$$\mathcal{P}(k) = \frac{\mathcal{V}_k}{\mathcal{V}_\Lambda}. \quad (13)$$

We call EFTs predictive (non-predictive), whenever their theory-space volume decreases (increases) along the flow. Non-predictive EFTs tend to formally result in $\mathcal{P}(k) \rightarrow \infty$ at finite $k < \Lambda$ which signals that they have diverged beyond perturbative control. In the predictive case, however, $\mathcal{P}(k)$ provides a quantitative measure of how predictive the EFT is. We caution that, at present, we are not able to provide proof that the predictivity measure constitutes a scheme-independent statement. We hope to sharpen the scheme-independence of this definition in future work.

It will also prove useful to exclude specific couplings, e.g., the measured SM couplings, from the predictivity measure and instead match them to their experimentally known values at a specified low-energy scale, e.g., at $\Lambda_{\text{ew}} < \Lambda$. The resulting $\tilde{\mathcal{P}}(k)$ can measure partial predictivity, even if the overall EFT is classified as non-predictive.

Both, the predictivity and the partial predictivity measure do not necessarily rely on perturbation theory and can be applied to (sufficiently converged) non-perturbative truncations of theory space as well. However, they do require to define an initial volume in theory space in which the present truncation is sufficiently converged, i.e., a non-perturbative analog of Equation (12). Whenever such an initial volume in theory space can be defined, its evolution under the RG flow allows us to quantify the predictivity of *effective asymptotic safety* via the measure in Equation (13). In particular, this applies to truncations of the Reuter universality class [66], see [67–69] for introductory texts and [65, 70–72] for previous discussions in the *effective asymptotic safety* context. We leave such an analysis for future work.

To exemplify the above definitions, we will discuss the heavy gauge-Yukawa sector of the SM in section 5 before adding new matter degrees of freedom in section 6.

5. THE HEAVY-TOP LIMIT OF THE STANDARD MODEL

We focus on the heavy gauge-Yukawa sector of the SM, i.e., on the three gauge couplings $\alpha_{1,2,3}$ and the top-Yukawa coupling α_t . It is a very good approximation to assume all other fermions as being massless, i.e., to set their Yukawa couplings to zero. Similarly, we neglect contributions from the quartic coupling λ_4

which is also negligible with regards to the gauge-Yukawa sector, as long as all couplings remain within the perturbative regime because it only arises at 2-loop and 3-loop order for Yukawa and gauge couplings, respectively. Supplementary conditions implied by stability conditions of the Higgs potential [24, 38] are deferred to future studies.

5.1. Partial Predictivity Within the Standard Model

The heavy SM is non-predictive, i.e., $\mathcal{P}(k)$ quickly diverges below the cutoff scale. This is a result of the antiscreening nature of the non-Abelian gauge couplings realizing the CAF phase, cf. section 2 in the SM. Coupling values at the cutoff scale that lie close to the edge of the perturbative regime will quickly be driven to values beyond perturbative control toward the IR.

On the other hand, if one excludes the non-Abelian gauge couplings from the predictivity measure and instead fixes them to their known experimental values at the electroweak scale, the SM is partially predictive in the remaining theory space. This is a consequence of the screening nature of both the top-Yukawa and the U(1) gauge coupling. When excluding also the U(1) gauge coupling from the predictivity measure, the resulting partial predictivity $\tilde{\mathcal{P}}(k)$ reflects the pre-Tevatron situation in which all the gauge couplings had already been experimentally measured, while the top-Yukawa coupling α_t remained unknown. **Figure 3** shows the evolution of $\tilde{\mathcal{P}}(k)$ along the RG flow. In this simple one-dimensional slice of theory space, the predictivity measure simply amounts to the normalized evolution of the full perturbative range of top-Yukawa values below Λ_{Planck} . Hence, enforcing a perturbative origin at Λ_{Planck} bounds the top quark to be lighter than $M_t \lesssim 210 \text{ GeV}$. The underlying reason is the associated partial IR fixed point for Yukawa couplings in gauge-Yukawa theories previously uncovered in [109–111], cf. also Equation (3).

5.2. The Landau Pole Remains Transplanckian

As discussed in section 3.2, the triviality of the U(1) hypercharge cannot be cured within perturbation theory. On the other hand, the associated Landau pole remains above the Planck scale as long as the other SM couplings remain within the perturbative regime (and no BSM representations with hypercharges are added).

Even in the absence of any new states with hypercharge, NLO and NNLO contributions from the non-Abelian gauge and top-Yukawa couplings in a modified BSM RG flow can potentially further screen the U(1) gauge coupling and therefore result in a lowered Landau pole, cf. also [37]. However, for any perturbative extension of the SM that still matches the measured electroweak-scale value for α_1 , the Landau pole remains at transplanckian energies. One can numerically determine that $\alpha_3 \lesssim 0.15$, $\alpha_2 \lesssim 0.09$, and $\alpha_t \lesssim 0.53$ is required to conform to the perturbativity criterion in Equation (12), i.e., to $|\beta^{(\text{NNLO})}| < \frac{1}{2}|\beta^{(\text{NLO})}|$. These maximal values have been determined by a grid search at random α_1 . We then fix the non-Abelian gauge couplings and the top Yukawa coupling to these maximal values. By definition, any RG flow within the perturbative regime cannot outgrow these values. Numerical integration of the resulting RG flow of the U(1)

⁹In case of a single fixed point and a purely linear flow, the ratio of theory-space volumes $\mathcal{V}$ is indeed directly related to the critical exponents.

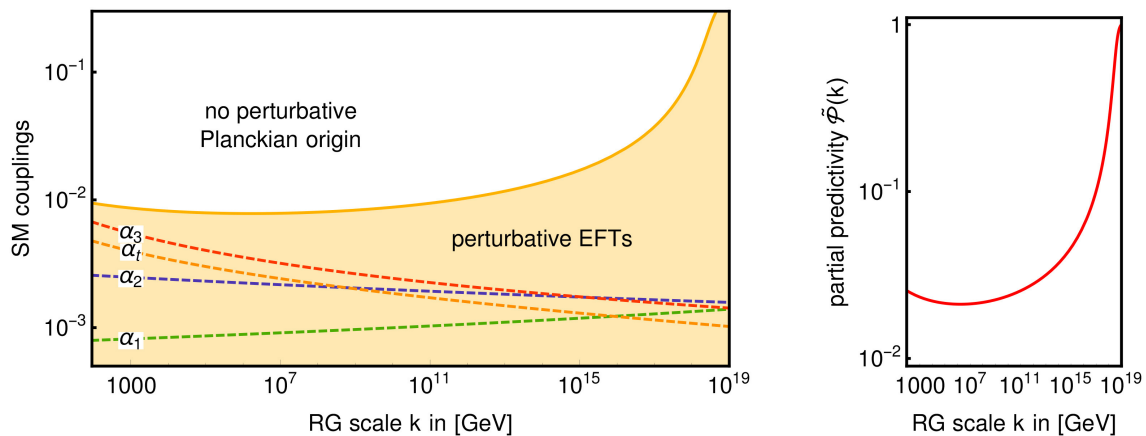


FIGURE 3 | (Left) RG flow of the SM gauge-Yukawa theory. The shaded region indicates values for α_t which can originate from a perturbative EFT at the cutoff scale Λ_{Planck} . The focusing of this region toward the lower scales exemplifies the partial predictive power of the SM as an EFT. The dashed trajectories indicate the RG flow of α_3 , α_2 , α_1 , and α_t , matching observed values at the electroweak scale. **(Right)** Evolution of the partial predictivity measure with the RG flow.

coupling shows that the $U(1)$ -Landau pole remains safely beyond the Planck scale.

The left panel in **Figure 4** shows the RG-flow of the Abelian gauge coupling matching to the observed electroweak-scale value for a random set of fixed values of the other heavy-SM couplings satisfying the perturbativity criterion. Subplanckian Landau poles are not present. Loosening the perturbativity criterion in Equation (12) to $|\beta^{(NNLO)}| < |\beta^{(NLO)}|$ allows for rare cases at the edge of the redefined perturbative regime for which the Landau pole is shifted slightly below the Planck scale, cf. right panel in **Figure 4**. In any case, all the perturbative BSM fixed points discussed in section 6 are much more perturbative than any of the above bounds.

We conclude that the persistence of a $U(1)$ -Landau pole—at least in any of the subsequently important BSM scenarios in which the BSM representations do not carry hypercharge—is no meaningful criterion in the search for physically interesting fixed points in the framework of perturbative EFTs below the Planck scale. Instead, one should merely verify that the Landau pole remains transplanckian. In this aspect, we advocate a different point of view, than, e.g., [40].

6. NEW MATTER DEGREES OF FREEDOM

In the following, we allow for additional fermionic matter in arbitrary representations (as well as for the associated uncharged scalars to facilitate Yukawa couplings). We have seen that, within perturbation theory, any $U(1)$ factor will remain trivial. Therefore, we do not attempt to modify the RG flow of the $U(1)$ gauge coupling and thus only add BSM fermions which are uncharged under the $U(1)$. We allow for an arbitrary number of different representations of BSM fermions, i.e., N_F^{Ra} fermions in the (d_2^{Ra}, d_3^{Ra}) -dimensional representation of $SU(2)$ and $SU(3)$, respectively. The Lagrangian (see Equation 25) and the β -functions for the three gauge couplings, the top-Yukawa

coupling, as well as additional BSM Yukawa couplings, i.e., for

$$\alpha_1 = \frac{g_1^2}{(4\pi)^2}, \quad \alpha_2 = \frac{g_2^2}{(4\pi)^2}, \quad \alpha_3 = \frac{g_3^2}{(4\pi)^2},$$

$$\alpha_{y_t} = \frac{y_t^2}{(4\pi)^2}, \quad \alpha_{y^{Ra}} = \frac{y_{Ra}^2}{(4\pi)^2}, \quad (14)$$

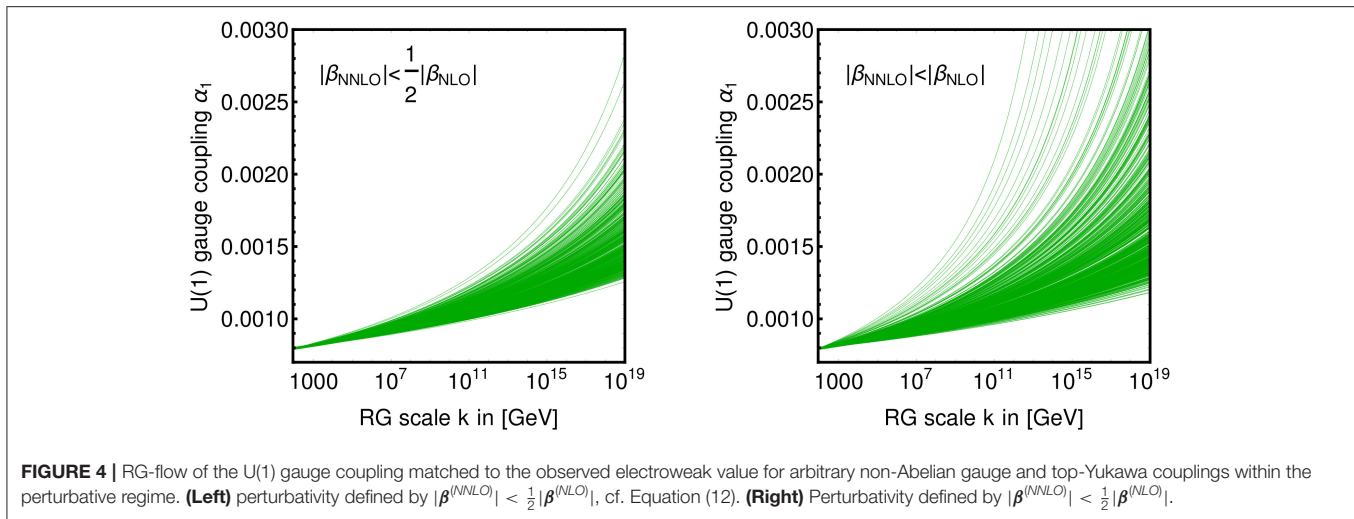
are generalized from [40] and collected in **Appendix B**. We emphasize that while the BSM scalars are uncharged, fluctuations of the charged SM Higgs scalar are always included.

With the intuition from the results in section 2 for simple non-Abelian gauge groups, we anticipate that, also in the semi-simple case, the non-Abelian subgroups cannot admit perturbatively controllable Litim-Sannino fixed points with an IR-repulsive (UV-attractive) direction. We confirm this expectation in the following explicit analysis. Fully IR-attractive gauge-Yukawa fixed points, on the other hand, can exist. From the viewpoint of effective asymptotic safety, these are the most predictive and in that sense most interesting fixed points, anyhow.

The larger the dimension of the BSM representations, the greater their screening effect on the 1-loop coefficient of the associated non-Abelian gauge coupling. Thus, there exists an upper dimension $d_{2,\text{crit}}^{Ra} = 4$ and $d_{3,\text{crit}}^{Ra} = 10$ beyond which even a single additional BSM representation will always push the associated non-Abelian SM gauge group into the completely trivial phase. Hence, the set of possible BSM representations for which perturbative non-vanishing gauge-Yukawa fixed points might exist is limited and easily tractable. With the help of computer algebra [117], we simply scan through all possibilities and identify those for which the NLO beta-functions exhibit a fixed point with

$$\alpha_{2*} > 0, \quad \alpha_{3*} > 0, \quad \alpha_{t*} > 0, \quad \text{and} \quad \alpha_{y_*^{Ra}} > 0 \quad \forall a. \quad (15)$$

We subsequently test the perturbativity of each of the resulting fixed points by initializing a numerical root search in the



NNLO beta-functions at the NLO fixed-point values. If the latter converges, we compare whether the signs of the critical exponents of the NLO and NNLO fixed points match. (If the root search does not converge, we discard the NLO fixed point). Thereby we can identify perturbative fixed points for which NNLO corrections are subleading¹⁰.

Irrespective of the specific representation $(d_2^{R_1}, d_3^{R_1})$ and the number of copies $N_F^{R_1}$, we find that a single type of BSM representation R_1 is insufficient to generate a fixed point at which both $\alpha_{2*} \neq 0$ and $\alpha_{3*} \neq 0$. IR-attractive gauge-Yukawa fixed points at which only one of the non-Abelian gauge couplings is non-vanishing are available in perturbation theory and have been identified in [40].

Proceeding to two different types of representations, i.e., R_1 and R_2 , we are able to identify a single combination of BSM representations for which a fixed point as in Equation (15) is possible, i.e.,

$$\begin{aligned} N_{R_1} &= 1 \text{ copy of the } (d_2^{R_1}, d_3^{R_1}) = (3, 1) \quad \text{and} \\ N_{R_2} &= 2 \text{ copies of the } (d_2^{R_2}, d_3^{R_2}) = (1, 6). \end{aligned} \quad (16)$$

Having specified the above representations, the respective BSM Lagrangian follows from Equation (25 in **Appendix B**). For this specific combination of BSM representations, both non-Abelian gauge groups are in the GY phase. Hence, all possible combinations of gauge-Yukawa fixed points exist. In particular, this includes a fully IR-attractive fixed point at

$$\begin{aligned} \alpha_{1*} &= 0, \quad \alpha_{2*} \approx 0.0131, \quad \alpha_{3*} \approx 0.0033, \quad \alpha_{t*} \approx 0.0124, \\ \alpha_{y*}^{R_1} &\approx 0.0082, \quad \alpha_{y*}^{R_2} \approx 0.0394. \end{aligned} \quad (17)$$

The fixed point persists at NNLO order.

To summarize, we find that by adding suitable matter content to the SM, the non-Abelian gauge-Yukawa sector of the SM can

transition from the CAF-phase to the GY-phase, and of course to the CT-phase. The explicit study supports that neither the BZ-phase nor the LS-phase is possible, cf. section 3. Within perturbation theory, the U(1) always remains in the CT-phase.

6.1. Predictivity Below the Planck Scale

For simple gauge-Yukawa theories in the CAF phase (BZ phase), the IR-complete region is reduced to the free theory (one-dimensional conformal window for vanishing Yukawa coupling), cf. **Figure 1**. Hence, these phases develop IR divergences for initial conditions that lie close to the edge of perturbativity at the cutoff scale. Put differently, they are non-predictive (as defined in section 4). On the contrary, the IR-complete region of theories in the GY or CT phase (and the LS) phase is two dimensional and covers all (or most) of the perturbative regime. Hence, these phases are predictive.

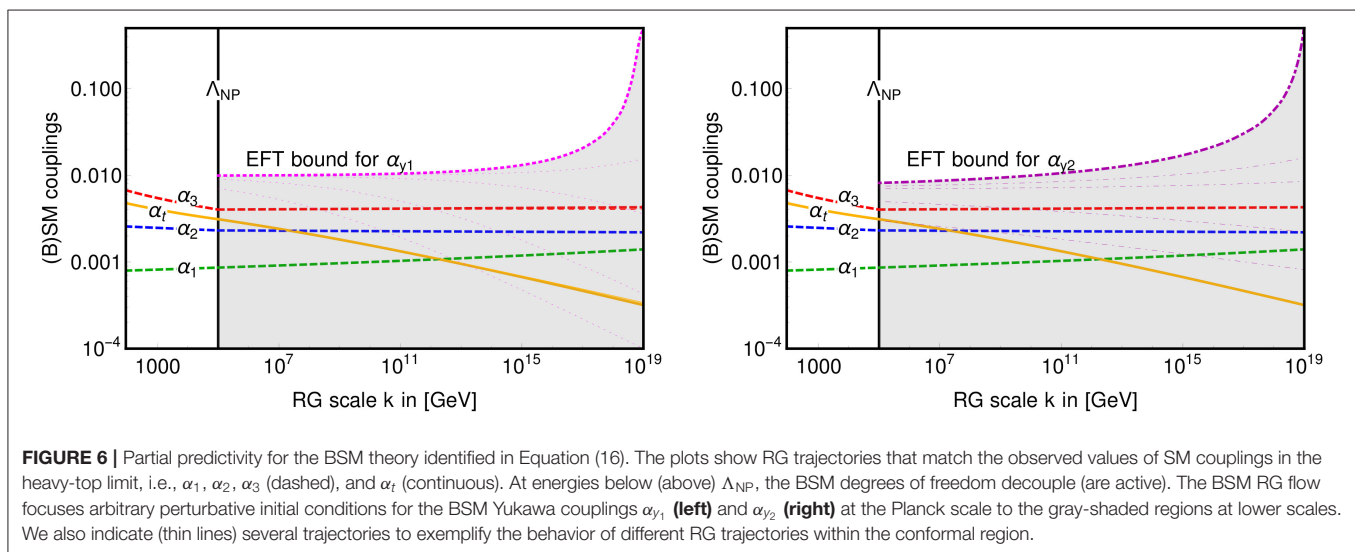
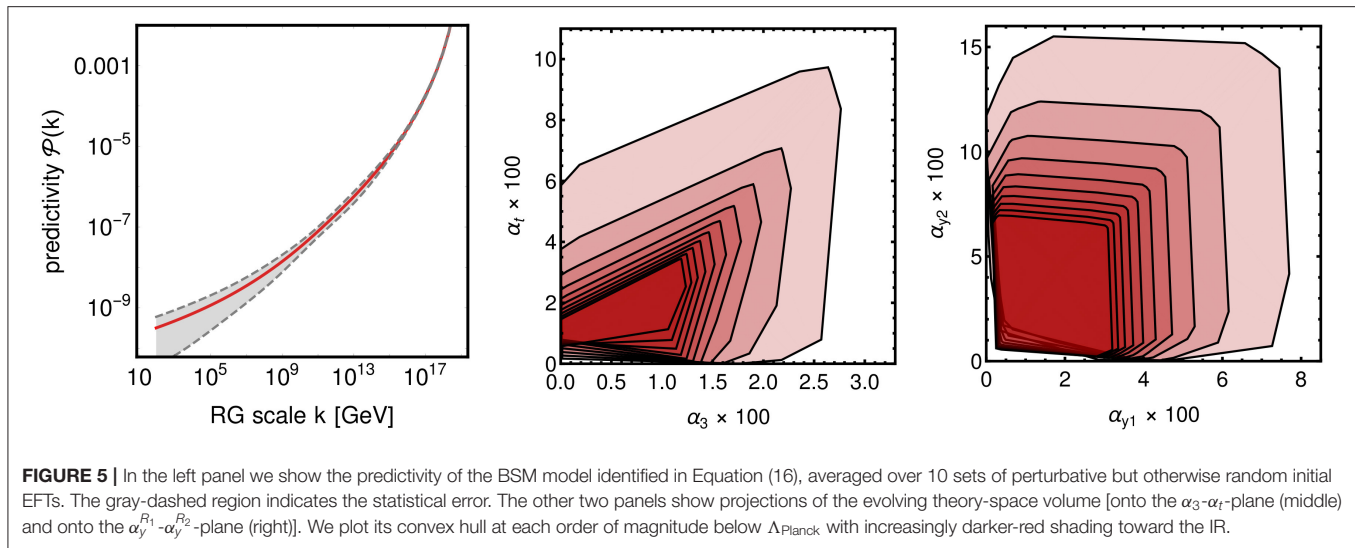
In Equation (16), we have identified a combination of BSM representations to push the non-Abelian SM subgroups into the predictive GY but not yet trivial phase. The two right-hand panels in **Figure 5** depict the associated decreasing volume in theory space as a function of the RG-flow toward the IR in two slices of the overall 6-dimensional theory space. The left-hand panel shows the corresponding evolution of the predictivity measure $\mathcal{P}(k)$. Specifying to $\Lambda_{NP} = 10^5$ GeV, the theory-space volume is reduced by a factor of $\mathcal{P}(k = \Lambda_{NP}) \sim 10^{-9}$ between Λ_{Planck} and Λ_{NP} .

Despite fixed-point values that depart significantly, i.e., by several 100%, from the measured SM values, predictivity is insufficient to exclude the BSM extension from matching to the SM electroweak scale. Put differently, the observed SM-coupling values lie within the “conformal” region of UV- and IR-complete theories (apart from the non-vanishing value of the Abelian gauge coupling, cf. section 3.2).

6.2. Partial Predictivity Below the Planck Scale

A phenomenologically more relevant question is that of partial predictivity under the condition of matching all the observed

¹⁰One might be able to construct more elaborate search algorithms and thereby potentially identify additional gauge-Yukawa BSM theories with perturbatively controlled interacting fixed points and we do not claim completeness.



SM couplings, i.e., α_1 , α_2 , α_3 , and α_t , to their measured electroweak-scale values. The resulting partial predictivity for the BSM Yukawa couplings—especially in α_{y_1} —is quite strong. **Figure 6** shows how the RG flow strongly focuses the BSM Yukawa couplings toward the IR, all the while enforcing that the SM couplings match to their electro-weak scale values. The full range of perturbative EFT values at Λ_{Planck} is mapped to values below the partial fixed point, i.e., $\alpha_{y_1}(k = \Lambda_{\text{ew}}) \lesssim 0.0165$ and $\alpha_{y_2}(k = \Lambda_{\text{ew}}) \lesssim 0.0083$. These values do not precisely match with the fixed-point values in Equation (17) because the SM couplings are matched to their electro-weak scale values, instead.

In general, any RG trajectory for the BSM Yukawa couplings in the gray region of **Figure 6** is possible. However, typical initial conditions, i.e., those which are not fine-tuned to values very close to zero, are all mapped to values very close to the partial fixed-point value, cf. thin lines in **Figure 6**. This is a result of the power-law scaling toward the interacting fixed point in Equation (17) (more specifically, toward its partial counterpart).

Quantitatively, the RG flow maps initial conditions within the perturbative but “natural” range of coupling values at the Planck scale $\alpha_{y_1}(k = \Lambda_{\text{Planck}}) \in [10^{-4}, 0.5]$ to a very narrow window at the electroweak scale, i.e., to $\alpha_{y_1}(k = \Lambda_{\text{ew}}) \in [1.646 \times 10^{-2}, 1.649 \times 10^{-2}]$. Assuming that the BSM Yukawa couplings should take such “natural,” i.e., $\mathcal{O}(1)$, values at the Planck scale, therefore predicts $\alpha_{y_1}(k = \Lambda_{\text{ew}}) \approx 1.65 \times 10^{-2}$. We caution that a correct matching to the SM values of α_t requires the latter to have an “unnatural” Planck scale value $\sim 10^{-5}$, thereby questioning the use of the above naturalness assumption. Similar arguments also apply to α_{y_2} , although partial predictivity is less pronounced, cf. **Figure 6**.

The above partial predictivity does *not* rely on the existence of a gauge-Yukawa fixed point like the one found in Equation (17). It is merely a consequence of the partial IR fixed-point for the BSM Yukawa couplings, cf. Equation (3). We list some explicit examples of BSM matter content to realize the CAF, GY, and CT phase (for both non-Abelian gauge groups) along

TABLE 2 | Predictivity $\mathcal{P}(\Lambda_{NP})$ and partial predictivity $\tilde{\mathcal{P}}(\Lambda_{NP})$ at a new-physics scale $\Lambda_{NP} = 10^5$ GeV for some selected BSM models in the three available phases characterized by their BSM matter content in the first two columns, see main text for further discussion.

$(N_F^{R_1}, d_2^{R_1}, d_3^{R_1})$	$(N_F^{R_2}, d_2^{R_2}, d_3^{R_2})$	Predictivity	Partial predictivity	Non-Abelian phase
(1, 2, 1)	(1, 1, 3)	$\mathcal{P}(\Lambda_{NP}) = \infty$	$\tilde{\mathcal{P}}(\Lambda_{NP}) \approx 2 \times 10^{-4}$	CAF
(1, 3, 1)	(2, 1, 6)	$\mathcal{P}(\Lambda_{NP}) \approx 10^{-9}$	$\tilde{\mathcal{P}}(\Lambda_{NP}) \approx 3 \times 10^{-4}$	GY
(1, 4, 1)	(11, 1, 3)	$\mathcal{P}(\Lambda_{NP}) \approx 10^{-11}$	$\tilde{\mathcal{P}}(\Lambda_{NP}) \approx 9 \times 10^{-4}$	CT
(1, 4, 1)	(1, 1, 10)	$\mathcal{P}(\Lambda_{NP}) \approx 10^{-12}$	Not viable	CT

The predictivity for the BSM model content in the last line is already high enough to exclude its validity when matched to SM values of couplings, i.e., by a sub-Planckian Landau pole in α_3 . (We refrain from listing further examples in mixed phases).

with predictivity and partial predictivity in **Table 2**. It can be concluded that while only theories in the GY and the CT phase are predictive, partial predictivity persists in all models. In particular, the partial predictivity in the absence of a gauge-Yukawa fixed point can outgrow the partial predictivity in the presence of one.

7. DISCUSSION

We have analyzed the fixed points of gauge-Yukawa theories and, in particular, the SM gauge group in the context of EFTs below the Planck scale. For the SM gauge groups, we have clarified why gauge-Yukawa fixed points with UV-attractive directions cannot occur within the perturbatively controlled regime. However, additional matter fields can result in a perturbative and fully IR-attractive gauge-Yukawa fixed point which realizes *effective asymptotic safety*. We have introduced a novel quantitative measure for the predictivity of general EFTs and have applied it to gauge-Yukawa BSM extensions. Concerning concrete BSM phenomenology, this allows us to make the following conclusions:

- The results highlight that the presence of an (Abelian) Landau pole, as long as it occurs at trans-Planckian scales, does *not* pose a strict no-go criterion in the search of perturbative interacting fixed points in non-gravitational and hence necessarily effective theories with a Planckian cutoff.
- We have identified a fully IR-attractive and (apart from the Abelian gauge coupling) fully interacting fixed point if suitable vector-like fermions without hypercharge, i.e., one $SU(3)$ singlet in the three-dimensional representation of $SU(2)$ and two $SU(2)$ singlets in the six-dimensional representation of $SU(3)$, are added to the SM, cf. Equations (16) and (25 in **Appendix B**) for the corresponding BSM Lagrangian. This particular theory is predictive along the RG flow toward the IR. We have quantified its predictive power and compared it to other BSM models without interacting fixed points. For all these models, partial predictivity restricts the range of coupling values of the BSM Yukawa couplings in dependence on the ratio between the BSM scale and the cutoff scale.
- In general, the predictive power of subplanckian effective asymptotic safety of gauge-Yukawa theories can be estimated by a simple argument: Let $\epsilon \ll 1$ be the perturbative parameter. For simple gauge-Yukawa theories, $\epsilon \lesssim 0.1$ has been found in [22] as the indicated regime of perturbative control. Perturbative fixed points that come about by the balance

of loop orders will necessarily result in critical exponents θ proportional to some power of ϵ , i.e., $\theta \lesssim \epsilon$. Extrapolating the linearized regime around the fixed point, one therefore expects $(\alpha(\Lambda_{IR}) - \alpha_*)/(\alpha(\Lambda_{UV}) - \alpha_*) = \epsilon \log(\Lambda_{IR}/\Lambda_{UV})$ for the associated coupling α . For the phenomenologically important case of $\Lambda_{NP}/\Lambda_{\text{Planck}} < \Lambda_{\text{ew}}/\Lambda_{\text{Planck}} \sim 10^{17}$, predictivity is thus expected to be limited to shrinking the allowed region of all perturbative coupling values by one or two orders of magnitude. This simple argument also motivates that predictivity can be further increased (i) for non-perturbative fixed points—as e.g., tentatively suggested in a toy model in [118]—because θ need not be small and (ii) for potential fixed points including gravitational fluctuations, see e.g., [119–122] since Λ_{UV} can be extended beyond the Planck scale.

More generally, the example of gauge-Yukawa theories suggests that the boundaries of all UV-complete and/or IR-complete theories constitute special hypersurfaces in the theory space. In particular, we have made the following observations.

- The boundary hypersurfaces separate theories on both sides. Whenever one is confident that such a boundary exists and one knows that experimentally observed values lie either inside or outside, one can exclude that the observed IR physics originates from UV physics on the other side of the boundary.
- Moreover, the boundary surfaces can inherit the IR-attractive properties of their delimiting fixed point. In such cases, generic EFTs at the cutoff scale—both UV complete and not UV complete—will converge to realize coupling values closer to the boundary surface toward the IR. This is a first step to generalize the local notion of fixed points to global IR-attractors in theory space.

These two points highlight that knowledge about such boundary surfaces can be of great value whenever one tries to relate theories at different scales. Of course, having all the information to exactly reconstruct the boundary surface amounts to knowing about all RG flows in its vicinity. One might, therefore, object that with this information one could directly evolve a theory between different scales and obtain its counterpart at other scales. However, this is true only if one knows about *all* the coupling values at a given scale which is typically not the case in the search for new physics. The constraints on BSM Yukawa couplings, that partial predictivity and perturbativity up to the Planck scale entail, provide for an example to emphasize this more general point.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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SUPPLEMENTARY MATERIAL

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Canonical Quantum Gravity, Constructive QFT, and Renormalisation

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The canonical approach to quantum gravity has been put on a firm mathematical foundation in the recent decades. Even the quantum dynamics can be rigorously defined, however, due to the tremendously non-polynomial character of the gravitational interaction, the corresponding Wheeler–DeWitt operator-valued distribution suffers from quantisation ambiguities that need to be fixed. In a very recent series of works, we have employed methods from the constructive quantum field theory in order to address those ambiguities. Constructive QFT trades quantum fields for random variables and measures, thereby phrasing the theory in the language of quantum statistical physics. The connection to the canonical formulation is made via Osterwalder–Schrader reconstruction. It is well known in quantum statistics that the corresponding ambiguities in measures can be fixed using renormalisation. The associated renormalisation flow can thus be used to define a canonical renormalisation programme. The purpose of this article was to review and further develop these ideas and to put them into context with closely related earlier and parallel programmes.

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1. INTRODUCTION

The canonical approach to quantum gravity has been initialised long time ago [1–14]. However, the mathematical foundations of the theory remained veiled due to the tremendous non-linearity of the gravitational interaction. This has much changed with the reformulation of general relativity as a Yang–Mills type gauge theory in terms of connection, rather than metric variables [15,16], and has culminated in a research programme now known as loop quantum gravity (LQG) (see e.g., Refs. 17–21 for monographs and recent reviews on the subject). The qualifier ‘loop’ stems from the fact that for gauge theories of Yang–Mills type, it has proved useful to formulate the theory in terms of holonomies of the connection along closed paths (loops) in order to maintain manifest gauge invariance. Such so-called (Wilson) loop variables are widely used, for instance, in (lattice) QCD [22].

LQG has succeeded in providing a rigorous mathematical framework: The representation theory of the canonical commutation relations and the $*$ relations has been studied and a unique representation has been singled out [23–27] that allows for a unitary representation of the spatial diffeomorphism group. Moreover, the generators of temporal diffeomorphisms, sometimes referred to as Wheeler–DeWitt operators, could be rigorously quantised on the corresponding Hilbert space [28–32], and in contrast to the perturbative approach to quantum gravity [33, 34], no ultraviolet divergences were found. It should be emphasised that this was achieved 1) in the continuum, rather than on a lattice, that is, there is no artificial cut-off left over; 2) for the physical Lorentzian signature, rather than unphysical Euclidian one; and 3) non-perturbatively

and background independently, that is, one does not perturb around a classical background metric and then quantises the fluctuations which thus manifestly preserves the diffeomorphism covariance of all constructions.

However, the theory is not yet completed: Due to the tremendously non-polynomial nature of the gravitational interaction, the usual factor ordering ambiguity in the quantisation of operator-valued distributions which are non-linear in the fields is much more severe. Thus, the operators defined in Refs. 28–32 suffer from those ambiguities. Moreover, the following problem arises: In the classical theory, the canonical generators of space-time diffeomorphisms (i.e., their Hamiltonian vector fields) form a Lie algebroid (i.e., a Lie algebra except that the structure constants are replaced by structure functions on the phase space) known as the hypersurface algebroid [35]. The structure functions are themselves promoted to operator-valued distributions upon quantization; thus, it becomes even harder to find quantization of those generators such that the algebroid is represented without anomalies than it would be for an honest Lie algebra. Specifically, the commutator between two temporal diffeomorphism generators is supposed to 1) be proportional to a linear combination of spatial diffeomorphism generators with operator-valued distributions as coefficients and 2) in an ordering, such that the following holds: The image of any such commutator of a dense domain of vectors in the Hilbert space must be in the kernel of the space of spatially diffeomorphism-invariant distributions on that domain. In Ref. 37, it is shown that both conditions 1) and 2) hold; however, the coefficients in that linear combination do not qualify as quantisations of their classical counterpart. Thus, while the quantisation of the hypersurface algebroid closes, it does so with the wrong operator-valued distributions as coefficients.

Thus, the status of LQG can be summarised as follows:

As compared to Refs. 1–14, it is now possible to ask and answer precise questions about the mathematical consistency of the whole framework. As compared to the perturbative approach, the framework does not suffer from ultraviolet divergences and one does not have to worry about the convergence of a perturbation series due to the manifestly non-perturbative definition of LQG. However, just as in the perturbative approach, one needs further input in order to draw predictions from the theory, although of a different kind: In the perturbative approach, there are an infinite number of counter terms necessary due to non-perturbative non-renormalisability all of which come with coefficients that have to be measured, but one can argue that only a finite number of them is of interest for processes involving energies not exceeding a certain threshold (effective field theory point of view). In LQG, there are in principle infinitely many quantisation ordering prescriptions possible, each of which comes with definite coefficients in order to yield the correct naive continuum limit, but it is not clear which ordering to choose so that presently one resorts to the principle of least technical complexity.

Various proposals have been made in order to improve the situation. In Ref. 38, one exploits the fact that classically one can

always trade a set of first-class constraints by a single weighted sum of their squares (called the master constraint). Since a single constraint always closes with itself and the weights can be chosen such that the master constraint commutes with spatial diffeomorphisms, one can now focus on the quantisation ambiguities involved in the master constraint without having to worry about anomalies. In Ref. 39, the case of general relativity coupled to perfect fluid matter was considered, which allows solving the constraints before quantisation so that the remaining quantisation ambiguity now only rests in the corresponding physical Hamiltonian that drives the time evolution of the physical (i.e., space-time diffeomorphism-invariant) observables. In Refs. 40–42, the constraints are quantised on a suitable space of distributions with respect to a dense domain of the Hilbert space, rather than the Hilbert space itself in order to find a representation of the hypersurface algebroid directly on that space of distributions which would at least partially fix the aforementioned ordering ambiguity.

It transpires that additional input is necessary in order to fix the quantisation ambiguity in the dynamics of LQG and thus to complete the definition of the theory. This would also put additional faith in applications of LQG, for instance to quantum cosmology [43–46] (where the amount of ambiguity is drastically reduced) which are believed to be approximations of LQG by enabling to make the connection between LQG and those approximations precise including an error control (see Refs. 47–53 for recent progress in that respect). In the recent proposal [54–57] which we intend to review in this article, the authors were inspired by Wilson's observation [54–57] that renormalisation methods help identify among the principally infinitely many interaction terms in Hamiltonians relevant for condensed matter physics the finitely many relevant ones that need to be measured. This insight implies that a theory may be perturbatively non-renormalisable but non-perturbatively renormalisable, also known as asymptotically safe [58]. The asymptotic safety approach to quantum gravity for Euclidian [59–68] and Lorentzian signature [69, 70] precisely rests on that idea and has received much attention recently. In fact, there is much in common between our proposal and asymptotically safe quantum gravity (especially for Lorentzian signature), and we will have the opportunity to spell out more precisely points of contact in the longer version of this article [196].

Also, there is a large body of work on renormalisation [71–75] in the so-called spin foam approach [85–92] and the related group field theory [76–81] and tensor model¹ [82–84] approach to quantum gravity. The spin foam approach is loosely connected to LQG in the following sense: The states of the Hilbert space underlying LQG are labelled by collections of loops, that is, 3D

¹In principle, any field theory with a polynomial Lagrangian can be written as a (coloured) tensor model as follows: Pick any orthonormal basis with respect to the measure appearing in the action, expand the field in that basis, call the expansion coefficients a coloured (by the space-time or internal indices) tensor in an infinite-dimensional ℓ_2 space, and call the integral over polynomials in those basis functions that appear in the action upon expanding the fields interaction terms of those tensors. If the basis carries labels in $\mathbb{N}^{nd}$, we obtain a coloured tensor model with tensors of rank n .

graphs. A spin foam is an operator that maps such states excited on a graph to states excited on another graph. The operator depends on a specific class of 4D cell complex (foam) such that its boundary 3D complex is dual to the union of the two graphs corresponding to the incoming and outgoing Hilbert spaces. The operator is supposed to form the rigging map [93] of LQG, that is, a generalised projector onto the joint kernel of the Wheeler–DeWitt constraints. We say that the connection is loose because the rigging nature of current spin foams in 4D is not confirmed yet. In any case, a spin foam operator can be formulated as a state sum model, and thus, renormalisation ideas apply. (For applications of renormalisation group ideas in the cosmological sector of LQG, see Refs. 94–96.)

Most of the work on renormalisation is either within classical statistical physics (e.g., Ref. 97) or the Euclidian (also called constructive) approach to the quantum field theory [98–100]. In the Euclidian approach, the quantum field, which is an operator-valued distribution on Minkowski space, is replaced by a distribution-valued random variable on Euclidian space. While the dynamics in the Minkowski theory is given by Heisenberg equations, in the Euclidian theory, it is encoded in a measure on the space of random variables. We are then back in the realm of statistical physics because loosely speaking, the measure can be considered as a Gibbs factor for a Hamiltonian (sometimes called Euclidian action) in four spatial dimensions. How then should one use renormalisation ideas for quantum gravity? Quantum gravity is not a quantum field theory on Minkowski space (unless one works in the perturbative regime, but then it is non-renormalisable). Also, while the Minkowski and Euclidian signature of metrics are related by simple analytic rotation in time from the real to the imaginary axis, this does not even work for classical metrics with curvature, not to mention the quantum nature of the metric (in ordinary QFT, the metric is just a non-dynamic background structure). One can, of course, start with Euclidian signature GR and try to build a measure theoretic framework, but then the relation to the Lorentzian signature theory is unclear. Moreover, while as an ansatz for the Euclidian signature measure, we can take the exponential of the Euclidian Einstein–Hilbert action, that action is not bounded from below, and thus, the measure cannot be a probability measure which is one of the assumptions of constructive QFT. Finally, in contrast to constructive QFT, in quantum gravity expectation, values (operator language) or means (measure language) of basic operators (or random variables) such as the metric tensor have no direct physical meaning because coordinate transformations are considered as gauge transformations; hence, none of the basic fields correspond to observables.

In our approach [54–57], we will use the framework [39], that is, we do not consider vacuum GR but GR coupled to matter which acts as a dynamical reference field. This enables us 1) to solve the spatial diffeomorphism and Hamiltonian constraints classically, 2) to work directly on the physical Hilbert space (i.e., the generalised kernel of all constraints equipped with the inner product induced by the rigged Hilbert space structure, 3) to have at our disposal immediately the gauge-invariant degrees of freedom such that the physical Hilbert space is the representation space of a $*$ representation of those observables, and 4) to be

equipped with a physical Hamiltonian that drives the physical time evolution of those observables. Concretely and out of mathematical convenience, we use the perfect fluid matter suggested in Refs. 101 and 102, but for what follows, these details are not important. Important is only that it is possible to rephrase GR coupled to matter as a conservative Hamiltonian system and that all the machinery that was developed for LQG can be imported. Now, the quantisation ambiguity rests, of course, in the physical Hamiltonian and it is that object and its renormalisation on which we focus our attention.

As we just explained, we can bring GR coupled to matter somewhat closer to the usual setting of ordinary QFT or statistical physics, but still we cannot apply the usual path integral renormalisation scheme because we work in the canonical (or Hamiltonian) framework. The idea is then to make use of Feynman–Kac–Trotter–Wiener-like ideas in order to generate a Wiener measure theoretic framework from the Hamiltonian setting and vice versa to use Osterwalder–Schrader reconstruction to map the measure theoretic (or path integral) framework to the Hamiltonian one. This way we can map between the two frameworks and thus import path integral renormalisation techniques into the Hamiltonian framework which are strictly equivalent to those employed in path integral renormalisation. In order that this works one needs to check, of course, that the Wiener measure constructed obeys at least a minimal subset [103] of Osterwalder–Schrader axioms [104] in order for the reconstruction to be applicable, most importantly reflection positivity.

This was one of the goals of [54–57], namely, to define a renormalisation group flow directly within the Hamiltonian setting with strict equivalence to the path integral flow. Specifically, the flow is a flow of Osterwalder–Schrader triples $(\mathcal{H}, H, \Omega)$ consisting of a Hilbert space $\mathcal{H}$, a self-adjoint Hamiltonian H thereon bounded from below, and a vacuum vector $\Omega \in \mathcal{H}$ annihilated by H . While physically well-motivated, of course, one does not need to do this. Indeed, renormalisation techniques for Hamiltonians and vacua directly within the Hamiltonian setting were invented before, and we devote the next section for putting our framework into context with schemes closely related to ours. The fact that we have a precise relation between Hamiltonian and path integral renormalisation makes it possible to bring Hamiltonian formulations of quantum gravity such as LQG and path integral formulations, such as asymptotically safe quantum gravity, into closer contact.

The architecture of this article is as follows:

In the second section, we give an incomplete overview over and sketch Hamiltonian renormalisation frameworks closely related to ours and point out differences and similarities.

In the third section, we review how classical general relativity coupled to suitable matter can be brought into the form of a conservative Hamiltonian system and the LQG quantisation thereof. The necessity to remove quantisation ambiguities will be highlighted.

In the fourth section, we recall some background material on constructive QFT, the Feynman–Kac–Trotter–Wiener construction, and Osterwalder–Schrader reconstruction.

In the fifth section, we derive the natural relation between families of cylindrically defined measures, coarse graining, renormalisation group flows, and their fixed points. We then

use Osterwalder–Schrader reconstruction to map the flow into the Hamiltonian framework. This section contains new material as compared to [54–57] in the sense that we 1) develop some systematics in the choice of coarse graining maps that are motivated by naturally available structures in the classical theory, 2) clarify the importance of the choice of random variable or stochastic process when performing OS reconstruction, and 3) improve the derivation of the Hamiltonian renormalisation flow by adding the uniqueness of the vacuum as an additional assumption (also made in the OS framework of Euclidian QFT [98–100]) as well as some machinery concerning degenerate contraction semi-groups and associated Kato–Trotter formulae.

In the sixth section, we summarise, spell out implications of the renormalisation programme for the anomaly-free implementation of the hypersurface algebroid, and outline the next steps when trying to apply the framework to interacting QFT and finally canonical quantum gravity such as LQG.

The paper is supplemented by the following appendices:

In **Supplementary Appendix A**, we prove some properties for a coarse graining scheme appropriate for non-Abelian gauge theories; in **Supplementary Appendix B**, we prove a lemma on the existence of certain Abelian C^* -algebras needed for the construction of stochastic processes during OS reconstruction; in **Supplementary Appendix C**, we collect some renormalisation terminology for readers more familiar with actions, rather than measures; in **Supplementary Appendix D**, we give a proof for the Kato–Trotter product formula for semi-groups and projections in the simple case that the semi-group has a bounded generator; and in **Supplementary Appendix E**, we prove a strong limit identity between projections needed in **Section 5.3**.

In **Supplementary Appendix F**, we mention concrete points of contact between the scheme developed here and others in the context of density matrix, entanglement, and projective renormalisation.

In **Supplementary Appendix G**, we sketch a relation between Hamiltonian renormalisation via Osterwalder–Schrader reconstruction and the functional renormalisation group which is the underlying technique of the asymptotic safety programme. This article is the journal version of Ref. 196 which is organised slightly differently in the sense that Appendices F, G of this article are part of the main text of Ref. 196.

2. OVERVIEW OVER RELATED HAMILTONIAN RENORMALISATION SCHEMES

The purpose of this section is not to give a complete scan of the vast literature on the subject of Hamiltonian renormalisation but just to give an overview over those programmes that we believe are closest to ours. Also, we leave out many finer details as we just want to sketch their relation to our framework in broad terms. In sections 6 and 7 of Ref. 196, we will give a few more details on the connection between our approach and the density matrix and functional renormalisation group.

The starting point is, of course, the seminal works by Kadanoff [105] and Wilson [106, 107]. Kadanoff introduced the concept of a block spin transformation in statistical physics, that is, a coarse

graining transformation in *real space* (namely, on the location of the spin degrees of freedom on the lattice), rather than in some more abstract space (e.g., momentum space blocking/suppressing as used, e.g., in the asymptotically safe quantum gravity approach). This kind of real-space coarse graining map is widely used not only in statistical physics but also in the path integral approach to QFT as, for instance, in lattice QCD [108]. On the other hand, Wilson introduced the concept of *Hamiltonian diagonalisation* to solve the Kondo problem (the low-temperature behaviour of the electrical resistance in metals with impurities). This defines a renormalisation group flow directly on the space of Hamiltonians and its lowest lying energy eigenstates. More precisely, one considers a family of Hamiltonians labelled by an integer-valued cut-off on the momentum mode label of the electron annihilation and creation operators. The renormalisation group flow is defined by diagonalising the Hamiltonian given by a certain cut-off label, and to use the eigenstates so computed to construct the matrix elements of the Hamiltonian at the next cut-off label. To make this practical, Wilson considered a *truncation*, at each renormalisation step, of the full energy spectrum to the 10^3 *lowest lying energy levels* which was sufficient for the low-temperature Kondo problem. This is in fact nothing but the concrete application of the Rayleigh–Ritz method. The concept of truncation plays an important role also in most other renormalisation schemes, as otherwise the calculations become unmanageable.

The next step was done by Wegner [109, 110] as well as Glazek and Wilson [111] which can be considered as a generalisation of the Hamiltonian methods of Refs. 106 and 107. It could be called *perturbative Hamiltonian block diagonalisation* and was applied in QFT already (e.g., Refs. 112 and 113 and references therein). Roughly speaking, one introduces a momentum cut-off on the modes of the annihilation and creation operators involved in the free part of the Hamiltonian, then perturbatively (with respect to the coupling constant) constructs unitarities which at least *block diagonalise* that Hamiltonian with respect to a basis defined by modes that lie below half the cut-off and those that lie between half and the full cut-off, and then projects the Hamiltonian onto the Hilbert space defined by the modes below half of the cut-off to define a new Hamiltonian at half the cut-off. This can be done for each value of the cut-off and thus defines a flow of Hamiltonians (and vacua defined as their ground states). Another branch of work closely related to this is the projective programme due to Kijowski [114, 115]. Here, a flow of Hamiltonians on Hilbert spaces for different resolutions is given by the partial traces of the corresponding density matrices given by minus their exponential (Gibbs factors, assuming that these are trace class). (See also Refs. 116–123 for more recent work on renormalisation building on this programme.)

In these developments, the spectrum of the Hamiltonian was directly used to define the flow. Another proposal was made by White [124] who defined the *density matrix renormalisation group*. This is a real-space renormalisation group flow which considers the reduced density matrix corresponding to the tensor product split of a vector (e.g., the ground state of a Hamiltonian) of the total Hilbert space into two factors corresponding to a block and the rest (or at least a much larger ‘superblock’). This

density matrix is diagonalised, and then, the Hilbert space is *truncated* by keeping only a certain fixed number of *highest lying eigenvalues* of the reduced density matrix. Finally, the Hamiltonian corresponding to the block is projected, and then, the resulting structure is considered as the new structure on the coarser lattice resulting from collapsing the blocks to new vertices (we are skipping here some finer details). This method thus makes use of entanglement ideas since the reduced density matrix defines the degree of entanglement via its von Neumann entropy.

A variant of this is the *tensor renormalisation group approach* due to Levin and Nave [125]. It is based on the fact that each vector in a finite tensor product of *finite-dimensional Hilbert spaces* can be written as a *matrix product state*, that is, the coefficients of the vector with respect to the tensor product base can be written as a trace of a product of matrices of which there are, in general, as many as the dimensionality of the Hilbert space. One now performs a real-space renormalisation scheme directly in terms of those matrices which are considered to be located on a lattice with as many vertices as tensor product factors. Importantly, this work connects renormalisation to the powerful numerical machinery of tensor networks [126].

Finally, as observed by Vidal [127] and Evenbly and Vidal [128, 129], one can improve [124, 125] by building in an additional unitary disentanglement step into the tensor network renormalisation scheme. This is quite natural because a tensor network can also be considered as a quantum circuit with the truncation steps involved considered as isometries, but a quantum circuit in quantum computing [130] consists of a network of unitary gates, some of which have a disentangling nature depending on the state that they act upon. The resulting scheme is called multi-scale entanglement renormalisation ansatz (MERA).

As this brief and incomplete discussion reveals, there are numerous proposals in the literature for how to renormalise quantum systems. They crucially differ from each other in the choice of the coarse graining map. There are various aspects that discriminate between these maps, such as the following:

(1) Real space vs. other labels

The degrees of freedom to be coarse grained are labelled by points in space-time or else (momentum, energy, etc.).

(2) Kinematic vs. dynamical

Real-space block spin transformations are an example of a kinematic coarse graining, that is, the form of the action, a Hamiltonian, its vacuum vector, its associated reduced density matrix, and the corresponding degree of entanglement do not play any role. By contrast, Hamiltonian block diagonalisation, density matrix, and entanglement renormalisation take such dynamical information into account.

(3) Truncated vs. exact

In principle, any renormalisation scheme can be performed exactly, for example, in real-space path integral renormalisation,

one can just integrate the excess degrees of freedom that live on the finer lattice but not on the coarser, thus obtaining the measure (or effective action) on the coarser lattice from that of the finer one. The same is true, for example, for the procedure followed in asymptotically safe quantum gravity. However, in practice, this may quickly become unmanageable, and thus, one resorts to approximation methods, for example, by truncation in the space of coupling constants, energy eigenstates, or reduced density matrix eigenstates.

For the newcomer to the subject, this plethora of suggestions may appear confusing. Which choice of coarse graining is preferred? Do different choices lead to equivalent physics? What can be said about the convergence of various schemes and what is the meaning of the fixed point(s) if it (they) exist(s)? The physical intuition is that different schemes should give equivalent results if 1) the corresponding fixed point conditions capture necessary and sufficient properties that the theory should have in order to qualify as a continuum theory and 2) *when performed exactly*. The first condition is obvious: we start from what we believe to be an initial guess for how the theory looks at different resolutions and then formulate a coarse graining flow whose fixed points are such that they qualify to define a continuum theory. The second condition entails that the coarse graining maps just differ in the separation of the total set of degrees of freedom into subsets corresponding to coarse and fine resolution, hence corresponds to choices of coordinate systems which, of course, can be translated into each other. However, when truncations come into play, this equivalence is lost because different schemes truncate different sets of degrees of freedom which are generically no longer in bijection. It is conceivable therefore that dynamically driven truncation schemes perform better at identifying the correct fixed point structure of the theory in the sense that they may converge faster and are less vulnerable to truncation errors or automatically pick the truncation of *irrelevant* couplings. This seems to be confirmed in spin system examples, but we are not aware of a general proof. Recently, the importance of the kinematic vs. dynamic issue has also been emphasised for the LQG and spin foam approach [131–133].

In our work, we currently are not concerned with issues of computability, that is, we consider an exact scheme. Next, as far as the coarse graining map is concerned, we currently favour a kinematic scheme. The reason for doing this is that kinematic schemes are *naturally suggested by measure theoretic questions*, namely, measures on spaces of infinitely many degrees of freedom are never of the type of the exponential of some action times a normalisation constant times Lebesgue measure. Neither of these three ingredients is well defined. What is well defined are integrals of certain probe functions of the field with respect to that measure. These probe functions, in turn, are naturally chosen to depend on test functions that one integrates the field against. Thus, these test functions provide a natural notion of resolution, discretisation, and coarse graining. By integrating the measure against probe functions, one obtains a family of measures labelled by the test functions involved. The relation between test functions at different resolution induces a corresponding relation between members of the family of measures which *must hold exactly* for a

true measure of the continuum QFT. In turn, such consistency relations called *cylindrical consistency* can be used to *define* a measure on a space of infinitely many degrees of freedom [134], called a *projective limit*. The idea is then to formulate measure renormalisation in such a way that its fixed points solve the consistency relations. This approach has been advocated in Refs. 135 and 136 for Euclidian Yang–Mills theory and in Refs. 137 and 138 for spin foams. Note that spin foams, strictly speaking, do not construct measures but rather are supposed to construct a rigging map so that Hamiltonian methods come also into play. Indeed, in Refs. 131–133, it was shown that the cylindrically consistent coarse graining of the rigging map and its underlying space-time lattice, thought of as an anti-linear functional on the kinematical Hilbert space, induce a coarse graining of the spatial lattice on its boundary and thus the Hilbert space thereon, equipping it with a system of consistent embeddings, a structure similar to inductive limits of Hilbert spaces (an inductive structure requires in addition the injections to be isometric). That latter structure underlies the kinematical Hilbert space of LQG, and a renormalisation procedure based on inductive limits was already proposed in Refs. 139–141 due to the similarity of LQG to the lattice gauge theory.

Another reason for why picking real-space coarse graining schemes as compared to, say, momentum space-based ones is their background independence, which is especially important for quantum gravity. In our work, as we consider the version of LQG in which the constraints already have been solved, we will work with probability measures. As we will see, the connection between inductive limits of Hilbert spaces and projective limits of path integral measures can be made crystal clear in this case. The price we pay by using an exact, kinematical scheme is that the fixed point (or renormalised) Hamiltonian becomes spatially non-local at finite resolution. However, in the free QFT examples studied [54–57], which are spatially local in the continuum, by blocking the known fixed point theory from the continuum, one can see that this is natural and *must happen* for such schemes; hence, it is not a reason for concern but, in fact, physical reality. The degree of spatial non-locality, in fact, decreases as we increase the resolution scale.

When applying the framework to interacting QFT, one will have to resort to some kind of approximation scheme, and possibly, tools from entanglement renormalisation combined with tensor network techniques may prove useful. However, note that QFT of bosonic fields (gravity is an example) deals with infinite-dimensional Hilbert spaces even when the theory depends only on a finite number of degrees of freedom, say, by discretising it on a lattice and confining it to finite volume. Thus, to apply tensor network techniques which, to the best of our knowledge, require the factors in the tensor product to be finite-dimensional Hilbert spaces, one would have to cut off the dimensions of those Hilbert spaces right from the beginning, that is, one would have to work with *three* cut-offs, rather than two (see, e.g., Refs. 142 and 143 where quantum group representations are used in gauge theories, rather than classical group representations, and perform real-space renormalisation or [144] where one combines both the UV and the dimension cut-off into one by turning the dimension of tensor spaces in tensor

models into a finite coarse graining parameter and otherwise performs the asymptotic safety programme which is often formulated in the presence of a cut-off anyway).

Some sort of truncation or approximation has to be made in practice when treating complex systems numerically. The physical insight behind the tensor network and density matrix/entanglement renormalisation developments, namely, the dynamically interesting vectors in a Hilbert space appear to lie in a ‘tiny’ subspace thereof is presumably a profound one, and the truncation of the Hilbert space to the corresponding subspaces appears to be well-motivated by the model (spin) systems studied so far. Still, what one would like to have is some sort of error control or convergence criteria on those truncations. We appreciate that this is a hard task for the future. For the time being, we phrase our framework without incorporating a cut-off on the dimension of Hilbert spaces as we are not yet concerned with numerical investigations; however, we may have to use some of these ideas in the future.

3. CANONICAL QUANTUM GRAVITY COUPLED TO REFERENCE MATTER

The physical idea is quite simple and goes back to Ref. 145: General relativity is a gauge theory, the gauge group being the space-time diffeomorphism group. Thus, the basic tensor and spinor fields in terms of which one writes the Einstein–Hilbert action and the action of the standard model coupled to the metric (or its tetrad) are not observable. However, the value of, say, a scalar field ϕ at that space-time point X_y , at which four reference scalar fields $\phi^0, \dots, \phi^3$ take values $y^0, \dots, y^3$, that is, $\phi(X_y)$; $\phi^\mu(X_y) = y^\mu$ is space-time diffeomorphism-invariant. For this to work, the relation $\phi^\mu(X) = y^\mu$ must, of course, be invertible, in particular the reference scalar fields must not vanish anywhere or anytime. This seems to be a property of dark matter [146].

These kinds of relational observables have been further developed by various authors, in particular [147–153]. When one couples general relativity and such reference matter preserving general covariance, it becomes possible to formulate the theory in a manifestly gauge-invariant way. The form of that gauge-invariant formulation, of course, strongly depends on the type of reference matter used and its Lagrangian. In what follows, we use the concrete model [39] out of mathematical convenience, but we emphasise that the same technique works in a fairly general context. In the next subsection, that model will be introduced and the classical gauge-invariant formulation will be derived. After that, we quantise it using LQG methods which will be introduced in tandem.

3.1. Gaussian Dust Model

The Lagrangian of the theory takes the form

$$L = L_{EH} + L_{SM} + L_D, \quad (3.1)$$

where L_{EH} is the Einstein–Hilbert Lagrangian, L_{SM} is the standard model Lagrangian coupled to GR via the metric, its tetrad or its spin connection, and L_D is the Gaussian dust Lagrangian [101, 102]

$$L_D = -\frac{1}{2} \sqrt{|\det(g)|} \left\{ g^{\mu\nu} \left[\rho (\nabla_\mu T) (\nabla_\nu T) + 2(\nabla_\mu T)(W_j \nabla_\nu S^j) \right] + \rho \right\}, \quad (3.2)$$

where g is the Lorentzian signature metric tensor, ∇_μ its Levi-Civita covariant differential, $\phi^0 := T$, $\phi^j := S^j$; $j = 1, 2, 3$ are the reference scalar fields introduced above, and ρ , W_j are additional four scalar fields. The latter four fields appear without derivatives and thus give rise to primary constraints in addition to those present even in vacuum GR. One can easily show that the contribution of L_D to the energy momentum tensor is of perfect fluid type. Further physical properties and motivations are discussed in Refs. 101 and 102. For what follows, it suffices to know that the equations of motion for T , S^j , say that $\nabla_\mu T$ is a time-like geodesic cotangent and that S^j is constant along the geodesic spray. Thus, those geodesics can be interpreted as world lines of dynamically coupled test observers.

The full constraint analysis of Eq. 3.1 is carried out in Ref. 39. There are secondary constraints, and the full set of constraints contains those of the first and second classes (see Ref. 153 for a modern treatment of Dirac's algorithm [154]). One has to introduce a Dirac bracket and solve the second-class constraints in the course of which the variables ρ , W_j are eliminated. The remaining constraints are then of first class and read as

$$C^{\text{tot}} = C + \frac{P - q^{ab} T_{,a} C_b}{\sqrt{1 + q^{ab} T_{,a} T_{,b}}}, \quad C_a^{\text{tot}} = C_a + P T_{,a} + P_j S^j_{,a} \quad (3.3)$$

Here, C is the Wheeler–DeWitt constraint function (including standard matter) and C_a , $a = 1, 2, 3$ are the spatial diffeomorphism functions (including standard matter). The Dirac bracket reduces to the Poisson bracket on all the variables involved in Eq. 3.3, and P , P_j are the momenta conjugate to T , S^j , for example, $\{P(x), T(y)\} = \delta(x, y)$. Here, $a = 1, 2, 3$ are tensorial indices on the spatial hypersurface σ of the Arnowitt–Deser–Misner foliation underlying the Hamiltonian formulation of GR [155] with intrinsic metric tensor q_{ab} . For the moment, it is just necessary to know that C and C_a do not involve the variables T , P , S^j , and P_j .

The constraints (Eq. 3.3) encode the space-time diffeomorphism gauge symmetry in Hamiltonian form, in particular they represent the hypersurface deformation algebra [111]. It is possible to solve these remaining constraints to determine the complete set of gauge-invariant (the so-called Dirac) observables and to determine the physical Hamiltonian H that drives their physical time evolution [39]. Equivalently, we may gauge fix Eq. 3.3. The above interpretation of T and S^j suggest to use the gauge conditions $G = T - t$ and $G^a = \delta^a_j S^j - x^a$. The stabilisation of these gauge conditions fixes the Lagrange multipliers λ and λ^a in the gauge generator

$$K := C^{\text{tot}}(\lambda, \vec{\lambda}) := \int_\sigma d^3x [\lambda C^{\text{tot}} + \lambda^a C_a^{\text{tot}}], \quad (3.4)$$

namely,

$$\dot{G}(t, x) = \{K, G(x)\} + \partial_t G(t, x) = \frac{\lambda(x)}{\sqrt{1 + q^{ab} T_{,a} T_{,b}}} + \lambda^a T_{,a} - 1 = 0,$$

$$\dot{G}^a(t, x) = \{K, G^a(x)\} + \partial_t G^a(t, x) = \lambda^b S^j_{,b} \delta^a_j = 0 \quad (3.5)$$

which when evaluated at $G = G^a = 0$ yields the unique solution $\lambda = 1$, $\lambda^a = 0$. Likewise, in this gauge, the constraints can be uniquely solved for $P = -C$ and $P_j = -\delta^a_j C_a$ while T and S^j are pure gauge. This shows that the physical degrees of freedom are those not involving T , P , S^j , and P_j .

For any function F independent of these variables, the reduced or physical Hamiltonian is that function on the phase space coordinatised by the physical degrees of freedom which generates the same time evolution as K when the constraints, gauge conditions, and stabilising Lagrange multipliers are installed

$$\{H, F\} := \{K, F\}_{C^{\text{tot}}=\vec{C}} = \vec{G} = \vec{G} = \vec{\lambda} = 0} = \left\{ \int_\sigma d^3x C, F \right\}, \quad (3.6)$$

which shows that

$$H = \int_\sigma d^3x C \quad (3.7)$$

Thus, the final picture is remarkably simple: The physical phase space is simply coordinatised by all metric and standard matter degrees of freedom (and their conjugate momenta), while the physical Hamiltonian is just the integral of the usual Wheeler–DeWitt constraint. The influence of the reference matter now only reveals itself in the fact that H is not constrained to vanish as it only involves the geometry and standard matter contribution C of C^{tot} and that the number of physical degrees of freedom has increased by four as compared to the system without reference matter. This phenomenon is, of course, well-known from the electroweak interaction: One can solve the three isospin SU(2) Gauss constraints for three of the four degrees of freedom sitting in the complex-valued Higgs isodoublet, leaving a single scalar Higgs field and three massive, rather than massless, vector bosons. (See Refs. 156 and 157 for further discussion.)

We close this subsection with three remarks:

First, a complete discussion requires to show that the gauge cut $G = G^a = 0$ on the constraint surface of the phase space be reachable from anywhere on the constraint surface. As Eq. 3.5 shows, this requires that $S^j_{,a}$ be invertible. We thus impose this as an anholonomic constraint on the total phase space. One easily verifies from Eq. 3.5 that this condition is gauge-invariant, that is, compatible with the dynamics.

Second, the simplicity of the final picture is due to the particular choice of reference matter. Other reference matter most likely will increase the complexity (see, e.g., Ref. 158), which produces a square root Hamiltonian! One may argue that the dust is a form of cold dark matter [146], but it is unclear whether this is physically viable. Nevertheless, the present model serves as a proof of principle, namely, that GR coupled to standard matter and reference can be cast into the form of a conservative Hamiltonian system.

Third, it should be appreciated that the reference matter helps us accomplish a huge step in the quantum gravity programme: It frees us from quantising and solving the constraints and constructing the physical inner product, the gauge-invariant observables, and their physical time evolution. All of these steps are of tremendous technical difficulty [17–21]. All we are left to do is to quantise the physical degrees of freedom and the physical Hamiltonian.

3.2. Loop Quantum Gravity Quantisation of the Reduced Physical System

In order to keep the technical complexity to a minimum, we consider just the contribution to H coming from the gravitational degrees of freedom (see [17–21, 28–32] for more detail on standard matter coupling). The Hamiltonian directly written in terms of $SU(2)$ gauge theory variables reads (we drop some numerical coefficients that are not important for our discussion)

$$\begin{aligned} H &= H_E + H_L \\ H_E &= \int_{\sigma} \text{Tr}(F \wedge \{V, A\}) \\ V &= \int_{\sigma} \sqrt{|\det(E)|} \\ H_L &= \int_{\sigma} \text{Tr}(\{H_E, V\}, A) \wedge \{H_E, V\}, A \wedge \{V, A\} \end{aligned} \quad (3.8)$$

Here, A is an $SU(2)$ connection and E an $SU(2)$ non-Abelian electric field that one would encounter also in an $SU(2)$ Yang–Mills theory. However, the geometric interpretation of A and E is different, namely, $e_j^a := E_j^a / \sqrt{|\det(E)|}$ is a triad, that is, $q^{ab} = \delta^{jk} e_j^a e_k^b$ is the inverse spatial metric. Here, as before, $a, b, c, \dots = 1, 2, 3$ denote spatial tensor indices, while now $j, k, l, \dots = 1, 2, 3$ denote $\mathfrak{su}(2)$ Lie algebra indices. Further, let Γ_a^j be the spin connection of e_j^a . Then, $K_{ab} := (A_a^j - \Gamma_a^j) e_b^k \delta_{jk}$ has the meaning of the extrinsic curvature of the ADM slices [155] on the kernel of the $SU(2)$ Gauss constraint

$$C_j := \partial_a E_j^a + \epsilon_{jkl} A_a^k E_m^a \delta_l^m \quad (3.9)$$

The important quantity V is recognised as the total volume of the hypersurface σ , and H_E and H_L are known as the Euclidian and Lorentzian contributions to H . (See Refs. 28–32 for further details.) The Poisson brackets displayed are with respect to the standard symplectic structure

$$\begin{aligned} \{A_a^j(x), A_b^k(y)\} &= \{E_j^a(x), E_k^b(y)\} \\ &= \{E_j^a(x), A_b^k(y)\} - \kappa \delta_b^a \delta_j^k \delta(x, y) = 0, \end{aligned} \quad (3.10)$$

where $\hbar \kappa = \ell_P^2$ is the Planck area. The definition of the phase space is completed by the statement that the elementary fields A and E are real-valued

$$[A_a^j(x)]^* - A_a^j(x) = [E_j^a(x)]^* - E_j^a(x) = 0 \quad (3.11)$$

The traces involved in 3.2 are carried out by introducing the Lie algebra-valued 1-forms $A = A_a^j \tau_j dx^a$, where $2i\tau_j$ are the Pauli matrices and $F = 2(dA + A \wedge A)$ is the curvature of A . The non-polynomiality of GR is hidden in the Poisson brackets that appear

in Eq. 3.8. The reason why we use these particular Poisson bracket structure will become clear only later.

To quantise the theory, we start from functions on the phase space that are usually employed in the lattice gauge theory (see, e.g., Ref. 159), namely, non-Abelian magnetic holonomy and electric flux variables

$$A(c) := \mathcal{P} \exp \left(\int_c A \right), \quad E_f(S) = \int_S \text{Tr}(f * E), \quad (3.12)$$

where $\mathcal{P}$ denotes path ordering, c is a piecewise analytic real curve, S is a piecewise real analytic surface, f is an $\mathfrak{su}(2)$ -valued function, and $*E = \epsilon_{abc} E^a dx^b \wedge dx^c / 2$ is the pseudo 2-form corresponding to the $\mathfrak{su}(2)$ -valued vector density E . Note that $A(c)$ is $SU(2)$ -valued, while $E_f(S)$ is $\mathfrak{su}(2)$ -valued

$$A(c)^* = (A(c)^{-1})^T = A(c^{-1})^T, \quad E_f(S)^* = -E_f(S)^T, \quad (3.13)$$

where c^{-1} is the same curve as c but with the opposite orientation. The simplest non-trivial Poisson brackets are

$$\{E_f(S), A(c)\} = \kappa A(c_1) f(c \cap S) A(c_2), \quad (3.14)$$

in case that $S \cap c$ is a single point in the interior of both c and S , see Refs. 23–27 for a complete discussion. The relations (3.13) and (3.14) are the defining relations of a non-commutative abstract $*$ -algebra $\mathfrak{A}$ generated by fluxes and complex-valued smooth functions F of a finite number of holonomy variables [23–27]. It is the free algebra generated by them and divided by the two-sided ideal generated by the canonical commutation relations $E_f(S)A(c) - A(c)E_f(S) = i\hbar\{E_f(S), F\}$ and the adjointness relations (Eq. 3.13). (See Refs. 23–27 for more details.)

Interestingly, the physical Hamiltonian H has a large symmetry group, namely, it is invariant under the group $\mathfrak{G} = SU(2)_{\text{loc}} \rtimes \text{Diff}(\sigma)$, where $SU(2)_{\text{loc}}$ denotes the group of local $SU(2)$ -valued gauge transformations and $\text{Diff}(\sigma)$ denotes the group of (piecewise real analytic) diffeomorphisms of σ . An element of $\mathfrak{G}$ is given by a pair $\mathbf{g} = (g, \varphi)$, which acts on the basic variables as

$$\alpha_{(g, \varphi)}(A) = -dgg^{-1} + g[\varphi^* A]g^{-1}, \quad \alpha_{(g, \varphi)}(*E) = g[\varphi^* (*E)]g^{-1}, \quad (3.15)$$

where φ^* denotes the pull-back action of diffeomorphisms on differential forms. This action lifts to the algebra $\mathfrak{A}$, specifically

$$\begin{aligned} \alpha_{(g, \varphi)}(A(c)) &= g(b(c)) A(\varphi(c)) g(f(c))^{-1}, \quad \alpha_{(g, \varphi)}(E_f(S)) \\ &= E_{[g^{-1}fg] \circ \varphi^{-1}}(\varphi(S)), \end{aligned} \quad (3.16)$$

where $b(c)$ and $f(c)$ denote the beginning and final points of c , and this simple covariant transformation behaviour was part of the reason why the particular ‘smearing’ of A along curves involved in holonomies is used. Note also the different character of the two groups: While we still have to find the gauge-invariant observables with respect to the Gauss constraint, the diffeomorphism constraint is already solved. The

diffeomorphisms in $\mathfrak{G}$ are thus to be considered as active diffeomorphisms, rather than passive ones.

The mathematical problem in quantising the theory consists in constructing a $*$ representation of $\mathfrak{A}$, that is, a representation $(\pi, \mathcal{H})$ of elements $a \in \mathfrak{A}$ as operators $\pi(a)$ densely defined on a common, invariant domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ such that the $*$ relations are implemented as adjointness relations and such that the canonical commutation relations are implemented as commutators between them. Thus, we want, in particular, that

$$\begin{aligned}\pi(a^*) &= [\pi(a)]^\dagger, \quad \pi(a+b) = \pi(a) + \pi(b), \quad \pi(ab) \\ &= \pi(a)\pi(b), \quad \pi(za) = z\pi(a), \quad [\pi(a), \pi(b)] = \pi(c); \end{aligned} \quad (3.17)$$

for all $a, b, c \in \mathfrak{A}$, $z \in \mathbb{Z}$ if $ab - ba = c$. In QFT, this problem is known to typically have an uncountably infinite number of unitarily inequivalent solutions; there is no Stone–von Neumann uniqueness theorem when the number of degrees of freedom is infinite. Hence, to make progress, we must use additional physical input. That input can only come from the Hamiltonian. Thus, we require in addition that the representation supports H as a self-adjoint operator (H is real-valued) also densely defined on $\mathcal{D}$ and such that $\mathcal{H}$ carries a unitary representation U of $\mathfrak{G}$ (such that its generators are self-adjoint by Stone's theorem). Using the powerful machinery of the Gel'fand–Naimark–Segal construction [160], the representation property and the unitarity property can be granted if we find a positive linear and $\mathfrak{G}$ invariant functional $\omega: \mathfrak{A} \rightarrow \mathbb{C}$ on $\mathfrak{A}$, that is,

$$\omega \circ \alpha_g = \omega, \quad \omega(a^*a) \geq 0 \quad (3.18)$$

In Refs. 23–27, it was found that there is a *unique* ω satisfying (3.18). While the derivation is somewhat involved, the final result can be described in a compact form. The dense domain $\mathcal{D}$ consists of functions of the form

$$\psi(A) = \psi_\gamma \left(\{A(c)\}_{c \in E(\gamma)} \right); \quad \psi_\gamma \in C^\infty \left(SU(2)^{|E(\gamma)|}, \mathbb{C} \right), \quad (3.19)$$

that is, ψ_γ is complex-valued, smooth functions of a finite number of holonomy variables. The union of the curves of these holonomies forms a finite graph γ , where $E(\gamma)$ denotes the set of its edges. Note that the elements of $\mathfrak{A}$ that just depend on the connection are themselves of the form (Eq. 3.19), and thus, their action by multiplication

$$[\pi(f)\psi](A) := f(A)\psi(A) \quad (3.20)$$

is densely defined. The fluxes are densely defined when acting by derivation

$$[\pi(E_f(S))\psi](A) := i\hbar \{E_f(S), \psi(A)\} \quad (3.21)$$

which also solves the canonical commutation relations.

To see that the adjointness conditions hold, we need the inner product. To define it, we note that graphs defined by finitely many piecewise analytic curves are partially ordered by set theoretic inclusion, and they are directed in the sense that for any two graphs γ_1, γ_2 , there exists γ_3 with $\gamma_1, \gamma_2 \subset \gamma_3$, for instance

$\gamma_3 = \gamma_1 \cup \gamma_2$. Then, we can decompose all edges of γ_1, γ_2 with respect to the edges of γ_3 and use the algebraic relations of the holonomy $A(c^{-1}) = A(c)^{-1}$ and $A(c \circ c_i) = A(c)A(c_i)$, where $c \circ c_i$ is the composition of curves $f(c) = b(c_i)$ in order to write ψ_1 and ψ_2 excited over γ_1 and γ_2 , respectively, as functions excited over γ_3 . Thus, it is sufficient to know the inner product of functions excited over the same graph γ which is given by

$$\langle \psi, \psi' \rangle_{\mathcal{H}} := \int_{SU(2)^{|E(\gamma)|}} \prod_{k=1}^{|E(\gamma)|} d\mu_H(h_k) \overline{\psi(\{h_k\})} \psi'(\{h_k\}), \quad (3.22)$$

where μ_H is the Haar measure on $SU(2)$. One can check that the adjointness relations are indeed satisfied, in fact $\pi(E_f(S))$ is an unbounded but essentially self-adjoint operator (i.e., a symmetric operator with unique self-adjoint extension).

In fact, Eq. 3.22 defines a cylindrical family of measures μ_γ , one for every graph γ . One has to check that Eq. 3.22 is well-defined because a function excited on γ can be written also as a function excited over any finer graph γ' by extending it trivially to the additional edges. This is, in fact, the case [23–27]. Then, the Kolmogorov-type extension theorems grant that the family extends to an honest continuum measure μ on the quantum configuration space $\overline{\mathcal{A}}$ of distributional connections. We will not go into the details here which can be found in Refs. 23–27 but just mentioning for the interested reader that this space coincides with the so-called Gel'fand spectrum of the Abelian C^* algebra that one obtains by completing the space of functions (Eq. 3.19) in the sup norm. It follows that the Hilbert space is given by $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu)$.

By construction, the Hilbert space $\mathcal{H}$ carries a unitary representation U of $\mathfrak{G}$ given by

$$(U(\mathfrak{g})\psi)(A) = \psi_\gamma \left(\{\alpha_g(A(c))\}_{c \in E(\gamma)} \right) \quad (3.23)$$

To check this, one uses the properties of the Haar measure (translation invariance) and the diffeomorphism invariance of Eq. 3.22 which does not care about the location and shape of the curves involved.

The Hilbert space comes equipped with an explicitly known orthonormal basis called spin network functions (SNWFs). This makes use of harmonic analysis on compact groups G [161], in particular the Peter and Weyl theorem which states that the matrix element functions of the irreducible representations of G , which are all finite-dimensional and unitary without loss of generality, are mutually orthogonal, unless equivalent, with respect to the inner product defined by the Haar measure on G ; moreover, they span the whole Hilbert space. As the irreducible representations of $SU(2)$ are labelled by spin quantum numbers, the name SNWF comes at no surprise. More in detail, an SNWF $T_{\gamma, j, \iota}$ is labelled by a graph γ , a tuple $j = \{j_c\}_{c \in E(\gamma)}$ of spin quantum numbers decorating the edges, and a tuple $\iota = \{\iota_v\}_{v \in V(\gamma)}$ of intertwiners decorating the vertices v in the vertex set $V(\gamma)$ of γ . Here, an intertwiner ι_v projects the tensor product of irreducible representations corresponding to the edges incident at v onto one of the irreducible representations appearing in its

decomposition into irreducibles (Clebsch–Gordan theory). Besides providing an ONB convenient for concrete calculations, SNWFs make it easy to solve the Gauss constraint: A detailed analysis [17–21] shows that **Eq. 3.9** can be quantised in the given representation and just imposes that the space of intertwiners be restricted to those projecting on the trivial (spin zero) representation. We call such intertwiners gauge-invariant. Hence, the joint kernel of the Gauss constraints is a closed subspace of $\mathcal{H}$ which is explicitly known. We will abuse the notation and will not distinguish between that subspace and $\mathcal{H}$ and henceforth consider the Gauss constraint as solved. All operators considered in what follows are manifestly gauge invariant and preserve that subspace.

As a historical remark, solutions of the Gauss constraint are excited on closed graphs since there is no non-trivial intertwiner between the trivial representation and a single irreducible one; hence, open ends are forbidden. For closed graphs, one can alternatively label SNWFs by homotopically independent closed paths (loops) with a common starting point (vertex) on that graph. Originally, one used loops as labels, hence the name loop quantum gravity (LQG).

One of the many unfamiliar features of $\mathcal{H}$ is that it is not separable which easily follows from the uncountable cardinality of the set of graphs. This is a direct consequence of the diffeomorphism invariance of the inner product: Two graphs that are arbitrarily close but disjoint are simultaneously also arbitrarily far apart under the inner product. Thus, if the measure clusters for far apart support of the smearing functions (here the graphs), then the orthogonality of the corresponding spin network functions comes at no surprise. A direct consequence of this is that the diffeomorphism operators $U(\varphi)$ do not act (strongly) continuously; hence, a generator of infinitesimal diffeomorphisms generated by the integral curves of vector fields cannot exist. Yet another direct consequence is that the connection operator A itself does not exist; only its holonomies do.

The remaining task is to quantise the Hamiltonian, and it is at this point where the aforementioned quantisation ambiguities arise. The strategy followed in Refs. 28–32 is as follows: It turns out that the volume operator appearing in **Eq. 3.2** can be quantised on $\mathcal{H}$ as an essentially self-adjoint operator whose spectrum is pure point (discrete) [162–164]. It is densely defined on the span of the SNWF, and it acts vertex-wise, with no contribution from gauge-(in)variant vertices that are not at least three (four) valent or from vertices whose incident edges have tangents in a common two-dimensional or one-dimensional space. Next, the holonomy along an open curve c can be expanded as $A(c) = 1_2 + \int_c A + \dots$ and along a closed curve α as $A(\alpha) = 1_2 + \int_\alpha \partial S = \alpha F$ so that the functions A and F that appear in **Eq. 3.8** can be approximated by suitable holonomies where the approximation is in terms of the ‘length’ of the curves involved which are matched with the coordinate volume assigned by the Lebesgue measure d^3x appearing in **Eq. 3.8**, approximating the integral by a Riemann sum (this is a regularisation step). Suppose then that somehow a well-defined operator H_E can be defined by replacing the classical functions by operators and the Poisson brackets by commutator times $\hbar$. Then, the same argument can be applied to the Lorentzian piece. As a final piece of information, one uses the

observation that a spatially diffeomorphism-invariant operator, densely defined on the span of SNWF, cannot have non-trivial matrix elements between SNWF excited over different graphs [36]. This has the following consequence: Let $\mathcal{H}_\gamma$ be the closed linear span of SNWF excited precisely over γ . Then, if H is supposed to preserve its classical diffeomorphism invariance upon quantisation, we necessarily have

$$H = \oplus_\gamma H_\gamma, \quad \mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma, \quad (3.24)$$

where each H_γ is self-adjoint on $\mathcal{H}_\gamma$, in particular it preserves this space. Let now $P_\gamma: \mathcal{H} \rightarrow \mathcal{H}_\gamma$ be the orthogonal projection. Then, the following concrete expression for H can be given [39] (again we drop some numerical coefficients and set $\hbar = 1$)

$$H_{E,\gamma} = P_\gamma H'_{E,\gamma} P_\gamma$$

$$H'_{E,\gamma} = i \sum_{v \in V(\gamma)} \sum_{c_1, c_2, c_3 \in E(\gamma); c_1 \cap c_2 \cap c_3 = v}$$

$$\epsilon^{IJK} \text{Tr} \left(\left[A(\alpha_{\gamma, v, c_I, c_J}) - A(\alpha_{\gamma, v, c_I, c_J})^{-1} \right] A(c_K) [V, A(c_K)^{-1}] \right) + h.c.$$

$$H_{L,\gamma} = P_\gamma H'_{L,\gamma} P_\gamma$$

$$H'_{L,\gamma} = i \sum_{v \in V(\gamma)} \sum_{c_1, c_2, c_3 \in E(\gamma); c_1 \cap c_2 \cap c_3 = v}$$

$$\epsilon^{IJK} \text{Tr} \left(A(c_I) \left[[H'_{E,\gamma}, V], A(c_I)^{-1} \right] A(c_J) \left[[H'_{E,\gamma}, V], \right. \right.$$

$$\left. A(c_J)^{-1} \right] A(c_K) [V, A(c_K)^{-1}] \left. \right) + h.c.$$

$$(3.25)$$

The sum is over vertices of γ and triples of edges incident at them (taken with outgoing orientation). For each vertex v and pairs of edges c, c' outgoing from v , one defines α_γ, v, c, c' as that loop within γ starting at v along c and ending at v along $(c')^{-1}$ with the minimal number of elements of $E(\gamma)$ used (if that loop is not unique, we average over them). It has been shown that the concrete expression (**Eq. 3.25**) has the correct semi-classical limit in terms of expectation values with respect to semi-classical coherent states [165–168] on sufficiently fine graphs of cubic topology [166–172].

Remarkably, **Eq. 3.25** defines an essentially self-adjoint, diffeomorphism-invariant, continuum Hamiltonian operator for Lorentzian quantum gravity in four space-time dimensions, densely defined on the physical continuum Hilbert space $\mathcal{H}$ which is manifestly free of ultraviolet divergences, that is, while for each given graph γ , the theory looks like a lattice gauge theory on γ ; the theory is defined on all lattices simultaneously, which makes it a continuum theory. Moreover, note that the vector $\Omega = 1$ has norm unity and that $H\Omega = 0$.

Yet, one cannot be satisfied with **Eq. 3.25** for the following reasons:

1. While it is true that one can give a better motivated derivation than we could sketch here for reasons of space, there are some ad hoc steps involved.
2. There are several ordering ambiguities involved in **Eq. 3.25**: Not only could we have written the factors in different orders but instead of using the fundamental representation to approximate connections in terms of holonomies, we could have used higher spin representations [173] or an average over several of them, and in each case, we would have different coefficients appearing in front of these terms.
3. Of particular concern is definition of the minimal loop. While this gives good semi-classical results on sufficiently fine lattices, the theory lives on all lattices, also those which are very coarse, and on those, expression **Eq. 3.25** is doubtful because the Riemann approximation mentioned above would suggest to use a much finer loop. In fact, one is supposed to take the regulator (i.e., the coordinate volume ϵ of the Riemann approximants) away, and in that limit, the loop would shrink to zero. One can justify that this does not happen by using a sufficiently weak operator topology [28–32], namely, there exist diffeomorphism-invariant distributions (linear functionals) l on the dense span of SNWF ψ [36], and we define an operator O_ϵ to converge to an operator O in that topology if $l([O_\epsilon - O]\psi) \rightarrow 0$ for all l, ψ . Now, due to diffeomorphism invariance, we can deform for any ϵ the small loop to any diffeomorphic one as long as we do not cross other edges of the graph, in particular we can deform it as close as we want to the minimal one. Then, the result mentioned above about the matrix elements of diffeomorphism-invariant operators, in fact, forces us to choose that loop precisely, not only approximately. Of course, while the diffeomorphism symmetry of H makes the space of diffeomorphism-invariant distributions a natural space to consider, it is still not perfectly justified to use it in order to define a topology.
4. The naive dequantisation of **Eq. 3.25** will perform poorly on very coarse graphs and will be far from the continuum expression **Eq. 3.8**, but one could argue that that vectors supported on coarse graphs simply do not qualify as good semi-classical states.
5. Using the same argument as in (3), there is nothing sacred about the minimal loop, and one could take again other loops and/or average of over them with certain weights. However, then the locality of **Eq. 3.25** is lost.
6. The block diagonal or superselection structure (**Eq. 3.24**) which is forced on us by the non-separability of the Hilbert space and its spatial diffeomorphism covariance appears unphysical, and one would expect that the Hamiltonian creates also new excitations.

It transpires that we must improve **Eq. 3.25**, and the discussion has indicated a possible solution: Blocking free QFT from the continuum (i.e., restricting the Hilbert space to vectors of finite spatial resolution) with respect to a kinematic real-space

coarse graining scheme exactly produces such a high degree of non-locality at finite resolution even if the continuum measure or the continuum Hamiltonian is local [54–57, 71–75, 108]. This bears the chance that what we see in **Eq. 3.25** is nothing but a naive guess of a continuum Hamiltonian which is blocked from the continuum but whose off-block diagonal form we cannot determine with the technology used so far. Accordingly, this calls for shifting our strategy which was already started in Refs. 169–172, 174 (in the sense that the block diagonal structure was dropped, but only one infinite graph was kept):

We take the above speculation serious and consider the operators H_γ as projections onto the subspaces $\mathcal{H}_\gamma$ of $\mathcal{H}$ of a continuum Hamiltonian H , but we will drop the unphysical block diagonal structure 3.24 which arises from the non-separability of $\mathcal{H}$. Rather the relation between H_γ is to be imposed by a renormalisation scheme induced by the path integral renormalisation scheme adopted in quantum statistical physics. To do this, we must first derive a path integral measure μ_γ from the OS data, $\mathcal{H}_\gamma$, H_γ , and Ω_γ where Ω_γ is the vacuum of H_γ by the usual Feynman–Kac–Trotter–Wiener formalism. Then, we can compute the flow of μ_γ in the usual way and then translate into a flow of OS data by OS reconstructing them from the measures. The fixed points of the flow will then define the possible continuum theories, and these may be ‘phases’ quite different from **Eq. 3.25**. The details of this programme will be the subject of the following sections.

4. CONSTRUCTIVE QFT, FEYNMAN–KAC–TROTTER–WIENER CONSTRUCTION AND OSTERWALDER–SCHRADER RECONSTRUCTION

The purpose of this section is to provide some background information on constructive QFT and related topics such as the Feynman–Kac–Trotter–Wiener construction of measures (path integrals) from a Hamiltonian formulation (operator formulation) and vice versa the Osterwalder–Schrader reconstruction of a Hamiltonian framework from a measure. Our description will be minimal. The prime textbook references are [98–100, 175].

4.1. Measure Theoretic Glossary

Let S be a set. A collection B of the so-called measurable subsets of S is called a σ -algebra if i. it is closed under taking complements with respect to S , ii. closed under taking countable unions, and iii. B contains the empty set $\emptyset$. The pair (S, B) is called a measurable space. A measure space is a triple (S, B, μ) , where (S, B) is a measure space and μ is a positive set function $\mu: B \rightarrow \mathbb{R}_0^+ \cup \{+\infty\}$ $s \mapsto \mu(s)$ which is σ -additive, that is, for any pairwise disjoint $s_I \cap s_J = \emptyset$, $I \neq J$; $I, J = \mathbb{N}$, we have

$$\mu(\cup_I s_I) = \sum_I \mu(s_I) \quad (4.1)$$

The measure μ is called a probability measure if $\mu(S) = 1$. One uses the notation

$$\mu(s) = \int_s d\mu(p) = \int_s d\mu(p) \chi_s(p), \quad (4.2)$$

where $\chi_s(p) = 1$ if $p \in s$ and $\chi_s(p) = 0$, else is called the characteristic function of $s \in \mathcal{B}$.

Consider now a second measurable space $(\tilde{S}, \tilde{\mathcal{B}})$. A function $X: S \rightarrow \tilde{S}$ is called measurable or a random variable if the pre-images $X^{-1}(\tilde{s}) = \{p \in S; f(p) \in \tilde{s}\}$ of measurable sets $\tilde{s} \in \tilde{\mathcal{B}}$ are measurable in S . Let $\mathcal{F}$ be the set of random variables $X: S \rightarrow \tilde{S}$; then for $X \in \mathcal{F}$, the set function

$$\tilde{\mu}(\tilde{s}) := \mu(X^{-1}(\tilde{s})), \quad \tilde{s} \in \tilde{\mathcal{B}} \quad (4.3)$$

defines also a probability measure called the distribution of X . We consider real-valued functions $f: \tilde{S} \rightarrow \mathbb{R}$ of the simple form

$$f(\tilde{p}) = \sum_n z_n \chi_{\tilde{s}_n}(\tilde{p}); \quad z_n \in \mathbb{R}, \quad \tilde{s}_n \in \tilde{\mathcal{B}}, \quad (4.4)$$

where the sum is over at most finitely many terms and define their integral as

$$\begin{aligned} \tilde{\mu}(f) &= \sum_n z_n \mu\left(s'_n\right) = \int_{\tilde{S}} d\tilde{\mu}(\tilde{p}) \left[\sum_n z_n \chi_{\tilde{s}_n}(\tilde{p}) \right] = \int_{\tilde{S}} d\tilde{\mu}(\tilde{p}) f(\tilde{p}) \\ &= \sum_n z_n \mu(X^{-1}(\tilde{s}_n)) = \int_S d\mu(p) \left[\sum_n z_n \chi_{X^{-1}(\tilde{s}_n)}(p) \right] \\ &= \int_S d\mu(p) \left[\sum_n z_n \chi_{\tilde{s}_n}(X(p)) \right] = \int_S d\mu(p) (f \circ X)(p) = \mu(f \circ X) \end{aligned} \quad (4.5)$$

One can show that this identity extends from simple functions to Borel functions that is, measurable functions $f: \tilde{S} \rightarrow \mathbb{R}$, where $\mathbb{R}$ is equipped with the Borel σ -algebra (the smallest σ algebra containing all open intervals). We can then also extend it to those complex functions whose real and imaginary parts are Borel by linearity.

A stochastic process indexed by an index set $\mathcal{I}$ is a family $\{X_i\}_{i \in \mathcal{I}}$ of random variables $X_i: S \rightarrow \tilde{S}$. For any finite subset $I = \{i_1, \dots, i_N\} \subset \mathcal{I}$, we have the joint distribution

$$\tilde{\mu}_I(\tilde{s}_1 \times \dots \times \tilde{s}_N) := \mu\left(\bigcap_{k=1}^N X_{i_k}^{-1}(\tilde{s}_k)\right) \quad (4.6)$$

The probability measures μ_I are called cylinder measures. For any complex-valued Borel function $f: \tilde{S}^N \rightarrow \mathbb{C}$, we have similarly as in Eq. 4.4

$$\int_S d\mu(p) f(\{X_{i_k}(p)\}_{k=1}^N) = \int_{\tilde{S}^N} d\tilde{\mu}_I(\tilde{p}_1, \dots, \tilde{p}_N) f(\tilde{p}_1, \dots, \tilde{p}_N) \quad (4.7)$$

Functions on S of the form $f_I(p) = f(\{X_{i_k}(p)\}_{k=1}^N)$ are called cylinder functions.

In what follows, we assume that for each $N \in \mathbb{N}_0$, there exists a distinguished system $\mathcal{W}_N$ of complex-valued, bounded elementary functions W on N copies of $\tilde{S}$ such that the corresponding cylinder functions enjoy the following properties:

- (1) They generate an Abelian $*$ algebra, that is, for all $I, I' \in \mathcal{I}$, the product $W_I W_{I'}$ is a finite, complex linear combination

of suitable $W_{I''}$, $I'' \in \mathcal{I}$, $W_{I''} \in \mathcal{W}_{|I''|}$ and also $\overline{W_I}$ is of that form.

- (2) $\mathcal{W}_N$ contains the constant function.
- (3) For each $I \in \mathcal{I}$, the moments $\mu(W_I)$, $W \in \mathcal{W}_{|I|}$ determine $\tilde{\mu}_I$ uniquely.
- (4) These properties show that W_I are $L_2(d\tilde{\mu}_I, \tilde{S}^{|I|})$ functions. We require their span to be dense.
- (5) We saw that a probability measure μ together with a stochastic process gives rise to a family of cylindrical probability measures $(\tilde{\mu}_I)_{I \in \mathcal{I}}$ on $\tilde{S}^{|I|}$. The converse question is under which circumstances a cylindrical family of cylinder probability measures determines a measure μ . A necessary criterion is as follows: The set $\mathcal{I}$ is partially ordered and directed by inclusion, that is, for each $I, J \in \mathcal{I}$, we find $K \in \mathcal{I}$ such that $I, J \subset K$ (for instance, $K = I \cup J$). Suppose that $I \subset J$. Then,

$$\mu(X_I^{-1}(\tilde{s}_I)) = \tilde{\mu}_I(\tilde{s}_I) = \mu\left(X_J^{-1}\left(\tilde{s}_I \times (\tilde{S})^{|J|-|I|}\right)\right) = \tilde{\mu}_J\left(\tilde{s}_I \times (\tilde{S})^{|J|-|I|}\right), \quad (4.8)$$

where $X_I = \{X_i\}_{i \in I}$, $\tilde{s}_I \in \tilde{S}^{|I|}$. Furthermore, for any permutation π on $N = |I|$ elements set, $\pi \cdot I = \{i_{\pi(1)}, \dots, i_{\pi(N)}\}$ and $\pi \cdot \tilde{s}_I = \{(\tilde{p}_{\pi(1)}, \dots, \tilde{p}_{\pi(N)}); (\tilde{p}_1, \dots, \tilde{p}_N) \in \tilde{s}_I\}$. Then,

$$\mu(X_{\pi \cdot I}^{-1}(\pi \cdot \tilde{s}_I)) = \tilde{\mu}_{\pi \cdot I}(\pi \cdot \tilde{s}_I) = \mu(X_I^{-1}(\tilde{s}_I)) = \tilde{\mu}_I(\tilde{s}_I) \quad (4.9)$$

Even more generally, a partial order on the set $\mathfrak{S}$ of finite subsets I of $\mathcal{I}$ is a transitive, reflexive, and antisymmetric relation, that is, $I < J \wedge J < K \Rightarrow I < K$ and $I < I$ and $I < J \wedge J < I \Rightarrow I = J$ for all $I, J, K \in \mathfrak{S}$. The set $\mathfrak{S}$ is called directed with respect to $<$, provided that for all $I, J \in \mathfrak{S}$, we find $K \in \mathfrak{S}$ such that $I, J < K$. For $I < J$, we may have surjective maps $P_{JI}: \tilde{S}^{|J|} \rightarrow \tilde{S}^{|I|}$ such that $X_I(p) = P_{JI}(X_J(p))$ and such that for $I < J < K$, we have $P_{JI} \circ P_{KJ} = P_{KI}$. Then, similar as in Eq. 4.8, we necessarily must have for $I < J$

$$\tilde{\mu}_I(\tilde{s}_I) = \tilde{\mu}_J(P_{JI}^{-1}(\tilde{s}_I)) \quad (4.10)$$

It turns out that these two conditions, Eqs 4.8, 4.9, or 4.10 is also sufficient in fortunate cases (for instance, if $\tilde{S} = \mathbb{R}$, which is the classical Kolmogorov theorem, see Ref. 134), that is, we can then reconstruct the measure space $(S, \mathcal{B}, \mu)$ and a stochastic process $\{X_i\}_{i \in \mathcal{I}}$ such that $\tilde{\mu}_I$ are the cylinder measures of μ . It follows that the $W_I \in \mathcal{W}_{|I|}$, $I \in \mathcal{I}$ lie dense in $L_2(S, d\mu)$.

Physical meaning: We consider the elements $p \in S$ to be space-time fields Φ or spatial fields ϕ , respectively. The index set $\mathcal{I}$ will have the meaning of a set of test functions or more generally distributions whose elements i label the random variable X_i . These map the fields smeared with test functions to a finite-dimensional manifold (usually copies of $\mathbb{R}$ or more generally of a Lie group). For instance, for a scalar field Φ , we may consider the random variable $X_F(\Phi) = \exp(i \int_{\mathbb{R} \times \sigma} d^4 x F(x) \Phi(x))$ which takes

values in $\tilde{S} = U(1)$. It is also customary to consider the field $p = \Phi$ itself as a random variable indexed by the same index set or to simply write $X_i(p) = p(i)$ as an abbreviation.

4.2. Constructive QFT

The application of interest of the previous subsection is a stochastic process indexed by either $\mathbb{R} \times L$ or just by L , where the label set L is a certain set of distributions on the spatial manifold. We distinguish between random variables ϕ indexed by a pair $(t, f) \in \mathbb{R} \times L$ and random variables ϕ indexed by $f \in L$. Some examples are as follows:

(1) Real quantum scalar fields with smooth smearing:

Consider $L = \mathcal{S}(\mathbb{R}^3)$, the space of smooth test functions of rapid decrease and $\tilde{\mathcal{S}} = \mathbb{R}$ equipped with the Borel σ -algebra. Then, $\phi(f) = \langle f, \phi \rangle = \int d^3x f(x) \phi(x)$ and $\Phi(t, f) = \langle f, \Phi(t, \cdot) \rangle$. Given $F := (f_1, \dots, f_N) \in L^N$, consider $\phi(F) = (\phi(f_1), \dots, \phi(f_N)) \in \mathbb{R}^N$. The space $\mathcal{W}_N$ of elementary functions on N copies of $\mathbb{R}$ can be chosen to be generated by the exponentials

$$w_{r_1, \dots, r_N}(\phi(F)) = \exp\left(i \sum_{k=1}^N r_k \langle f_k, \phi \rangle\right) \quad (4.11)$$

with $r_1, \dots, r_N \in \mathbb{R}$ labelling the (necessarily one-dimensional) unitary irreducible representations of $U(1)$.

In fact, since in this case, the space L is a vector space, it is sufficient to consider the functions $w(\phi(f)) = \exp(i\phi(f))$, $f \in L$. Analogously, the space of elementary functions for the time-dependent fields can be chosen as $(w_k \in \mathcal{W}_{N_k})$

$$W(\Phi(t_1, F_1), \dots, \Phi(t_T, F_T)) = w_T(\Phi(t_T, F_T)) \cdot w_1(\Phi(t_1, F_1)), \quad (4.12)$$

which, of course, reduces to

$$\exp\left(i\Phi(t_T, f'_T)\right) \cdot \exp\left(i\Phi(t_1, f'_1)\right), f'_k \in L \quad (4.13)$$

for certain $f'_k \in L$. Obviously, the Abelian $\ast$ -algebra and boundedness conditions are satisfied. That these elementary functions suffice to determine the cylindrical measures requires a more involved argument (Bochner's theorem, [134]).

(2) Real quantum scalar fields with distributional smearing:

Consider a subset $L = \subset \mathcal{S}'(\mathbb{R}^3)$ of the tempered distributions and $\tilde{\mathcal{S}} = U(1)$ equipped with the Borel σ -algebra. In applications to scalar fields coupled to general relativity elements, $f \in L$ are typically δ -distributions supported at a single point.

Then, $\phi(f) := \exp(i\langle f, \phi \rangle)$ and $\Phi(t, f) = \exp(i\langle f, \Phi(t, \cdot) \rangle)$, where $\langle f, \cdot \rangle$ is the evaluation of $f \in L$ on ϕ . Given $F := (f_1, \dots, f_N) \in L^N$, consider $\phi(F) = (\phi(f_1), \dots, \phi(f_N)) \in U(1)^N$. The space $\mathcal{W}_N$ of elementary functions on N copies of $U(1)$ can be chosen to be generated by the exponentials

$$w_{r_1, \dots, r_N}(\phi(F)) = \exp\left(i \sum_{k=1}^N r_k \langle f_k, \phi \rangle\right) \quad (4.14)$$

with $r_1, \dots, r_N \in \mathbb{R}$ labelling the (necessarily one-dimensional) unitary irreducible representations of $U(1)$. Analogously, the space of elementary functions for the time-dependent fields can be chosen as $(w_k \in \mathcal{W}_{N_k})$

$$W(\Phi(t_1, F_1), \dots, \Phi(t_T, F_T)) = w_T(\Phi(t_T, F_T)) \cdot w_1(\Phi(t_1, F_1)) \quad (4.15)$$

In this case, we could still equip L with the structure of a real vector space if we extend L to the finite real linear combinations $\tilde{L}$ of its generating set L . Since this is no longer possible for the non-Abelian gauge theory example below, we will refrain from doing this, in order to highlight the structural similarity between the examples.

(3) Non-Abelian gauge fields for compact gauge groups G :

A form factor is a distribution

$$f_c^a(x) = \int_c dy^a \delta^{(3)}(x, y), \quad (4.16)$$

where c is a one-dimensional path in σ . We take $\tilde{\mathcal{S}} = G$ equipped with the natural Borel σ -algebra and

$$\begin{aligned} \phi(c) &:= \phi(f_c) := \mathcal{P} \exp\left(\int_c \phi\right) = \mathcal{P} \exp(\phi(f_c)); \phi(f_c) \\ &= \int_\sigma d^3x f_c^a(x) \phi_a(x), \end{aligned} \quad (4.17)$$

where we have identified ϕ as a G connection and $\mathcal{P}$ denotes path ordering. Thus, **Eq. 4.17** is the direct analogue of the scalar field construction (note that the Lie generators are anti-self-adjoint since G is compact so that **Eq. 4.17** is unitary) and $\phi(f_c)$ is simply the holonomy of ϕ along c . Likewise,

$$\Phi(t, c) := \mathcal{P} \left(\exp\left(\int_c \Phi(t, \cdot)\right) \right) \quad (4.18)$$

Note that the form factors do not form a vector space; in general, they cannot be added (unless two curves share a boundary point), and they can never be multiplied by a non-integer real scalar (there is a certain groupoid structure behind this [17–21]). Accordingly, our space of generating set of elementary functions $\mathcal{W}_N$ on N copies of G need to be more sophisticated. We consider the space L of form factors, and for each $F = (f_{c_1}, \dots, f_{c_N}) \in L^N$, the 'pairing' $\phi(F) = (\phi(c_1), \dots, \phi(c_N)) \in G^N$. Then, a possible choice of generating set $\mathcal{W}_N$ of elementary functions is

$$w_\delta(\phi(F)) = \prod_{k=1}^N \sqrt{d_{j_k}} [\pi_{j_k}(\phi(c_k))]_{m_k, n_k} \quad (4.19)$$

with $\delta := \{(j_1, m_1, n_1), \dots, (j_N, m_N, n_N)\}$. In fact, it is sufficient to consider mutually disjoint (up to end points), piecewise real analytic curves c_k . Here, j labels an irreducible representation π_j of G of dimension d_j and $[\pi_j(g)]_{m, n}$, $m, n = 1, \dots, d_j$ its matrix element functions. By the Peter and Weyl theorem, these functions suffice to determine the cylindrical measures uniquely at least if they are absolutely continuous with respect to the product Haar measure. Likewise, we consider the elementary functions

$$W_{\delta_1, \dots, \delta_T}(\Phi(t_1, F_1), \dots, \Phi(t_T, F_T)) = w_{\delta_N}(\Phi(t_T, F_T)) \cdot w_{\delta_1}(\Phi(t_1, F_1)) \quad (4.20)$$

The fact that these functions satisfy all requirements is the statement of Clebsch–Gordan decomposition theory together with the properties of the holonomy to factorise along segments of a curve (note the piecewise analyticity condition).

This ends our list of examples. We will denote the measure related to the stochastic process $\{\phi(t, f)\}$ by μ and the measure related to the stochastic process $\{\phi(f)\}$ by ν . As the notation suggests, ϕ is a field defined on space-time $M = \mathbb{R} \times \sigma$, while ϕ is a field defined on space σ . Note that $M = \mathbb{R} \times \sigma$ with σ any 3D manifold is a consequence of the requirement of global hyperbolicity [176, 177].

The measures μ underlying a relativistic QFT are not only probability measures. In addition, they need to satisfy a set of axioms [98–100, 104] called Osterwalder–Schrader axioms which, however, are tailored to $M = \mathbb{R}^4$, stochastic processes with L being a vector space and with an Euclidean background metric at one's disposal. In quantum gravity and more generally in non-Abelian gauge theories, one typically must or may want to drop some of these structures. As a consequence, we will only keep those axioms that can also be applied in this more general context.

Some of them generalise to stochastic processes not indexed by a vector space, and some do not. Some generalise from the manifold $\mathbb{R}^4$ to the general space-time manifold $\mathbb{R} \times \sigma$ allowed by global hyperbolicity, and some do not. Fortunately, those that do generalise are sufficient for the reconstruction process [103]. We call them the minimal OS axioms, and we call a probability measure that satisfies them an OS measure.

An important remark is that the measures for gauge theories (such as general relativity) are to be formulated in terms of observable (gauge-invariant) fields which are typically composites of the elementary fields. That is why we work in a manifestly gauge (diffeomorphism)-invariant (equivalently, gauge-fixed) context as outlined in Section 3. In fact, in Ref. 39, we find an explicit formula that relates the observable composite fields to the elementary ones. The crucial condition is that the algebra of those observable fields is under sufficient mathematical control in order that Hilbert space representations can be found. This is the case for the construction sketched in Section 3.

The minimal set of OS axioms can be phrased as follows:

Let $\theta(t, x) := (-t, x)$ and $T_s(t, x) := (t + s, x)$ denote time reflection and time translation, respectively. Let $w_k \in \mathcal{W}_{N_k}$, $k = 1, \dots, T$, $F_k \in L^{N_k}$, $t_k \in \mathbb{R}$ and

$$\begin{aligned} W_{(t_1, F_1), \dots, (t_T, F_T)} &:= w_T(\Phi(t_T, F_T)) \dots w_1(\Phi(t_1, F_1)) \\ R \cdot W_{(t_1, F_1), \dots, (t_T, F_T)} &= W_{(-t_1, F_1), \dots, (-t_T, F_T)}, \\ U(s) \cdot W_{(t_1, F_1), \dots, (t_T, F_T)} &= W_{(t_1+s, F_1), \dots, (t_T+s, F_T)} \end{aligned} \quad (4.21)$$

Then, we have the following conditions on the generating functional

$$\mu(W(t_1, F_1), \dots, (t_T, F_T)) \quad (4.22)$$

I. Time reflection invariance:

$$\mu(W_{(-t_1, F_1), \dots, (-t_N, F_T)}) = \mu(W_{(t_1, F_1), \dots, (t_T, F_T)}) \quad (4.23)$$

II. Time translation invariance

$$\mu(W_{(t_1+s, F_1), \dots, (t_N+s, F_T)}) = \mu(W_{(t_1, F_1), \dots, (t_N, F_T)}). \quad (4.24)$$

III. Time translation continuity

$$\begin{aligned} \lim_{s \rightarrow 0} \mu \left([W_{(t_1, F_1), \dots, (t_T, F_T)}]^* W_{(t'_1+s, F'_1), \dots, (t'_T+s, F'_T)} \right) \\ = \mu \left([W_{(t_1, F_1), \dots, (t_T, F_T)}]^* W_{(t'_1, F'_1), \dots, (t'_T, F'_T)} \right) \end{aligned} \quad (4.25)$$

IV. Reflection positivity

Consider the vector space V of the complex span of functions of the form $W(t_1, F_1), \dots, (t_T, F_T)$ with $t_1, \dots, t_T > 0$. Then, for any $\Psi, \Psi' \in V$,

$$\langle \Psi, \Psi' \rangle := \mu(\bar{\Psi} R \cdot \Psi') \geq 0 \quad (4.26)$$

Note that the stochastic process indexed by $\mathbb{R} \times L$ considers random variables $\phi(t, f)$ at sharp points of time. It is often argued that this index set provides an insufficient ‘smearing’ in the time direction and fails to cover interacting QFT at least in 3 + 1 space-time dimensions (in 1 + 1 and 2 + 1 dimensions, there are examples for which this works [178–180]). However, this argument rests on perturbative results as on 3 + 1-dimensional Minkowski space; so far, no interacting QFT (obeying the Wightman axioms) has been rigorously constructed. It is still conceivable [181] that in a non-perturbative construction of the theory, for which constructive QFT is designed, one can deal with fields at sharp time. One could, of course, be more general and consider stochastic processes indexed by some L which now also includes smearing in the time direction, and the formulation of reflection positivity will then constrain to elements of L with positive time support; however, then the Wiener measure construction sketched below will not work. Our viewpoint is that this more general situation can be obtained from the sharp time construction because integrals of smearing functions with respect to time can be approximated by Riemann sums, which in turn are nothing but integrals with respect to sharp time smearing functions.

At the moment, it is rather unclear how and why μ and ϕ define a relativistic QFT. This will become clear in the next subsection.

4.3. Osterwalder–Schrader (OS) Reconstruction

The following abstract argument is standard [98–100]. (See Refs. 54–57 for a proof adapted to the notation in this article.) Due to reflection positivity, Eq. 4.26 defines a positive semi-definite

sesquilinear form on V . We compute its null space N and complete the quotient of equivalence classes V/N in the inner product **Eq. 4.24** to a Hilbert space. Given $\Psi \in V$, we denote its equivalence class $\Psi + N$ by $[\Psi]_\mu$, where we keep track of the measure dependence of the quotient construction. By construction the $D = [V]_\mu$ is dense in $\mathcal{H}$. Since the constant function $\Psi = 1 \in V$, we define a ‘vacuum’ vector by $\Omega := [1]_\mu$. Finally, we define for $s \geq 0$

$$K(s)[\Psi]_\mu := [U(s)\Psi]_\mu \quad (4.27)$$

The constraint $s \geq 0$ is due to the time support condition in the definition of V . One must show that this is well-defined (independent of the representative) [98–100]. By virtue of their definition (**Eq. 4.2**), the $U(s)$ forms a one-parameter Abelian group of operators $U(s)U(s') = U(s+s')$ on $L_2(S, d\mu)$. This implies that the $K(s)$ forms a one-parameter Abelian semi-group due to the constraint $s \geq 0$ (again one must show that the definition is well-defined). Time translation continuity (**Eq. 3.23**) translates into weak continuity of the semi-group. Furthermore, by time translation invariance 4.24, $U(s)$ defines unitary, in particular bounded operators, on $L_2(S, d\mu)$ which translates into the statement that $K(s)$ forms a contraction semi-group. Thus [98–100], there exists a positive self-adjoint operator H , called ‘Hamiltonian’ on $\mathcal{H}$ such that $K(s) = e^{-sH}$. Obviously, $K(s)\Omega = \Omega$; thus, Ω is a ground state for H which justifies the name ‘vacuum’.

This elegant argument is deceivingly simple. To actually compute the *Osterwalder–Schrader triple* $(\mathcal{H}, \Omega, H)$ from μ and to relate it to the fields and Hamiltonian in terms of which one would construct the quantum theory using canonical quantisation is not clear yet. However, one can again use the following abstract argument [54–57]. Suppose that there is an Abelian C^* -algebra $\mathfrak{B}$ of bounded operators on $\mathcal{H}$ such that $\mathfrak{B}\Omega$ is dense (the C^* -norm is inherited from the uniform operator topology). It is not difficult to show that this is always the case when $\mathcal{H}$ is separable which is the only case that we will consider in our application to renormalisation, but it also holds in many non-separable situations (see appendix B of Ref. 196 for a proof). Let $\Delta(\mathfrak{B})$ be its Gelfand spectrum [182] (which is a compact space), that is, the space of all $*$ homomorphisms $\phi: \mathfrak{B} \rightarrow \mathbb{C}$. Then, by Gelfand’s theorem, $\mathfrak{B}$ can be thought of as the space $C(\Delta(\mathfrak{B}))$, that is, the continuous functions on the spectrum which is an Abelian C^* -algebra with respect to the sup norm. The correspondence (Gelfand isomorphism) is given by $\hat{b}(\phi) = \phi(b)$ for all $\phi \in \Delta(\mathfrak{B})$, and in fact, this is an isometric isomorphism of C^* -algebras. Consider now the linear functional

$$\nu(\hat{b}) := \langle \Omega, b\Omega \rangle \quad (4.28)$$

which by construction is positive $\nu(|\hat{b}|^2) = \|b\Omega\|^2$. By the Riesz–Markov theorem [175], there exists a (regular Borel) probability measure on $S_I := \Delta(\mathfrak{B})$ which by abuse of notation we also denote by ν such that

$$\nu(\hat{b}) = \int_{\Delta(\mathfrak{B})} d\nu(\phi) \hat{b}(\phi), \quad (4.29)$$

that is, to say, the Hilbert space $\mathcal{H}$ obtained from OS reconstruction can be thought of as $L_2(\Delta(\mathfrak{B}), d\nu)$ under the isomorphism $b\Omega \mapsto \hat{b}$,

in particular Ω corresponds to the constant function equal to 1. We thus have managed to cast $\mathcal{H}$ into the language of measure theory on the set $S' = \Delta(\mathfrak{B})$. The fields ϕ that come out of this construction are random variables indexed by some index set L' , that is, we have shown that we can always construct such a measure and a corresponding stochastic process. We think of the field ϕ as the spatial configuration fields underlying a canonical quantisation approach. A priori, however, it is not clear what L' is, although it must be related in some way to $\mathbb{R}^+ \times L$. In the case of free fields, one can show that, in fact, one can choose $\mathfrak{B}$ in such a way that $L' = L$ due to the quotient construction involved in $\mathcal{H}$ but even then it is a priori not clear how $\Phi(t, f)$ and $\phi(f)$, $f \in L$ are related. Again, in the case of free fields, one shows that $\phi(f)$ can be thought of as $\Phi(0, f)$, the space-time field at sharp time zero. However, in general, the relation between the stochastic processes underlying Φ and ϕ may be more complex. In any case, the operator H translates in this language into the operator

$$\widehat{Hb} := \widehat{Hb}\Omega \quad (4.30)$$

4.4. Feynman–Kac–Trotter–Wiener (FKTW) Construction

Given an OS triple $(\mathcal{H}, \Omega, H)$, we saw at the end of the previous subsection that without loss of generality, we can assume that $\mathcal{H} = L_2(S_I, d\nu)$, where ν is a probability measure on S equipped with a Borel σ -algebra and that we are given a stochastic process $\phi(f)$, $f \in L$ indexed by some index set L , at least when $\mathcal{H}$ is separable (which will be the case in our applications). Moreover, $\Omega = 1$ in this presentation of $\mathcal{H}$ is cyclic for some C^* -algebra of functions on S_I . We pick some set $\mathcal{W}_N$, $N \in \mathbb{N}_0$ of elementary functions $w \in \mathcal{W}_N$ subject to the conditions 1.–4. spelled out just after (4.7) and for $F = (f_1, \dots, f_N) \in L^N$ have $\phi(F) = (\phi(f_1), \dots, \phi(f_N)) \in (S_I)^N$ as well as

$$w_F(\phi) = w(\phi(F)) \quad (4.31)$$

Let now $T \in \mathbb{N}_0$, $t_1 < t_2 < \dots < t_T$ and $F_k \in L^{N_k}$, $w_k \in \mathcal{W}_{N_k}$. We consider the expectation value functional

$$\langle \Omega, w_{T, F_T} e^{-(t_T - t_{T-1})H} w_{T-1, F_{T-1}} e^{-(t_{T-1} - t_{T-2})H} \dots e^{-(t_2 - t_1)H} w_{1, F_1} \Omega \rangle \quad (4.32)$$

Consider now a stochastic process $\Phi(s, f)$ indexed by $(s, f) \in \mathbb{R} \times L$ and the elementary functions

$$W_{(t_k, F_k)_{k=1}^T}(\Phi) = w_{T, F_T}(\Phi(t_N, \cdot)) \dots w_{1, F_1}(\Phi(t_1, \cdot)) \quad (4.33)$$

Then, the Wiener measure μ , if it exists, evaluated on **Eq. 4.33**

$$\mu\left(W_{(t_k, F_k)_{k=1}^T}\right) \quad (4.34)$$

is supposed to equal **Eq. 4.31**. The non-trivial question is why this should be the case, under which circumstances, and how to construct μ . For this, we consider the integral kernel K_β of the operator $e^{-\beta H}$, $\beta > 0$, that is,

$$[e^{-\beta H} \psi](\phi) =: \int_S d\nu(\phi') K_\beta(\phi, \phi') \psi(\phi') \quad (4.35)$$

Note the semi-group property

$$\int_{S'} d\nu(\phi) K_{\beta_1}(\phi_1, \phi) K_{\beta_2}(\phi, \phi_2) = K_{\beta_1 + \beta_2}(\phi_1, \phi_2) \quad (4.36)$$

Define $S := \prod_{t \in \mathbb{R}_0} S'_t$. For each $T \in \mathbb{N}_0$, consider $t_1 < \dots < t_T$ and measurable sets $s'_{t_k} \subset S'$ and define the set function

$$\mu \left(\left[\times_{k=1}^T s'_{t_k} \times \left[\times_{t \notin \{t_1, \dots, t_T\}} S' \right] \right] \right) := \int_{[S']^T} d\nu(\phi_1) \dots d\nu(\phi_T) \prod_{k=1}^T \chi_{s'_{t_k}}(\phi_k) \prod_{k=1}^{T-1} K_{t_{k+1}-t_k}(\phi_{k+1}, \phi_k) \quad (4.37)$$

It is not clear that this is a positive set function, but when it is, it is called the Wiener measure generated by the OS triple. For sufficient criteria for this property called Nelson-Symanzik positivity in the case of scalar fields (see Refs. 183 and 184). Basically, one needs to show that matrix elements of $e^{-\beta H}$ between positive functions are positive. Note that for $s'_k = S'$ for all k , we get

$$\mu(S) = \langle \Omega, e^{-(s_T - s_1)H} \Omega \rangle = 1 \quad (4.38)$$

This shows that μ is a probability measure on S . For quantum mechanical Schrödinger Hamiltonians, one can use the Trotter product formula and the Wiener measure of the heat kernel to prove positivity [185] (Feynman–Kac formula).

One can now show the following [54–57]:
Theorem.

- i. Suppose that OS data $(\mathcal{H}, H, \Omega)$ are given and that the corresponding Wiener measure μ exists. Then, μ is an OS measure and its OS reconstruction reproduces the given OS data up to unitary equivalence.
- ii. Suppose that an OS measure μ is given thus producing OS data $(\mathcal{H}, \Omega, H)$. Then, the corresponding Wiener measure exists and reproduces μ up to equivalence of measure spaces.

Here, measure spaces (S_j, B_j, μ_j) ; $j = 1, 2$ are called equivalent if there exists a bijection $F: S_1 \rightarrow S_2$ such that both F and F^{-1} are measurable and such that $\mu_1 = \mu_2 \circ F$. The reason why we generically only reproduce an equivalent and not an identical starting point lies in the large freedom in the choice of the stochastic process ϕ when performing the OS reconstruction step.

5. RENORMALISATION

5.1. Motivation

Our motivation for renormalisation comes from the current state of affairs with respect to the definition of the quantum dynamics in LQG as outlined in Section 3. In that case, the Hilbert space $\mathcal{H} = L_2(S, d\nu)$ is precisely of the form we envisage here. Moreover, we have a vacuum Ω for a candidate Hamiltonian

H that, however, we are not sure whether all steps of the quantisation process that led to H are justified, namely, we have defined H as H_γ on certain mutually orthogonal subspaces $\mathcal{H}_\gamma$ preserving it using a choice of discretisation of the classical continuum expression which has naively the correct dequantisation if the graph γ fills the spatial manifold σ sufficiently densely. The definition of elementary functions in Eq. 4.19 precisely reproduces the SNWF, and thus, the spatial connection defines a stochastic process indexed by graphs.

As already mentioned at the end of Section 3, we would like to take a fresh look at the problem. As usual in constructive QFT, if σ is not already compact, we replace it with a compact manifold σ_R , where R is an infrared (IR) cut-off which we remove in the end $R \rightarrow \infty$ (thermodynamic limit). In order not to clutter the notation, the dependence on R of all considerations that follow will be suppressed. Next, we do not consider all finite graphs γ (taking all finite graphs leads to a non-separable Hilbert space) but only a controllable countable family $\mathcal{M}$, therein which, however, is such that the discretised classical variables (configuration and momentum fields) in terms of which we perform the quantisation separate the points of the classical phase space when all the graphs in $\mathcal{M}$ are at our disposal. The set $\mathcal{M}$ is supposed to be partially ordered and directed. The motivation for doing so stems from the spatial diffeomorphism invariance of the classical LQG Hamiltonian: The algebraic form of the Hamiltonian discretised on diffeomorphic graphs is identical. This is precisely the starting point of the algebraic quantum gravity proposal [169–172], where it was emphasised that one can quantise gravity in terms of abstract graphs which gain their physical meaning only after choosing an embedding supplied, for instance, by a semi-classical state.

To have some intuitive picture in mind, consider $\sigma = \mathbb{R}^3$ with toroidal compactification $\sigma_R = T^3$ (where each direction has length R with respect to the Euclidian background metric on $\mathbb{R}^3$ and with periodic boundary conditions installed) and $\mathfrak{r}$ the set of all finite graphs σ_R of cubic topology. This is still an uncountable set which we now restrict to a countable one as follows. Each element of $\mathfrak{r}$ is uniquely labelled by $M \in \mathbb{N}$, where M^3 is the number of vertices of the graph (one could generalise this and have different numbers of vertices in each direction). We pick once and for all a coordinate system and locate the vertices of γ_M at the points

$$m \epsilon_M, m \in \mathbb{Z}_M^3, \mathbb{Z}_M = \{0, 1, \dots, M-1\}, \epsilon_M = \frac{R}{M}, \quad (5.1)$$

where the edges of the graph are straight lines in the coordinate directions between the vertices. We equip $\mathcal{M} := \mathbb{N}$ with the following partial order: $M < M'$ iff $\frac{M'}{M} \in \mathbb{N}$. Note that this implies $\gamma_M \subset \gamma_{M'}$ since

$$m \epsilon_M = m \frac{M'}{M} \epsilon_{M'} =: m' \epsilon_{M'} \quad (5.2)$$

with $m' \in \mathbb{Z}_{M'}^3$ and because the edges of the graphs are straight lines in the coordinate directions. This is certainly not a linear order because not all natural numbers are in relation but still

equips Γ with a direction: Given M and M' take, for instance, $M'' = MM'$, then $M, M' < M''$ (more efficiently take M'' as the least common multiple). It is clear that for M sufficiently large discretised phase space variables obtained by integrating continuum variables over 0- or 1-dimensional subsets of γ_M (vertices or edges) or by integrating momentum variables over 3- or 2-dimensional subsets of the cell complex corresponding to γ_M (faces and cubes) will separate the points of the continuum phase space. Instead of γ_M , one could also use the cubic cell complex γ_M^* dual to γ_M defined by saying that the barycentres of the cubes of γ_M^* coincide with the vertices of γ_M . However, in the spirit of economy, we will not use the additional structure γ_M^* in what follows.

5.2. Discretisation of Phase Space

In canonical quantisation, we start with a continuum phase space coordinatised by configuration fields ϕ^J and canonically conjugate momentum fields π_J in terms of which the classical continuum Hamiltonian H is formulated. Here, the index J corresponds to an internal symmetry and is typically Lie algebra valued. Now, we consider a discretisation of both the phase space and the Hamiltonian, one for each lattice M , while keeping track of how these fields ϕ_M^J and π_M^J are related to the continuum fields ϕ^J and π_J . The idea for how to do this stems from the observation that by construction of generally covariant field theories, the fields ϕ^J and π_J are *dual* in the sense that there is a natural bilinear form $\langle \pi, \phi \rangle'_{I \times K} = \sum_J \langle \pi_J, \phi^J \rangle_{I \times K}$ on the phase space (usually a cotangent bundle T^*K) $I \times K$ of momentum and configuration fields, respectively, where $\langle \cdot, \cdot \rangle$ is spatially diffeomorphism-invariant. Note that $\langle \cdot, \cdot \rangle'$, $\langle \cdot, \cdot \rangle$ just differ by tracing over the internal directions in field space, that is, $I = I^d$, $K = K^d$, where d is the number of internal directions in field space.

For instance, the momentum of a scalar field is geometrically a scalar density of weight one, so that $\langle \pi, \phi \rangle' = \langle \pi, \phi \rangle = \int_\sigma d^3x \pi(x) \phi(x)$. The momentum of a G connection is geometrically a Lie algebra-valued vector field density so that $\langle \pi, \phi \rangle' = \sum_J \int_\sigma d^3x \pi_J^a(x) \phi_a^J(x)$. This also holds for higher p -forms as they occur in some supergravity theories as well as for (standard model or Rarita–Schwinger) fermions. Note that the bilinear form is in general not invariant under the internal symmetry group, but this will not be important for what follows. The fact that π and ϕ are conjugate is the statement, that their canonical brackets are

$$\{\langle \pi, k' \rangle', \langle i', \phi \rangle'\} = \langle i', k' \rangle' \quad (5.3)$$

for all $(i', k') \in I \times K$.

The fact that the bilinear form $\langle \cdot, \cdot \rangle$ is at our disposal motivates a natural choice for the index set L and L^* of the stochastic process ϕ, π . Namely, we choose L to be a certain distributional extension of I and likewise L^* as a certain distributional extension of K . These extensions should be such that $\langle i, k \rangle$ remains well-defined for $i \in L$, $k \in L^*$. For instance, for a scalar field we may choose L as the set of δ distributions $f_p(x) = \delta_p(x)$ with support at single points $p \in \sigma$ and L^* as the set of characteristic functions $g^R(x) = \chi_R(x)$ of connected D -dimensional submanifolds R of σ . For a compact G-connection, we can choose L as the set of form factors $f_c^a(x) = \int_c dy^a \delta(x, y)$ with support on (piecewise analytic) curves c . For L^* ,

we would consider the set of dual form factors of the form $g_a^S(x) = 1/(D-1)! \int_S \epsilon_{ab_1 \dots b_{D-1}} dy^{b_1} \wedge \dots \wedge dy^{b_{D-1}} \delta(x, y)$ with support on (piecewise analytic) $D-1$ submanifolds S . We may also have opportunity to consider their Lie algebra-valued versions $f_c^{aj}(x) = \tau^j f_c^a(x) \in L^*$, $g_{aj}^S(x) = \tau_j g_a^S(x) \in (L^*)^*$, where τ^j and τ_j are dual bases in the defining representation of the Lie algebra of G such that $\text{Tr}(\tau^j \tau_K) = \delta_K^j$. Note that we deliberately do not make use of the fact that these distributions are elements of vector spaces. This is because we aim at a uniform description of both linear and non-linear theories. In the case of linear theories, the description can be significantly simplified as we have done in Refs. 54–57.

The connection to **Section 4.2** is then as follows: For each $f \in L$, $\phi \in K$, we consider a map $(f, \phi) \mapsto \phi(f) \in \tilde{S}$. For linear theories, one usually takes $\tilde{S} = U(1)$, and for a G gauge theory, one takes $\tilde{S} = G$. The object $\phi(f)$ exploits the existence of the natural bilinear form $\langle \cdot, \cdot \rangle$. For instance, for a scalar field, one considers $\phi(f_p) = \exp(i \langle f_p, \phi \rangle)$, while for a G connection, we consider the holonomy $\phi(f_c) = \mathcal{P} \exp(\langle f_c, \phi^J \rangle_{\tau_j})$. For each $N \in \mathbb{N}$, we consider $F = (f_1, \dots, f_N) \in L^N$ and define $\phi(F) = (\phi(f_1), \dots, \phi(f_N)) \in \tilde{S}^N$. The space of elementary functions $\mathcal{W}_N$ consists of maps $\tilde{S} \rightarrow \mathbb{C}$ subject to the conditions listed in the beginning of **Section 4.2**. We may generate $\mathcal{W}_N$ from monomials labelled by matrix element functions of finite-dimensional unitary representations of $\tilde{S}$ (see **Eq. 4.19**).

For each $M \in \mathbb{N}$, let L_M be the space of discrete functions on the lattice consisting of M^D points with values in $\mathbb{R}^t$, where t is tensorial number of configuration (or momentum) degrees of freedom per spatial point ($t = 1$ for scalar fields, $t = D$ for a G Yang–Mills theory in $D+1$ space-time dimensions, etc.). That is, an element $l_M \in L_M$ assigns to each point $m \in \mathbb{Z}_M^D$ a vector in $\mathbb{R}^t$. The space L_M carries an auxiliary real Hilbert space structure (L_M is, of course, a finite-dimensional vector space), for example, for a G Yang–Mills theory,

$$\langle l_M, \tilde{l}_M \rangle_{L_M} = \sum_{m \in \mathbb{Z}_M^D} \sum_{a=1}^t l_M(m, a) \tilde{l}_M(m, a) \quad (5.4)$$

for any $l_M, \tilde{l}_M \in L_M$, and we wrote $[l_M]^a(m) =: l_M(m, a)$.

Definition.

A discretisation of the continuum phase space $I \times K$ subordinate to $M \in \mathbb{N}$ is a pair of linear maps

$$I_M: L_M \rightarrow L; K_M: L_M \rightarrow L^* \quad (5.5)$$

with the following properties:

- i. For any $l_M, l_M' \in L_M$

$$\langle I_M l_M, K_M l_M' \rangle_{I \times K} = \langle l_M, l_M' \rangle_{L_M} \quad (5.6)$$

That is, to say $I_M' K_M = K_M' I_M = \text{id}_{L_M}$ where $I_M': I \rightarrow L_M$, $K_M': K \rightarrow L_M$ are the dual maps defined by

$$\langle I_M l_M, \phi \rangle_{I \times K} = \langle l_M, I_M' \phi \rangle_{L_M}, \quad \langle \pi, K_M l_M \rangle_{I \times K} = \langle K_M' \pi, l_M \rangle_{L_M}, \quad (5.7)$$

ii. For any $M < M'$ define the injection maps

$$I_{MM'} := K_{M'}' I_M; K_{MM'} := I_{M'}' K_M: L_M \rightarrow L_{M'} \quad (5.8)$$

Then, we require

$$I_M' I_{MM'} = I_M, K_M' K_{MM'} = K_M \quad (5.9)$$

To see how this gives rise to discretised configuration and momentum variables let $\delta_M^m a, \delta_m, a^M \in L_M$ with $m \in \mathbb{Z}_M^D, a = 1, \dots, t$ be the Kronecker functions $[\delta_M^m a]_b(\tilde{m}) := \delta_b^a \delta_m, \tilde{m}$ and $[\delta_m, a^M]^b(\tilde{m}) := \delta_a^b \delta_m, \tilde{m}$. Then, the following functions on the continuum phase space

$$(\pi_M)_j^a(m) := \langle \pi_j, K_M \delta_M^{m,a} \rangle_{I \times K}, (\phi_M)_a^j(m) := \langle I_M \delta_M^{m,a}, \phi^j \rangle_{I \times K} \quad (5.10)$$

enjoy canonical brackets

$$\{(\pi_M)_j^a(m), (\phi_M)_b^K(\tilde{m})\} = \delta_b^a \delta_j^K \delta_{m,\tilde{m}} \quad (5.11)$$

where the first condition (Eq. 5.6) was used. Thus, Eq. 5.6 makes sure that the discretisations Eq. 5.10 enjoy canonical brackets, so we call Eq. 5.6 the *symplectomorphism property*. The motivation for the second condition Eq. 5.9 will become clear only later; however, we note that it implies that for all $M < M' < M''$

$$I_{M'M''} I_{MM'} = K_{M'}' [I_M' I_{MM'}] = I_{MM''} \quad (5.12)$$

which we thus call *cylindrical consistency property*. Likewise, $K_{M'M''} K_{MM'} = K_{MM''}$. It says that injecting a function into the continuum is independent from which resolution scale M this is done.

Finally, we will impose a further restriction on the maps I_M, K_M , which amounts to a convenient choice of normalisation and thus is called *normalisation property*. Namely, we require that for all $M < M'$, the map $I_{MM'}: L_M \rightarrow L_{M'}$ restricts to $B_M \rightarrow B_{M'}$, where B_M is the set of functions on $\mathbb{Z}_M^D$ with values in the *bit space* $\{0, 1\}^t$. This condition is only necessary in the non-Abelian case, and there avoids overcounting.

We note that Eq. 5.9 defines elements $\pi_M = K_M' \pi, \phi_M = I_M' \phi$ of L_M^{dt} that we can now use to try to define a discretisation $H_M = H_M[\{(\pi_M)_j^a(m), (\phi_M)_a^j(m)\}_{a,j,m}]$ of the Hamiltonian $H = H[\pi, \phi]$. For instance, if the Hamiltonian depends only quadratically on the fields, then one may try (including discretisations of spatial derivatives and some spatial averages)

$$H_M := H[\pi = I_M \pi_M, \phi = K_M \phi_M] \quad (5.13)$$

For interacting Hamiltonians, more sophisticated approximations must be used. Certainly, the expression for H_M is in general plagued by a large amount of discretisation ambiguity beyond the choice of discretised variables. On the other hand, the fact that $\pi_M = K_M' \pi$ and $\phi_M = I_M' \phi$ are conjugate will be convenient when constructing H_M , and it is efficient to construct them motivated by the naturally available bilinear form $\langle \cdot, \cdot \rangle$ on the phase space.

To see that there are non-trivial examples for such maps, consider a scalar field in D spatial dimensions compactified on a torus with Euclidian coordinate length R in all directions. Then (recall $\epsilon_M = R/M$),

$$\begin{aligned} (I_M l_M)(x) &:= \sum_{m \in \mathbb{Z}_M^D} l_M(m) \delta_{m \epsilon_M}(x), (K_M l_M)(x) := \\ &= \sum_{m \in \mathbb{Z}_M^D} l_M(m) \chi_{m \epsilon_M}(x), \end{aligned} \quad (5.14)$$

where

$$\chi_{m \epsilon_M}(x) = \prod_{a=1}^D \chi_{[m^a \epsilon_M, (m^a+1) \epsilon_M)}(x), \quad (5.15)$$

where the latter denotes the characteristic functions of left closed—right open—intervals. This clopen interval structure is very important in order that Eqs 5.6 and 5.9 are satisfied [54–57]. Similar constructions work for gauge fields (see appendix A or [196]). Note that we changed here the notation as compared to [54–57]: The maps I_M, E_M used there are called here K_M, I_M , respectively. The motivation for this change of notation is to make it manifest how much of the structure is in fact already canonically provided by the structure of the classical theory.

Given the lattice in D spatial dimensions labelled by $M \in \mathbb{N}$, we consider in general $N = M^D t$ degrees of freedom $\phi(I_M l_M) := \phi_M(l_M) := \{\phi(I_M l_M^{m,a})\}_{m \in \mathbb{Z}_M^D, a=1, \dots, t} \in \tilde{S}^N$, where $I_M^{m,a} = l_M \delta_M^{m,a}$ and l_M is restricted to the subset $B_M \subset L_M$ of functions $\mathbb{Z}_M^D \rightarrow F_2^t$, where $F_2 = \{0, 1\}$ is the field in two elements (*bit space*). Thus, $l_M(m, a) \in \{0, 1\}$ is restricted to the information whether the degree of freedom $\phi(I_M l_M^{m,a})$ is excited or not. This is justified because 1. the missing information about the strength of the excitation is encoded in the representation label (see below) and 2. because the maps $I_{MM'}$ restrict to maps $B_M \rightarrow B_{M'}$ by assumption.

The space of elementary functions $\mathcal{W}_M$ on the lattice labelled by M is then generated by

$$w_{j,n,\tilde{n}}^M(\phi_M(l_M)) = w_{j,n,\tilde{n}}^M(\phi(I_M l_M)) = \prod_{m,a} [\pi_{j_{m,a}}(\phi(I_M l_M^{m,a}))]_{n_{m,a}, \tilde{n}_{m,a}} \quad (5.16)$$

Here, $j_{m,a}$ labels an irreducible representation $\pi_{j_{m,a}}$ of G (one from each equivalence class), $d_{j_{m,a}}$ is its dimension, and $[\pi_{j_{m,a}}(\cdot)]_{n_{m,a}, \tilde{n}_{m,a}}$ denote its matrix elements with $n_{m,a}, \tilde{n}_{m,a} \in \{1, \dots, d_{j_{m,a}}\}$.

To see how Eq. 5.16 interacts with the map $I_{MM'}$ in the case of non-Abelian gauge theory, we note that the cylindrical consistency property of $I_{MM'}$ implies

$$w_{j,n,\tilde{n}}^M(\phi_M(l_M)) = w_{j,n,\tilde{n}}^M(\phi(I_M l_M)) = \sum_{\alpha} w_{j,n',\tilde{n}',\alpha}^{M'}(\phi_{M'}(I_{MM'} l_M)), \quad (5.17)$$

where the notation is as follows (see appendix A or [196]): $j_{m'a'} := [I_{MM'} l_M]_{[m'/M, M]}, a$, where $[\cdot]$ denotes the Gauss bracket, $n'_{m'}, a = n_m, a$ if $m' = M/Mm, \tilde{n}'_{m,a} = \tilde{n}_{m,a}$ if $m'^a + 1 = M'/M(m^a + 1), m'^b = M'/Mm^b; b \neq a$, and otherwise the sum over α denotes the sum over all $\tilde{n}_{m'a'} = n_{m'+\delta_a}, a' \in \{1, \dots, d_{j_{m'}, a}\}$ with $[\delta_a]^b := \delta_a^b$ that arise by writing the holonomy along the edge labelled by $m \in \mathbb{Z}_M^D, a = 1, \dots, D$ as products of holonomies along edges labelled by $m' \in \mathbb{Z}_{M'}^D, a$.

In general, therefore we see that for any generating function $w^M \in \mathcal{W}_M$, we have for all $M < M'$

$$w^M(\phi_M(l_M)) = \sum_{\alpha} z_{\alpha} w_{\alpha}^{M'}(\phi_{M'}(I_{MM'}l_{M'})), \quad (5.18)$$

where the sum over α involves a finite, unique set of generating functions $w^{M'} \in \mathcal{W}_{M'}$, and z_{α} are certain, definite complex numbers. Similar statements then, of course, hold for the stochastic process labelled by $\mathbb{R} \times L$ and for the functions

$$\begin{aligned} W^M(\Phi_M(t^{(1)}, l_M^{(1)}), \dots, \Phi_M(t^{(T)}, l_M^{(T)})) \\ = w_T^M(\Phi_M(t^{(T)}, l_M^{(T)})) \dots w_1^M(\Phi_M(t^{(1)}, l_M^{(1)})) \end{aligned} \quad (5.19)$$

5.3. Hamiltonian Renormalisation

Abstracting from the concrete lattice implementation and field content above, we are in the following situation: There is a partially ordered and directed label set $\mathcal{M}$, and for each $M \in \mathcal{M}$, we have a map $I_M: B_M \rightarrow L^{N(M)}$, where L is the index set of the stochastic process ϕ , $N(M) \in \mathbb{N}$ is the number of elements of L in the image of I_M , and $B_M = \{0, 1\}^{N(M)}$. Then, $\phi_M(l_M) := \phi(I_M l_M) \in \tilde{\mathcal{S}}^{N(M)} =: \tilde{\mathcal{S}}_M$, and we have a generating set of elementary functions $w^M: \tilde{\mathcal{S}}^{N(M)} \rightarrow \mathbb{C}$.

Suppose that for each $M \in \mathcal{M}$, we have discretised the system somehow as sketched above and picked some OS triple $(\mathcal{H}_M^{(0)}, \Omega_M^{(0)}, H_M^{(0)})$ with $\mathcal{H}_M^{(0)} = L_2(d\nu_M^{(0)}, \tilde{\mathcal{S}}_M)$. That is to say, we have a stochastic process $\{\phi_M(l_M)\}_{M \in \mathcal{M}}$ indexed by B_M and probability measures $\nu_M^{(0)}$ on $\tilde{\mathcal{S}}_M$. The Hamiltonian $H_M^{(0)}$ preserves $\mathcal{H}_M^{(0)}$ and annihilates the unit vector $\Omega_M^{(0)} \in \mathcal{H}_M^{(0)}$, which is cyclic. We consider a space of elementary functions $\mathcal{W}_M$ such that in particular $w^M(\phi_M(l_M))\Omega_M^{(0)}$; $w^M \in \mathcal{W}_M$, $l_M \in B_M$ lie dense in $\mathcal{H}_M^{(0)}$.

Using the Feynman–Kac–Trotter–Wiener (FKTW) construction, we obtain a family of OS measures $\mu_M^{(0)}$ on $S_M = \prod_{t \in \mathbb{R}} \tilde{\mathcal{S}}_M$, which can be probed using a stochastic process $\Phi_M(t, l_M)$ labelled by $\mathbb{R} \times B_M$. This measure family $\{\mu_M^{(0)}\}_{M \in \mathcal{M}}$ will generically not be cylindrically consistent and therefore does not define a continuum measure μ because of the discretisation ambiguities involved in the construction of $H_M^{(0)}$ which determines $\mu_M^{(0)}$. If it was, then we would have for $w_1, \dots, w_T \in \mathcal{W}_M$

$$\begin{aligned} \mu(w_T(\Phi(t_T, l_M^{t_T})) \dots w_1(\Phi(t_1, l_M^{t_1}))) \\ = \mu_M(w_T(\Phi_M(t_T, l_M^{t_T})) \dots w_1(\Phi_M(t_1, l_M^{t_1}))) \end{aligned} \quad (5.20)$$

Using $I_M = I_{M'}I_{MM'}$ for $M < M'$, we would find the identity

$$\begin{aligned} \mu_M(w_T(\Phi_M(t_T, l_M^{t_T})) \dots w_1(\Phi_M(t_1, l_M^{t_1}))) \\ = \mu_{M'}(w_T(\Phi_{M'}(t_T, I_{MM'}l_M^{t_T})) \dots w_1(\Phi_{M'}(t_1, I_{MM'}l_M^{t_1}))) \end{aligned} \quad (5.21)$$

called the condition of cylindrical consistency.

As reviewed in **Section 3**, condition **Eq. 5.21** grants the existence of μ under rather generic conditions. The strategy (see also Refs. 137 and 138) is therefore to construct an iterative sequence of measure families $\mathbb{N}_0 \ni n \mapsto \{\mu_M^{(n)}\}_{M \in \mathcal{M}}$ called *renormalisation (group) flow* with initial family as above such that the fixed point family satisfies **Eq. 5.21**. We refer to section

C of [196] for the reader interested in more notions of the renormalisation group in the language of measure theory.

The scheme that we will employ in fact does not make use of **Eq. 5.21** for all $M < M'$ but only $M' = p^n M$, where p is a prime. The simplest choice is $p = 2$, but we have tested the formalism also for $p = 3, 5$ [54–57] and mixtures thereof in the case of free scalar QFT. This, in fact, does cover all possible M because any natural number can be written as kp^l ; k, p relative prime, but the fixed point family could depend on k . Of course, one assumes that the fixed point family is independent of the choices of p, k as an expression of universality as confirmed again for simple systems [54–57]. Thus, we define as renormalisation flow

$$\begin{aligned} \mu_M^{(n+1)}(w_T(\Phi_M(t_T, l_M^{t_T})) \dots w_1(\Phi_M(t_1, l_M^{t_1}))) \\ = \mu_{M'}^{(n)}(w_T(\Phi_{M'}(t_T, I_{MM'}l_M^{t_T})) \dots w_1(\Phi_{M'}(t_1, I_{MM'}l_M^{t_1}))) \end{aligned} \quad (5.22)$$

for $M' = 2M$. Having then obtained μ^* from cylindrically consistent projections μ_M^* , we want to construct the OS triple $(\mathcal{H}^*, \Omega^*, H^*)$ using OS reconstruction. However, while we are sure that $\mu_M^{(0)}$ is an OS measure for each M by theorem 4.4, we are a priori not granted that $\mu_M^{(n)}$ is an OS measure, that is, that the flow preserves the OS measure class. This is, in fact, shown in Refs. 54–57.

Theorem.

The renormalisation flow (Eq. 5.22) preserves the OS measure class, and its fixed points define OS measures μ^* .

Responsible for this result is the fact that the time operations that define an OS measure commute with the spatial coarse graining operation. Thus, in principle, we can perform renormalisation in the measure (or path integral) language and then carry out OS reconstruction in order to find the continuum Hamiltonian theory that we are interested in. On the other hand, the fact that FKTW construction and OS reconstruction are inverses of each other (theorem 4.4) allows for the possibility to map the renormalisation flow of measures directly into a renormalisation flow of OS triples. In detail,

Step 1: Identifying the stochastic processes

We need to work out the null space of the reflection positive sesquilinear form determined by the measure $\mu_M^{(n)}$ from the vector space V_M of finite linear combinations of vectors of the form

$$w_T^M(\Phi_M(t_T, l_M^{t_T})) \dots w_1^M(\Phi_M(t_1, l_M^{t_1})) \quad (5.23)$$

with $t_T > t_{T-1} > \dots > t_1 > 0$ for $w_K^M \in \mathcal{W}_M$ (for coinciding points of time we can reduce the number of time steps by decomposing the products of elementary functions into linear combinations of those).

The Hilbert space $\mathcal{H}_M^{(n)}$ is then the completion of the span of equivalence classes $[\psi_M]_{\mu_M^{(n)}}$, $\psi_M \in V_M$, in particular the vacuum is $\Omega_M^{(n)} = [1]_{\mu_M^{(n)}}$. However, the abstract description in terms of equivalence classes is not very useful in practice, rather we wish to describe them concretely in terms of stochastic processes and measures $\nu_M^{(n)}$ as outlined in **Section 4.3**. As the Hilbert spaces we deal with are separable, this is always possible (see appendix B of [196]); however, that construction does not directly refer to the space-time stochastic process Φ we started from. The reason why

this happens is because of the appearing equivalence classes: To perform concrete calculations, one will work with representatives, which makes the construction non-canonical because the choice of such representatives is largely a matter of taste. In our setting, if $\mu_M^{(n)}$ is obtained by the FKWT construction from OS data, then, of course, $\Phi_M(0, \cdot) =: \phi_M(\cdot)$ is a possible choice. However, in the renormalisation step, we are to deduce the OS data at resolution M from the measure $\mu_M^{(n+1)}$ which was renormalised from $\mu_{M'}^{(n)}$, $M < M'$ via Eq. 5.22, and thus, it is not a priori clear how the stochastic process ϕ_M can be chosen, in particular it is not clear whether it can be chosen as $\phi_M(\cdot) = \Phi_{M'}(0, I_{MM'} \cdot)$ which appears to be a natural choice.

However, we are in a better situation than in the generic case because it is clear that $\mathcal{H}_M^{(n+1)}$ can be formulated in terms of the fields $\phi_M(t, l_M) = \Phi_{M'}(t, I_{MM'} l_M)$ for a minimal number of distinguished times $t \in \tau$, where the set τ is determined by the quotient process (see, e.g., Refs. 54–57). Alternatively, one can view the fields $\phi_M(t, l_M)$, $t \in \tau$ as fields at time zero $\tilde{\phi}_M(l_M)$, but in a larger space of fields, that is, a stochastic process $\tilde{\phi}_M$ with a larger index set $\tilde{B}_M = \tau \times B_M$ that still lives on the lattice labelled by M [54–57]. It follows that without further input, which will be provided below, $\phi_M(\cdot) = \Phi_{M'}(0, I_{MM'} \cdot)$ is in general a *compound field*, that is, a *collective degree of freedom* composed out of $\tilde{\phi}_M$ which together with its momentum π_M is insufficient to define the Hamiltonian $H_M^{(n+1)}$ which will generically depend on the larger set of variables $\tilde{\phi}_M$ and its conjugate momentum $\tilde{\pi}_M$. Note that this compound field is composite out of other gauge-invariant fields as an effect of renormalisation and not because of reasons of gauge invariance.

Step 2: Working out the flow of OS triples

Using the correspondence between the Wiener measures $\mu^{(n)}$ and the corresponding operator expressions, we have for $t_T > \dots > t_1$

$$\begin{aligned} & \mu_M^{(n+1)}(w_T^M(\Phi_M(t_T, l_M^T)) \cdot w_1^M(\Phi_M(t_1, l_1^M))) \\ &= \langle \Omega_M^{(n+1)}, w_T^M(\phi_M(l_M^T)) e^{-(t_T - t_{T-1})H_M^{(n+1)}} w_{T-1}^M(\phi_M(l_M^{T-1})) e^{-(t_{T-1} - t_{T-2})H_M^{(n+1)}} \dots \\ & e^{-(t_2 - t_1)H_M^{(n+1)}} w_1^M(\phi_M(l_1^M)) \Omega_M^{(n+1)} \rangle_{\mathcal{H}_M^{(n+1)}} \\ &= \mu_{M'}^{(n)}(w_T^M(\Phi_{M'}(t_T, I_{MM'} l_M^T)) \cdot w_1^M(\Phi_{M'}(t_1, I_{MM'} l_1^M))) \\ &= \langle \Omega_{M'}^{(n)}, w_T^M(\phi_{M'}(I_{MM'} l_M^T)) e^{-(t_T - t_{T-1})H_{M'}^{(n)}} w_{T-1}^M(\phi_{M'}(I_{MM'} l_M^{T-1})) e^{-(t_{T-1} - t_{T-2})H_{M'}^{(n)}} \dots \\ & e^{-(t_2 - t_1)H_{M'}^{(n)}} w_1^M(\phi_{M'}(I_{MM'} l_1^M)) \Omega_{M'}^{(n)} \rangle_{\mathcal{H}_{M'}^{(n)}} \end{aligned} \quad (5.24)$$

for all choices of $M \in \mathcal{M}$; $T \in \mathbb{N}_0$; $t_T > \dots > t_1$; $l_M^1, \dots, l_M^T$ (in practice, e.g., $M' = 2M$ is fixed).

We consider Eq. 5.24 as the *master equation* from which everything must be deduced. To avoid the compound field phenomenon mentioned above, we use that Eq. 5.24 i) is

supposed to hold for an arbitrary number of time steps and ii) we add as further input one more OS axiom, namely, *uniqueness of the vacuum* which is, in fact, a standard axiom to impose in QFT on Minkowski space [98–100]. In terms of measures, it can be stated as *ergodicity of time translations*

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T ds U(s) \Psi =_{\mu \text{ a.e.}} \mu(\Psi) \cdot 1, \quad \Psi \in L_2(S, d\mu) \quad (5.25)$$

We separate this axiom from the minimal ones because it enters in a crucial way only at this last stage of the renormalisation process. The subsequent discussion considerably extends the arguments of Refs. 54–57.

First of all, going back to Eq. 5.24 and picking $T = 1$, we find

$$\begin{aligned} & \langle \Omega_M^{(n+1)}, w_T^M(\phi_M(l_M)) \Omega_M^{(n+1)} \rangle_{\mathcal{H}_M^{(n+1)}} \\ &= \langle \Omega_{M'}^{(n+1)}, w^M(\phi_{M'}(I_{MM'} l_M)) \Omega_{M'}^{(n)} \rangle_{\mathcal{H}_{M'}^{(n)}} \end{aligned} \quad (5.26)$$

Using the fact that $w_M \in \mathcal{M}$ form a $-*$ algebra, we can formulate 5.26 as follows: Assuming by induction that up to renormalisation step n , the vectors $w^M(\phi_M(l_M)) \Omega_M^{(n)}$ and $w^M \in \mathcal{W}_M$ span a dense subspace of $\mathcal{H}_M^{(n)}$, consider the closed linear span $\widehat{\mathcal{H}_{M'}^{(n)}}$ of vectors of the form

$$w^M(\phi_{M'}(I_{MM'} l_M)) \Omega_{M'}^{(n)}, \quad (5.27)$$

which is a subspace of $\mathcal{H}_{M'}^{(n)}$. Then, Eq. 5.26 is the statement that the map

$$\begin{aligned} J_{MM'}^{(n)}: \mathcal{H}_M^{(n+1)} &\rightarrow \widehat{\mathcal{H}_{M'}^{(n)}}; w^M(\phi_M(l_M)) \Omega_M^{(n+1)} \mapsto w^M \\ &(\phi_{M'}(I_{MM'} l_M)) \Omega_{M'}^{(n)} \end{aligned} \quad (5.28)$$

is an *isometry*, that is,

$$[J_{MM'}^{(n)}]^\dagger J_{MM'}^{(n)} = 1_{\mathcal{H}_M^{(n+1)}} \quad (5.29)$$

which implies that

$$P_{MM'}^{(n)} := J_{MM'}^{(n)} [J_{MM'}^{(n)}]^\dagger: \widehat{\mathcal{H}_{M'}^{(n)}} \rightarrow \widehat{\mathcal{H}_{M'}^{(n)}} \quad (5.30)$$

is a projection.

Next for $T = 2$, $t_2 - t_1 = \beta$, we find from Eq. 5.24

$$\begin{aligned} & \langle \Omega_M^{(n+1)}, w(\phi_M(l_M)) e^{-\beta H_M^{(n+1)}} w(\phi_M(l_M)) \Omega_M^{(n+1)} \rangle_{\mathcal{H}_M^{(n+1)}} \\ &= \langle \Omega_{M'}^{(n)}, w(\phi_{M'}(I_{MM'} l_M)) e^{-\beta H_{M'}^{(n)}} w(\phi_{M'}(I_{MM'} l_M)) \Omega_{M'}^{(n)} \rangle_{\mathcal{H}_{M'}^{(n)}} \end{aligned} \quad (5.31)$$

and using again the $-*$ property of the algebra $\mathcal{W}_M$ and taking formally the first derivative of Eq. 5.31 at $\beta = 0$, we conclude

$$H_M^{(n+1)} = [J_{MM'}^{(n)}]^\dagger H_{M'}^{(n)} J_{MM'}^{(n)} \quad (5.32)$$

Note that (choose $w = 1$ in Eq. 5.28)

$$H_M^{(n+1)} \Omega_M^{(n+1)} = [J_{MM'}^{(n)}]^\dagger H_{M'}^{(n)} \Omega_{M'}^{(n)} = 0; \quad (5.33)$$

hence, the new vacuum is automatically annihilated by the new Hamiltonian.

We notice that for finite β , Eq. 5.31 is not implied by Eq. 5.32, unless $[H_{M'}^{(n)}, P_{MM'}^{(n)}] = 0$, and it is here where we use the condition

that the correspondence pt5.24 is to hold for an arbitrary number and choices of time as well as the uniqueness of the vacuum. Using the projection $P_{MM'}^{(n)}$ onto the closed linear span of the $w(\phi_{M'}(I_{MM'}l_M))\Omega_{M'}^{(n)}$, we see that the operators $w(\phi_{M'}(I_{MM'}l_M))$ on $\mathcal{H}_{M'}^{(n)}$ are block diagonal with respect to the decomposition

$$\mathcal{H}_{M'}^{(n)} = P_{MM'}^{(n)}\mathcal{H}_{M'}^{(n)} \oplus [P_{MM'}^{(n)}]^\perp \mathcal{H}_{M'}^{(n)} \quad (5.34)$$

since they together with their adjoints leave $P_{MM'}^{(n)}\mathcal{H}_{M'}^{(n)}$ invariant (the $w \in \mathcal{W}_M$ generate a $*$ -algebra). Thus, $P_{MM'}^{(n)}w(\phi_{M'}(I_{MM'}l_M)) [P_{MM'}^{(n)}]^\perp = [P_{MM'}^{(n)}]^\perp w(\phi_{M'}(I_{MM'}l_M))P_{MM'}^{(n)} = 0$, but in general, $[P_{MM'}^{(n)}]^\perp w(\phi_{M'}(I_{MM'}l_M)) [P_{MM'}^{(n)}]^\perp \neq 0$. Thus, it is not sufficient to insert w operators an arbitrary number of times and at arbitrary places into the correspondence Eq. 5.24 in order to deduce (Eq. 5.32) from Eq. 5.24.

Let $b_{M'I}^{(n)}$ be an orthonormal basis of $P_{MM'}^{(n)}\mathcal{H}_{M'}^{(n)}$. Then, since $\Omega_{M'}^{(n)}$ is cyclic for the algebra $W_{MM'}^{(n)}$ generated by the $w(\phi_{M'}(I_{MM'}l_M))$ with respect to $P_{MM'}^{(n)}\mathcal{H}_{M'}^{(n)}$, we find $w_I \in W_{MM'}^{(n)}$ such that $b_{M'I}^{(n)} = w_I \Omega_{M'}^{(n)}$ (or can be made at least arbitrarily close). Next, assume that $\Omega_{M'}^{(n)}$ is the unique ground state for $H_{M'}^{(n)}$, then

$$e^{-\beta H_{M'}^{(n)}} \rightarrow \Omega_{M'}^{(n)} \langle \Omega_{M'}^{(n)}, \cdot \rangle_{\mathcal{H}_{M'}^{(n)}} \quad (5.35)$$

becomes the projection on the ground state for $\beta \rightarrow \infty$. It follows in the limit $\beta \rightarrow \infty$

$$\begin{aligned} P_{MM'}^{(n)} &= \sum_I b_{M'I}^{(n)} \langle b_{M'I}^{(n)}, \cdot \rangle_{\mathcal{H}_{M'}^{(n)}} \\ &= \sum_I w_I' \Omega_{M'}^{(n)} \langle w_I' \Omega_{M'}^{(n)}, \cdot \rangle_{\mathcal{H}_{M'}^{(n)}} \rightarrow \sum_I w_I' e^{-\beta H_{M'}^{(n)}} (w_I')^\dagger \end{aligned} \quad (5.36)$$

Let w_I be the element in the algebra generated by the $w(\phi_{M'}(l_M))$ such that $J_{MM'}^{(n)} w_I \Omega_{M'}^{(n+1)} = w_I' \Omega_{M'}^{(n)}$ (which exists because $P_{MM'}^{(n)}\mathcal{H}_{M'}^{(n)}$ is the closure of the image of $J_{MM'}^{(n)}$). Then, due to isometry $[J_{MM'}^{(n)}]^\dagger J_{MM'}^{(n)} = 1_{\mathcal{H}_{M'}^{(n+1)}}$ (5.29), we have

$$\begin{aligned} \sum_I w_I \Omega_{M'}^{(n+1)} \langle w_I \Omega_{M'}^{(n+1)}, \cdot \rangle_{\mathcal{H}_{M'}^{(n+1)}} &= [J_{MM'}^{(n)}]^\dagger \sum_I w_I' \Omega_{M'}^{(n)} \langle w_I' \Omega_{M'}^{(n)}, \cdot \rangle_{\mathcal{H}_{M'}^{(n)}} J_{MM'}^{(n)} \\ &= [J_{MM'}^{(n)}]^\dagger P_{MM'}^{(n)} J_{MM'}^{(n)} = 1_{\mathcal{H}_{M'}^{(n+1)}} \end{aligned} \quad (5.37)$$

On the other hand, if $\Omega_{M'}^{(n+1)}$ is the unique ground state for $H_{M'}^{(n+1)}$, we have by the same argument as in Eqs 5.35 and 5.36 in the limit $\beta \rightarrow \infty$

$$1_{\mathcal{H}_{M'}^{(n+1)}} \rightarrow \sum_I w_I e^{-\beta H_{M'}^{(n+1)}} w_I^\dagger \quad (5.38)$$

Since the identity operator $1_{\mathcal{H}_{M'}^{(n+1)}}$ can be inserted an arbitrary number of times and at arbitrary places on the left hand side of Eq. 5.24 and since it can be written as (Eq. 5.38) which under the correspondence Eq. 5.24 translates into Eq. 5.36, the correspondence Eq. 5.24 is to hold also when we insert $P_{MM'}^{(n)}$

an arbitrary number of times and at arbitrary places on the right hand side of Eq. 5.24. In particular, this means that we must replace on the right hand side of Eq. 5.24 the operator $e^{-\beta H_{M'}^{(n)}}$ by

$$\lim_{N \rightarrow \infty} P_{MM'}^{(n)} \left(e^{-\frac{\beta}{N} H_{M'}^{(n)}} P_{MM'}^{(n)} \right)^N \quad (5.39)$$

To see this, we write in Eq. 5.24 for each $k = 2, \dots, T$ and for any $N \in \mathbb{N}$ on the lhs $e^{-(t_k - t_{k-1})H_{M'}^{(n+1)}} = (e^{-(t_k - t_{k-1})H_{M'}^{(n)}/(n+1)})^N$ and replace $1_{\mathcal{H}_{M'}^{(n)}}$ by the approximants (5.38) or more precisely the $P(n, k, \beta)$ of appendix E of [196] for $P = 1_{\mathcal{H}_{M'}^{(n+1)}}$. Using multi-

linearity of Eq. 5.24, we can rewrite the resulting expression in terms of Eq. 5.24 again, just that now we have not T insertions of w operators, but $T' = 2NT$ insertions at times $t'_1 < \dots < t'_T$ such that

$$t_{2kN+2l} - t'_{kN+2l-1} = \frac{t_k - t_{k-1}}{N}, \quad t'_{2kN+2l-1} - t'_{kN+2l-2} = \beta, \quad (5.40)$$

for $k = 1, \dots, T-1; l = 1, \dots, N$. By the correspondence 5.24, this translates into the corresponding expressions on the right hand side with approximants (5.36) or more precisely the $P(n, k, \beta)$ of appendix E of [196] for $P = P_{MM'}^{(n)}$. Then, one takes strong limits in the appropriate order (see appendix E of [196]), in particular $\beta \rightarrow \infty$, keeping $t_k - t_{k-1}$ fixed. As this is to hold for all N , we take $N \rightarrow \infty$.

Equation 5.39 is known in the mathematics literature [189–192] as a degenerate case of a Kato–Trotter product [188], of which there are many versions. One of them states that for contraction semi-groups generated by self-adjoint operators A, B such that $A + B$ is essentially self-adjoint on the dense domain $D(A) \cap D(B)$, we have strong convergence

$$\lim_{N \rightarrow \infty} [e^{-A/N} e^{-B/N}]^N = e^{-(A+B)} \quad (5.41)$$

In our case, the second contraction semi-group, $s \mapsto e^{-sB}$ is replaced by the degenerate one $K(s) = K(0) = P_{MM'}^{(n)}$. In [189–192], sufficient criteria for the existence of a degenerate semi-group $K(\beta)$, $K(0)$ an invariant projection, rather than the identity, are studied, such that in, say, the strong operator topology $K(\beta) = \lim_{N \rightarrow \infty} (e^{-\beta/NA} P)^N$. Assuming that existence $K(\beta)$ of the limit (5.39) is secured, we deduce

$$\begin{aligned} H_M^{(n+1)} &:= -[J_{MM'}^{(n)}]^\dagger \left[\frac{d}{d\beta} K(\beta) \right]_{\beta=0} J_{MM'}^{(n)}, \quad K(\beta): \\ &= \lim_{N \rightarrow \infty} P_{MM'}^{(n)} \left[e^{-\frac{\beta}{N} H_{M'}^{(n)}} P_{MM'}^{(n)} \right]^N \end{aligned} \quad (5.42)$$

In particular, if the solution of Eq. 5.42 is given by

$$K(\beta) = P_{MM'}^{(n)} e^{-\beta P_{MM'}^{(n)} H_{M'}^{(n)} P_{MM'}^{(n)}}, \quad (5.43)$$

we recover Eq. 5.32, since $P_{MM'}^{(n)} J_{MM'}^{(n)} = J_{MM'}^{(n)}$. In appendix D of Ref. 196, we prove Eq. 5.43 for the case that $H_{MM'}^{(n)}$ is bounded, that is, Eq. 5.43 is strictly true when replacing $H_{M'}^{(n)}$ by its bounded spectral projections $E_M^{(n)}(B)$, B Borel.

In what follows, we will assume this to hold also when $e^{-\beta H_{M'}^{(n)}}$ is a general contraction semi-group. In Refs. 189–192, we find proofs

for existence of a resulting degenerate semi-group under special circumstances, but no concrete formulae in terms of the original projection and semi-group are given. Thus, for the time being, we will use Eq. 5.32 as a plausible solution of the *exact relation* Eq. 5.42 but keep in mind that Eq. 5.42 may contain more information.

To conclude this step, under the assumption that uniqueness of the vacuum is preserved under the renormalisation flow and that the degenerate Kato–Trotter product formula applies to general contraction semi-groups, we can strictly derive Eq. 5.29 and Eq. 5.32 as equivalent to Eq. 5.24. Unfortunately, it is not possible to show that the uniqueness property is automatically preserved under the flow: Suppose that $H_{M'}^{(n)}$ has unique vacuum $\Omega_{M'}^{(n)}$ and that $H_M^{(n+1)}v_M = [J_{MM'}^{(n)}]^\dagger H_{M'}^{(n)}J_{MM'}^{(n)}v_M = 0$, then we can just conclude that $H_{M'}^{(n)}J_{MM'}^{(n)}v_M \in [P_{MM'}^{(n)}]^\perp \mathcal{H}_{M'}$. Hence, without further input, the uniqueness property must be checked self-consistently.

Step 3: Constructing the continuum theory from the fixed point data

Once we found a fixed point family $J_{MM'}$; $(\mathcal{H}_M, \Omega_M, H_M)$ with $M < M'$, $M, M' \in \mathcal{M}$, we have an inductive limit structure $(J_{MM'}, \mathcal{H}_M)$ of Hilbert spaces since $J_{M'M}J_{MM'} = J_{MM'}$ is inherited from $I_{M'M}I_{MM'} = I_{MM'}$ for $M < M' < M''$ and therefore can define the continuum Hilbert space $\mathcal{H}$ as its inductive limit which always exists [160]. Thus, there exist isometries $J_M: \mathcal{H}_M \rightarrow \mathcal{H}$ such that $J_M J_{MM'} = J_{M'}$, $M < M'$. Moreover, there exists a consistently defined quadratic form H (not necessarily an operator) such that $H_M = J_M^\dagger H J_M$. Note that we can compute matrix elements of H between the subspaces $J_M \mathcal{H}_M$, $J_{M'} \mathcal{H}_{M'}$ of $\mathcal{H}$ for any M, M' without actually knowing H , just its known finite resolution projections are needed, by using any $M, M' < M''$

$$\begin{aligned} \langle J_M \psi_M, H J_{M'} \psi_{M'} \rangle_{\mathcal{H}} &= \langle J_M J_{MM''} \psi_M, H J_{M'} J_{M'M''} \psi_{M'} \rangle_{\mathcal{H}} \\ &= \langle J_{MM''} \psi_M, H_{M'} J_{M'M''} \psi_{M'} \rangle_{\mathcal{H}_{M''}} \end{aligned} \quad (5.44)$$

We stress that H is *not* the inductive limit of H_M since that would require $H_M J_{MM'} = J_{MM'} H_{M'}$. This inductive limit condition is much stronger than the quadratic form condition $H_M = J_{MM'}^\dagger H_{M'} J_{MM'}$ which can be seen by multiplying the inductive limit condition from the left with $J_{MM'}^\dagger$ and using isometry. It is not possible to derive the inductive limit condition from the quadratic form condition because $J_{MM'}^\dagger$ has no left inverse.

We emphasise that this Hamiltonian renormalisation scheme can be seen as an *independent, real-space, kinematical* renormalisation flow different from the OS measure (or path integral) scheme even if the assumptions that were made during its derivation from the measure theoretic one are violated. Note that both schemes are *exact*, that is, make no truncation error. This is possible because we do not need to compute the spectra of the Hamiltonians (which is practically impossible to do analytically without error even at finite resolution), but only matrix elements which is computationally much easier and can often performed analytically, even if the Hilbert spaces involved are *infinite-dimensional* as is the case in bosonic QFT even at finite resolution.

As a final remark, recall that the reduction of Eqs 5.24–5.29 and Eq. 5.29 rests crucially on the assumption that the vacuum vectors $\Omega_M^{(n)}$ remain the unique ground states of the Hamiltonians $H_M^{(n)}$ in the course of the renormalisation, a condition which is difficult to keep track-off in practice and which, in fact, contains dynamical information. Is it possible that the OS measure flow and the Hamiltonian flow (Eqs 5.29 and 5.32) nevertheless deliver the same continuum theory, even if we drop the vacuum uniqueness assumption? In that respect, note that one arrives at Eqs 5.29 and (5.32 from Eq. 5.24 by deleting by hand the off-block diagonal terms in $H_{M'}^{(n)}$ with respect to the decomposition (Eq. 5.34). When deleting those terms by hand, then Eq. 5.24 indeed becomes equivalent to Eqs 5.29 and 5.32. This is reminiscent of the Raleigh–Ritz procedure of diagonalising a self-adjoint operator [188]: There the statement is that for any self-adjoint operator H bounded from below (which is precisely our situation) and any finite-dimensional projection P , $\dim(P)$ eigenvalues of PHP ordered by size are upper bounds of $\dim(P)$ eigenvalues, ordered by size, in the discrete part of the spectrum (i.e., isolated eigenvalues of finite multiplicity) of H . Here, we deal with an infinite projection, instead of a finite one, but the general setting is the same. The idea is that as we increase M , we approach the continuum Hamiltonian for which eventually there are no off-diagonal elements.

6. CONCLUSION

In this contribution, we have reviewed, extended, and clarified the proposal [54–57]. The extension consisted in i. an improved derivation of the renormalisation scheme (5.29) and (5.32) from OS reconstruction using an extended minimal set of OS axioms that also includes the uniqueness of the vacuum (which is, in fact, always assumed in QFT on Minkowski space) and ii. a much more systematic approach to the choice of coarse graining maps for a general QFT which are motivated by structures naturally provided already by the classical theory. The clarification consisted in separating off the null space quotient process imposed by OS reconstruction as an independent part of the renormalisation flow whose formulation naturally uses the language of stochastic processes.

We also had the opportunity to make several points of contact with other renormalisation programmes that are currently being further developed. For instance, the reduced density matrix approach on which entanglement renormalisation schemes rest occurs naturally in our scheme as well when looking at the flow of the vacuum and Hilbert space. Next, since we consider a real-space renormalisation scheme, when translated in terms of the flow of Wiener measures that we obtain from the flow of OS data, we are rather close to the asymptotic safety programme because our spatial lattices can, of course, be translated into momentum lattices by Fourier transformation that are used in the asymptotically safe quantum gravity programme. Finally, our proposal is obviously very close in language and methods to all other Hamiltonian renormalisation schemes, and while we currently focus

on a kinematical coarse graining scheme, our approach also contains dynamical components such as the flow of the vacuum.

In [54–57, 193], we have successfully applied our scheme to free QFT (scalar fields and Abelian gauge theories) exploiting their linear structure. Obviously, one should construct further solvable examples of interacting theories, for example, interacting 2D scalar QFT [178–180] or free Abelian gauge theories but artificially discretised in terms of non-linear holonomies in order to simulate the situation in loop quantum gravity (see [193] for further remarks).

Of course, the ultimate goal is to use Hamiltonian renormalisation to find a continuum theory for canonical quantum gravity. Here, we can use the LQG candidate as a starting point because it is rather far developed, but, of course, the flow scheme developed can be applied to any other canonical programme. However, using LQG and the concrete scheme that employs a fixed subset of graphs γ_M labelled by $M \in \mathbb{M}$ of cubical topology is, at each resolution M , precisely the algebraic quantum gravity (AQG) version of LQG [169–172]. Hence, we can already speculate on what can be expected from the renormalisation flow:

The Hamiltonian $H_M^{(0)}$ defined on the corresponding $\mathcal{H}_M^{(0)} = L_2(SU(2)^{3M^3}, d^{3M^3} \mu_H)$ (μ_H being the $SU(2)$ Haar measure) could be, but not needs to be, ordered in such a way as to annihilate the vacuum $\Omega_M^{(0)} = 1$ of a discretised volume operator $V_M^{(0)}$ as it is standard in current regularisations of the Hamiltonian constraint. In fact, it may be desirable to choose the vacuum of $H_M^{(0)}$ not to coincide with that of $V_M^{(0)}$ in order to imprint its algebraic structure. The operator $H_M^{(0)}$ preserves $\mathcal{H}_M^{(0)}$ but not each subspace defined by sub-lattices of γ_M and is thus not super-local in contrast to the definition [28–32]; for instance, it will use volume operators local to a vertex and holonomies along plaquettes incident at that vertex (next neighbour interaction). When running the renormalisation scheme, next-to-next neighbour interactions will be switched on (this is exactly what happens in the examples [54–57, 193]), and upon reaching the fixed point, the Hamiltonian H_M will involve all possible interactions *with precise coefficients* and thus be spatially non-local but hopefully quasi-local (i.e., the interactions die off exponentially with the distance between vertices defined by the 3D taxi driver metric on the graph (each edge counting one unit)). Note that this quasi-locality at finite resolution can be straightforwardly computed in the examples [54–57, 193] by using the spatially local continuum Hamiltonian and projecting it with $J_M, J_M^\dagger$ (blocking from the continuum) and is thus *physically correct*. In other words, spatial locality in the continuum is not in conflict with spatial non-locality at finite resolution. In fact, we even *expect* a high degree of spatial non-locality for very small M for which the naive dequantisation of $H_M^{(0)}$ at any phase space point p will be far off the classical value $H(p)$ which matches with the remarks made at the end of **Section 3**.

Several questions arise from this picture should the flow display any fixed points: First, for compact σ and if indeed we use a countable set of lattices γ_M as above, the resulting inductive limit Hilbert space could be separable (since there is a countable basis defined by vectors at finite resolution), thus would not be the standard LQG

representation space $\mathcal{H}_{\text{LQG}} = L_2(\bar{\mathcal{A}}, d\mu_{\text{AL}})$ of square integrable functions with respect to the Ashtekar–Lewandowski measure μ_{AL} on a space $\bar{\mathcal{A}}$ of distributional connections². In view of the uniqueness theorem [23–27], one of its assumptions will then be violated. The most likely possibility is that the corresponding vacuum expectation value functional is not spatially diffeomorphism-invariant since the diffeomorphism symmetry was explicitly broken in the renormalisation process. If the continuum Hamiltonian is still spatially diffeomorphism-invariant, we would be in the situation of spontaneous symmetry breakdown and could view this as a phase transition from the symmetric $\mathcal{H}_{\text{LQG}}$ phase to this broken phase. Note that in our gauge-fixed situation, the diffeomorphism group is considered as a continuous symmetry group and not as a gauge group.

Next, precisely due to this separability, the resulting theory may not suffer from the discontinuity of holonomy operators which otherwise gives rise to what has been called the ‘staircase problem’ in the literature [194]: The cubical graphs γ_M contain paths only along the coordinate axes. Since all $M \in \mathcal{M}$ are allowed, these paths separate the points of the classical configuration space but not of the distributional space $\bar{\mathcal{A}}$. In particular, any path that is not a ‘staircase’ path cannot be accommodated at any finite resolution. Yet, the continuum Hamiltonian in the example [193] does not care about the fact that it was defined as a fixed point of a flow of its finite resolution projections of cubical lattices only; it also knows how to act on states which are excited on non-‘staircase’ paths. The reason for why this happens is as follows: Consider any path c and some staircase approximant $\tilde{c}$ with the same end points as c which has zero winding number with respect to c so that $c \circ \tilde{c}^{-1} = \partial S$ bounds a surface. Then, for an Abelian connection A , we have $\int_V A - \int_{\tilde{c}} A = \int_S dA$, and in the classical theory, the surface integral converges to zero. In the quantum theory, a similar calculation can be made because the Hilbert space measure is supported on a different kind of distributional connections than $\bar{\mathcal{A}}$.

Finally, although the scheme, strictly speaking, was derived for theories with gauge-fixed space-time diffeomorphism constraints and a true physical Hamiltonian bounded from below, we may, of course, ‘abuse’ it and also consider constraint operators $C(f)$ as Hamiltonians, define their finite resolution expressions $C(f)_M^{(0)}$, and let them flow (here, f is a test function on the spatial manifold σ)³. This will involve as a new ingredient also a discretisation of the smearing function f which could be done using the maps I_M, K_M for scalar fields (see Ref. 193). Suppose then that for all f fixed point families, $\{C(f)_M\}_{M \in \mathbb{N}_0}$ can be obtained. Should we expect that the $C(f)_M$ represent a finite resolution version of the classical continuum constraint (hypersurface deformation) algebra $\{C(f), C(g)\} = C(h(f, g))$, where $h(f, g)$ is another (in general, phase space-dependent) smearing function? The answer is in the negative! Namely, what we want is that the continuum

²In the non-compact case, one may need to take the infinite tensor product extension [69] which is also non-separable but in a different sense, and there one regains separability by passing to irreducible representations of the observable algebra.

³In fact, the physical Hamiltonian of **Section 3** is not manifestly bounded from below, hence we to abuse the formalism in the sense that we assumed the semi-boundedness.

operators obey $[C(f), C(g)] = iC(h(f, g))$ (with appropriate orderings of $C, h(f, g)$ in place). But if $C(f)_M = J_M^\dagger C(f) J_M$, then

$$\begin{aligned} [C(f)_M, C(g)_M] &= J_M^\dagger [C(f) P_M C(g) - C(g) P_M C(f)] J_M \\ &= C(h(f, g))_M + J_M^\dagger [C(f) [P_M, C(g)]] J_M \end{aligned} \quad (6.1)$$

Thus, even if the continuum algebra closes, one *does not see this at any finite resolution*, unless $[C(f), P_M] = 0$ for all f, M . This will generically not hold because not even $C(f)_{M'}$ preserves $J_{MM'} \mathcal{H}_M$, $M < M'$, unless $J_{MM'} C(f)_M = C(f)_{M'} J_{MM'}$, that is, $C(f)$ forms an inductive family which is not expected. Of course, the correction term in Eq. 6.1 is expected to become ‘small’ in the limit $M \rightarrow \infty$ in which $P_M \rightarrow 1_{\mathcal{H}}$, and thus, an appropriate criterion for closure of the continuum algebra using only finite resolution projections can be formulated (see Refs. 54–57 for the simpler case of rotational invariance). Note that the quantisation performed for spatially diffeomorphism-invariant Hamiltonian operators on the Hilbert space $\mathcal{H}_{\text{LQG}}$ displayed in Section 3 was *forced* to have the unphysical property $[H, P_M] = 0$ (see the statement just before 3.24). But the underlying theorem exploits in a crucial way the non-separability of $\mathcal{H}_{\text{LQG}}$, and thus fortunately does not hold on separable Hilbert spaces.

Before closing, note that even if this approach of taking the UV limit can be completed and unless the manifold σ is compact, we still must take the thermodynamic or infrared limit and remove the IR cut-off R (compactification scale). As is well-known from statistical quantum field theory [160], interesting phenomena related to phase transitions can happen here. Moreover, constructible examples of low dimensional

interacting QFT show that the thermodynamic limit requires techniques that go beyond what was displayed here [178–180]. However, we consider this momentarily as a ‘higher order’ problem and reserve it for future research.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: <https://www.frontiersin.org/articles/10.3389/fphy.2020.548232/full#supplementary-material>.

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Hamiltonian Renormalization V: Free Vector Bosons

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In a recent proposal we applied methods from constructive QFT to derive a Hamiltonian Renormalization Group in order to employ it ultimately for canonical quantum gravity. The proposal was successfully tested for free scalar fields and thus a natural next step is to test it for free gauge theories. This can be done in the framework of reduced phase space quantization which allows using techniques developed earlier for scalar field theories. In addition, in canonical quantum gravity one works in representations that support holonomy operators which are ill defined in the Fock representation of say Maxwell or Proca theory. Thus, we consider toy models that have both features, i.e. which employ Fock representations in which holonomy operators are well-defined. We adapt the coarse graining maps considered for scalar fields to those theories for free vector bosons. It turns out that the corresponding fixed pointed theories can be found analytically.

Keywords: renormalization group, canonical quantum gravity, Hamiltonian renormalization, constructive QFT, lattice gauge field theory

1. INTRODUCTION

The construction of interacting four-dimensional Quantum Field Theories (QFTs) is an interesting and fundamentally important problem in modern physics. Despite several attempts it has not been satisfactorily completed as of today (Wightman and Gårding, 1964; Osterwalder and Schrader, 1973; Osterwalder and Schrader, 1975; Glimm and Jaffe, 1987; Froehlich, 1978; Rivasseau, 2000; Jaffe and Witten, 2000). Due to several challenges along the way, preliminary computations are often done in the presence of finite infrared and ultraviolet cut-offs, most prominently in the framework of Lattice Gauge Theories (LGT) (Creutz, 1983; Hashimoto et al., 2017). Especially considering approaches toward Quantum Gravity, it motivated proposals where the discretization of space(-time) was assumed to be fundamental (Loll, 1998; Giesel and Thiemann, 2007; Bahr and Dittrich, 2009; Bahr and Dittrich, 2009; Dupuis et al., 2012; Loll, 2019). This allowed to make a wide range of predictions by performing computations using established tools from LGT, see for example (Kogut and Susskind, 1975; Bahr et al., 2017; Dapor and Liegener, 2018; Glaser and Steinhaus, 2019; Han and Liu, 2020).

However, as it is not yet experimentally supported whether these discrete structures are fundamental, one can independently ask if they can be understood as coarse graining of some underlying continuum QFT and—of course—the construction of such a QFT is in itself an aspirational goal. A possible avenue for this comes in the form of *inductive limits* (Kadison and Ringrose, 1986; Janas, 1988; Saunders, 1998; Thiemann, 2007). This presents a construction by which a QFT described by a Hilbert space $\mathcal{H}$ supporting a Hamiltonian operator $\hat{H}$ can in principle be obtained from a consistent family of discretized theories described by a Hilbert space $\mathcal{H}_M$ supporting a Hamiltonian $\hat{H}_M$ where M labels the different discretization scales. The necessary condition for the existence of such an inductive limit is that there exists a family of isometric injection maps $J_{M \rightarrow M'} : \mathcal{H}_M \rightarrow \mathcal{H}_{M'}$ for $M < M'$ in the sense of M' describing finer resolution than M . $J_{M \rightarrow M'}$ must be subject

to a certain *compatibility condition* in order to enable the reconstruction of the inductive limit Hilbert space $\mathcal{H}$ and to allow an interpretation of the $\mathcal{H}_M$ as restrictions of $\mathcal{H}$ to coarse resolution M . Similarly, there exists a condition for a family of quadratic forms $\hat{H}_M$ which guarantees the existence of a corresponding limit quadratic form $\hat{H}$ on $\mathcal{H}$.

In a recent series of paper (Lang et al., 2018a; Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) we introduced a Hamiltonian formulation of the renormalization group which is rather close in methodology to density matrix renormalization (Brothier and Stottmeister, 2019; Brothier, 2019; Stottmeister et al., 2020) and projective renormalization (Okolow, 2013; Kijowski and Okolow, 2017; Lanéry and Thiemann, 2017a; Lanéry and Thiemann, 2017b; Lanéry and Thiemann, 2018; Lanéry, 2018; Lanéry, 2016; Yamasaki, 1985) which in turn are based on the seminal ideas of Wilson, Kadanov and Fisher (Fisher, 1974; Wilson, 1975; Kadanoff, 1977). The proposal is motivated by formulations of the renormalization group in the covariant setting (Fisher, 1974; Wilson, 1975; Kadanoff, 1977; Wilson and Kogut, 1974; Peter, 1998) which can be reformulated in Hamiltonian terms using Osterwalder-Schrader reconstruction and in fact gives rise to a natural flow of inductive structures and Hamiltonian quadratic forms (Lang et al., 2018a; Lang et al., 2018b). That the direct Hamiltonian Renormalization Group delivers the correct results has been demonstrated for the case of the massive, free scalar field in arbitrary dimensions (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d). The next challenge for this program is its extension to gauge theories, as the most interesting models of modern physics are phrased in this language, e.g. QCD. In this paper, we perform the first steps in this direction by considering a toy model which is a certain deformation of the reduced Hamiltonian of Maxwell theory. The deformation consists in adding a Proca like mass term to higher powers of the Laplacian in order that the Fock space defined by that Hamiltonian supports holonomy operators, which are exponentials of the connection smeared along one-dimensional curves. The motivation for considering such theories comes from an approach to canonical quantum gravity (Thiemann, 2007; Rovelli, 2004) for which holonomies play a fundamental role and are promoted to well defined operators upon quantisation.

A possible way to proceed is as follows: prior to quantization one can transcend to the reduced phase space, where the Gauss constraints have been implemented. Since the gauge-invariant (transversal) modes can be treated as scalars, the tools from (Lang et al., 2018a; Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) become applicable. With them, it is possible to analytically determine the fixed points which lead to the correct continuum theory.

Another approach is to implement the Gauss constraint after quantization. This involves adapting the coarse graining maps for scalar fields to vector bosons. In particular, this involves smearing the field against form factors rather than scalar smearing functions. In this paper we will incorporate the latter feature by considering a modification of Proca theory that allows for holonomy operators. The actual solution of the Gauss constraint after quantization combined with coarse graining will be subject

of a subsequent paper (Liegner and Thiemann, 2020). We will introduce the necessary coarse-graining maps for this procedure and present explicitly how fixed points can be computed in the new setting.

The architecture of the article is as follows:

In **Section 2** we follow the route of reduced phase space quantization. The first **Subsection 2.1** reviews the framework of our version of the Hamiltonian Renormalization Group for scalar fields to familiarize with the notation of this paper and to enable comparison with (Lang et al., 2018a). We start by first looking at “classical” discretisations and define injection and evaluation maps between theories of different resolution. These discretisations are built, e.g., with respect to cuboidal tessellations of our spatial manifold. The second **Subsection 2.2** introduces a $U(1)$ toy model with Gauss constraint $G(x)$, which is highly inspired by free Maxwell electrodynamics. As an alternative to implementing the Gauss constraint classically, one may introduce a Master Constraint of the form $\mathbf{M} = (1/2) \int d^3x K(x, x_i) G(x) G(x_i)$ and promote it to an operator on the Fock space with some positive kernel $K(x, x_i)$ analogously to (Dittrich and Thiemann, 2006). Determining the physical Hilbert space will reduce to the space of transversal modes. This is equivalent to first fixing the gauge on the classical level and then performing a reduced phase space quantization of the transversal modes. As both methods lead to the same result, we will employ here the latter strategy. In the third **Subsection 2.3** we briefly recall how the tools from (Lang et al., 2018a; Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) find application and lead to the correct fixed points.

In **Section 3** we go further into the direction of LGT: we are interested in the connection integrated along edges of the discretizing lattices. To bring this formulation close to (Lang et al., 2018a), in **Subsection 3.1** we define the discretized fields as the continuum fields smeared against (distributional) form factors. For refinement, we pick the factor 2 (i.e. $M \rightarrow 2M$) simply for illustrative purposes. Extensions to any other factor appear to be possible, and we will assume that their fixed points are independent of the refinement choice (see the discussion in (Lang et al., 2018c)). Similar to (Lang et al., 2018a; Lang et al., 2018b), from studying the discretized theories we deduce of how to define the discretized Hilbert spaces for the quantum theory and the coarse graining maps $J_{M \rightarrow M'}$ between Hilbert spaces of different resolution in the second **Subsection 3.2**. As the two introduced coarse graining maps—called deleting kernel and filling kernel—are fundamentally different, it is a priori not clear how the renormalization group behaves with respect to both of them and whether both produce physically viable fixed points. To investigate this, we test both of them in **Section 3.3**, where we study a gauge-variant version of the toy model from the previous section—hence not relying on a reduced phase space quantization. This model features a Proca like mass term and higher powers of the Laplacian in order that holonomy operators be well-defined in the Fock space defined by that Hamiltonian. Hence, it gives first insights into theories allowing for holonomies and their renormalization. The fixed points can be found analytically after one adapts the coarse graining maps and chooses a suitable discretization: While in the Fock

representation induced by the continuum Hamiltonian holonomy operators do exist, as a first step we do not express the lattice approximants of the Hamiltonian in terms of lattice holonomies in order to simplify the analysis. In future work (Liegner and Thiemann, 2020), in order to test the representation that is used in Loop Quantum Gravity, we aim at expressing the lattice Hamiltonians in terms of holonomies as well which makes the problem substantially more complicated as then the theory will be self-interacting.

In **Section 4** we summarize our findings and conclude with outlook for further research.

2. REDUCED PHASE SPACE QUANTIZATION FOR ABELIAN GAUGE THEORIES

We present a possible strategy to extend the framework of direct Hamiltonian renormalization developed in (Lang et al., 2018a; Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) to Abelian gauge theories via reduced phase space quantization. For this purpose, **Subsection 2.1** gives a short review of the framework as it was used for scalar fields. The second subsection motivates a toy model in order to test the Hamiltonian renormalization. To keep this preliminary study simple, we choose the Abelian gauge group $U(1)$ and define the classical, continuum Hamiltonian in **Subsection 2.2** such that it resembles free Maxwell electrodynamics.¹ The actual computation of the renormalization group flow is completely analogous to (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) and we will outline the general strategy in **Subsection 2.3**.

2.1. Review: Classical Discretisations of Scalar Fields

We consider an infinite dimensional, conservative Hamiltonian system defined on a globally hyperbolic spacetime of the form $\mathbb{R} \times \sigma$. If the spatial manifold σ is not compact we introduce an infrared (IR) cut-off R by restricting to smearing (i.e. test) functions which are defined on a compact submanifold, e.g. a torus $\sigma_R := [0, R]^D$ if $\sigma = \mathbb{R}^D$. We will assume this cut-off R to be implicit in all formulae below, but do not display it to keep them simple.

The dynamical variables of the system are the scalar field $\phi \in C^\infty(\sigma)$ and its canonical conjugated momentum π_ϕ , i.e.

$\{\pi_\phi(y), \phi(x)\} = \delta^D(x, y)$. We define their smearing against test functions $f \in L(\sigma)$, i.e. functions from σ to $\mathbb{R}$ whose properties we leave unspecified for the moment:

$$\begin{aligned}\phi[f] &:= \int_\sigma d^D x \phi(x) f(x), \\ \pi_\phi[f] &:= \int_\sigma d^D x \pi_\phi(x) f(x).\end{aligned}\quad (2.1)$$

Moreover, an ultraviolet cut-off M is introduced in the form of some cell complex $C_M = \{c(m)\}_m$. The elements of the cell complex are regions $c(m) \subset \sigma$ such that $c(m) \cap c(m') = \emptyset$ and $\cup_m c(m) = \sigma$ and there are only finitely many elements, i.e. $|C_M| < \infty$. Knowledge of the $c(m)$ can be translated into knowledge of the indicator (or characteristic) functions $\chi_M(m) : \sigma \rightarrow \{0, 1\}$ which are defined as

$$\chi_M(m)(x) := \begin{cases} 1, & \text{if } x \in c(m) \in C_M, \\ 0, & \text{else.} \end{cases} \quad (2.2)$$

Once a cell complex C_M is chosen, one can introduce discretisations of the scalar field by restricting the observables (with respect to which the field is probed) to finite spatial resolution given by C_M via the following choice of *evaluation* map:

$$E_M : L(\sigma) \rightarrow L_M, \quad (2.3)$$

$$f \mapsto f_M(m) := (E_M f)(m) = \epsilon_M^{-D} \int_\sigma d^D x f(x) \chi_M(m)(x),$$

with L_M being the set of finite sequences with $|C_M|$ many elements and $\epsilon_M^D = \int d^D x \chi_M(m)(x)$ which we assume to be independent of m in the following. On the other hand, given a $f_M : \mathbb{Z}_{|C_M|} := \{0, 1, \dots, |C_M| - 1\} \rightarrow \mathbb{R}$ we can embed it into the continuum via an *injection* map:

$$I_M : L_M \rightarrow L(\sigma), \quad (2.4)$$

$$f_M \mapsto (I_M f_M)(x) := \sum_m f_M(m) \chi_M(m)(x) =: f_M([x]_{\chi_M}). \quad (2.5)$$

We have introduced the map $[x]_{\chi_M} := m$ such that $\chi_M(m)(x) = 1$, which is always well-defined due to the properties of C_M . Defined in this way, E_M serves as the left inverse of I_M :

$$E_M \circ I_M = \text{id}_{L_M}. \quad (2.6)$$

Turning toward comparing discretisations of different resolutions with each other, we are mostly interested in families of cell complexes $\{C_M\}_M$ such that they define a partially ordered and directed set. This can happen, e.g., with defining $M < M'$ iff $\forall c'(m') \in C_{M'}$ there is $c(m) \in C_M$ such that $c'(m') \subset c(m)$.² It corresponds to viewing a function defined on coarse resolution as a function of finer resolution. Moreover, we restrict to finite partitions, meaning in particular that the number of cells $c'(m')$ contained in any $c(m)$ is finite: $N_{M',M}(m) < \infty$. (In (Lang et al., 2018b; Lang

¹In the previous work (Bahr et al., 2011) in addition to free scalar fields also free gauge theories such as Maxwell theory and linearized gravity were renormalized. While there are some similarities, the difference to the scalar field treatment of (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) and the present work is as follows: First, while (Bahr et al., 2011) is concerned with the renormalization of actions, we are concerned with renormalization of vacua, Fock representations and Hamiltonians. Next, (Bahr et al., 2011) provides explicit formulae for $1 + 1$ dimensions while we treat $1 + D$ dimensions for any D . Finally, (Bahr et al., 2011) adapts the coarse graining map to the gauge symmetry while we perform a manifestly gauge invariant reduced phase space quantization. With respect to the latter issue, see also (Liegner and Thiemann, 2020).

²By demanding that it is a proper subset, we guarantee that there are multiple elements in $C_{M'}$ forming a partition of $c(m)$.

et al., 2018c; Lang et al., 2018d) it was $N_{M,M}(m) = 2^D$ for all m but this simplification is merely to ease computations.). For the purpose of comparing different discretisations with each other, one introduces a map between the discretisations with respect to two cell complexes C_M and $C_{M'}$ called *coarse graining* $I_{M \rightarrow M'} : L_M \rightarrow L_{M'}$ if $M < M'$. The coarse graining map is a free choice of the renormalization group (RG) process whose flow it drives, and its viability can be tested only a posteriori. In (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) the main focus rested on choosing the concatenation of evaluation and injection for different discretisations as coarse graining:

$$I_{M \rightarrow M'} = E_{M'} \circ I_M. \quad (2.7)$$

However, let us mention that already in (Lang et al., 2018c) also a second choice, called deleting kernel, was investigated: Let $M < M'$ and choose for any $m \in M$ a representative $m'_o(m) \in M'$ where $c'(m'_o(m)) \subset c(m)$. Also, let r be a mapping such that $r_{m,m'} = \delta_{m,m'_o(m)}$, i.e. selecting for $m' \in M'$ the representative $m'_o(m)$ of the coarse cell $c(m)$. Then

$$(I_{M \rightarrow M'}^{\text{Del}} f_M)(m') = \sum_{m \in M} N_{M',M}(m) f_M(m) r_{m,m'}. \quad (2.8)$$

In the quantum theory of free scalar fields both maps could be used to build injections that led to physically viable fixed point theories. However, it was only choice (2.7) which turned out to be *cylindrically consistent*, i.e.

$$I_{M'} \circ I_{M \rightarrow M'} = I_M. \quad (2.9)$$

Basically, this means that injection into the continuum can be done independently of the discretization on which we consider the function to be defined, which is a physical plausible assumption.

We finish this section by presenting two examples for possible choices of cell complexes C_M in case of the torus $\sigma_R = [0, R]^D$:

(i) Discretization using regular cubes. The first example is the choice employed in (Lang et al., 2018a; Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) which introduced a cubic lattice of $M \in \mathbb{N}$ points in each direction and with spacing $\varepsilon_M = R/M$. Then, the characteristic functions of C_M take the following form:

$${}^0\chi_M(m)(x) = \prod_{k=1}^D \bar{\chi}_M[m_k](x_k) \text{ with} \\ \bar{\chi}_M[m_k](x_k) = \begin{cases} 1, & \text{if } x_k \in [\varepsilon_M m_k, \varepsilon_M(m_k + 1)), \\ 0, & \text{else.} \end{cases} \quad (2.10)$$

However, this is by far not the only possibility. In order to demonstrate that nothing is special about the choice of tessellation of σ , we will use in **Section 3** the following cell complexes:

(ii) Discretization using parallelepipeds. We consider D -dimensional tessellations of the following form: at least one axis of the parallelepiped is aligned with one of the coordinate axes and a second axis of the parallelepiped connects diametral corners of an elementary hypercube. Then the remaining axes are either aligned with the coordinate axes or explore all possibilities to connect diametral corners of lower dimensional hypercubes. This yields two possibilities in $D = 2$ and nine possibilities in $D = 3$. We can formalize this as follows: Let $i, j = 1, \dots, D$ and

$${}^{ij}\chi_M(m)(x) := \int d^D y \delta(y_i - \varepsilon_M m_i, \sum_{k \neq i} (y_k - \varepsilon_M m_k)) \left(\prod_{k \neq i} \bar{\chi}_M[m_k](y_k) \right) \\ \times \bar{\chi}_M[m_j](x_j - \sum_{k \neq j} (x_k - \varepsilon_M m_k)) \left(\prod_{k \neq j} \delta(y_k, x_k) \right). \quad (2.11)$$

In $D = 2$ the explicit form of the two possible parallelograms reads:

$${}^{1,1}\chi_M(m)(x) = {}^{2,1}\chi_M(m)(x) \\ = \bar{\chi}_M[m_2](x_2) \bar{\chi}_M[m_1 - m_2](x_1 - x_2), \quad (2.12)$$

$${}^{1,2}\chi_M(m)(x) = {}^{2,2}\chi_M(m)(x) \\ = \bar{\chi}_M[m_1](x_1) \bar{\chi}_M[m_2 - m_1](x_2 - x_1). \quad (2.13)$$

In $D = 3$ the fundamental cells take the form of parallelepipeds. While nine different cases exist, we display only the explicit expressions for $i = 1$:

$${}^{1,1}\chi_M(m)(x) = \bar{\chi}_M[m_1 - m_2 - m_3](x_1 - x_2 \\ - x_3) \bar{\chi}_M[m_2](x_2) \bar{\chi}_M[m_3](x_3), \quad (2.14)$$

$${}^{1,2}\chi_M(m)(x) = \bar{\chi}_M[m_2 - m_1 - m_3](x_2 - x_1 - x_3) \bar{\chi}_M[m_1 - m_3] \\ \times (x_1 - x_3) \bar{\chi}_M[m_3](x_3), \quad (2.15)$$

$${}^{1,3}\chi_M(m)(x) = \bar{\chi}_M[m_3 - m_1 - m_2](x_3 - x_1 - x_2) \bar{\chi}_M[m_1 - m_2] \\ \times (x_1 - x_2) \bar{\chi}_M[m_2](x_2), \quad (2.16)$$

and for $i = 2, 3$ similar functions with permutations of the indices are found.

2.2. Phase Space Reduction of a Continuum Toy Model

This subsection motivates and introduces a classical Hamiltonian system subject to the Gauss constraint for Abelian gauge group $U(1)$ in $D = 3$ on a compact torus $\sigma = [0, 1]^3$. The field content will be a $U(1)$ -connection A_a and the corresponding electric vector field P^a . Due to $U(1)$ being 1-dimensional, there is only one constraint per point, which reads:

$$G(x) = (\partial_a P^a)(x). \quad (2.17)$$

The most prominent example of a $U(1)$ gauge theory is free Maxwell electrodynamics:

$$H := \frac{\epsilon_0}{2} \int d^3x \left(P_a(x) P^a(x) + c^2 A_a^\perp(x) \delta^{ab} (\omega^2 A^\perp)_b(x) \right), \quad (2.18)$$

with A split into transversal and longitudinal part respectively:

$$\begin{aligned} A_a^\perp(x) &= A_a(x) - A_a^\parallel(x), \\ A_a^\parallel(x) &= \left(\partial_a \frac{1}{\Delta} \partial^b A_b \right)(x). \end{aligned} \quad (2.19)$$

Further, ϵ_0 is the electric constant of units $[J/(mV^2)]$, but in the following we set $c = \epsilon_0 = 1$. In Maxwell electrodynamics it is $\omega^2 = -\Delta$ with Δ being the Laplacian. We modify (2.18) by replacing Δ with

$$\omega^2 = \frac{1}{p^{2(n-1)}} (-\Delta + p^2)^n, \quad (2.20)$$

with some Proca like mass term $p > 0$ and $n > 0$. This is merely a generalization as standard Maxwell theory can be reobtained in the limit $p \rightarrow 0$ and $n = 1$.

Our goal is to go to the reduced phase space and therefore we also split the electric field P^a into $P_\perp^a$ and $P_\parallel^a$ defined similar to (2.19):

$$\begin{aligned} P_\perp^a(x) &= P^a(x) - P_\parallel^a(x), \\ P_\parallel^a(x) &= \left(\partial^a \frac{1}{\Delta} \partial_b P^b \right)(x). \end{aligned} \quad (2.21)$$

Due to the fact that the transverse modes are gauge-invariant, i.e. $\{G[\Lambda], A_a^\perp(x)\} = \{G[\Lambda], P^a(x)\} = 0$ for all $\Lambda \in L(\sigma)$, it follows that the Hamiltonian (2.18) is gauge-invariant, too.

The unreduced phase space is equipped with Poisson brackets $\{P^a(x), A_b(y)\} = \delta_b^a \delta^{(3)}(x, y)$. As is standard, we perform a canonical transformation to:

$$\begin{aligned} \{P_\parallel^a(x), A_b^\parallel(y)\} &= \frac{1}{\Delta} \partial^a \partial_b \delta^{(3)}(x, y), \\ \{P_\perp^a(x), A_b^\perp(y)\} &= \left(\delta_b^a - \frac{\partial^a \partial_b}{\Delta} \right) \delta^{(3)}(x, y), \\ \{P_\perp^a(x), A_b^\parallel(y)\} &= \{P_\parallel^a(x), A_b^\perp(y)\} = 0. \end{aligned} \quad (2.22)$$

Next, we reduce to the subspace $A^\parallel = P_\parallel = 0$ and go into Fourier space

$$\hat{P}_\perp^a(k) := \int d^3x e^{ik \cdot x} P_\perp^a(x), \quad \hat{A}_a^\perp(k) := \int d^3x e^{ik \cdot x} A_a^\perp(x), \quad (2.23)$$

which can be decomposed as

$$\begin{aligned} \hat{P}_\perp^a(k) &= \epsilon_1^a(k) \hat{P}_1(k) + \epsilon_2^a(k) \hat{P}_2(k), \\ \hat{A}_\perp^a(k) &= \epsilon_1^a(k) \hat{A}_1(k) + \epsilon_2^a(k) \hat{A}_2(k), \end{aligned} \quad (2.24)$$

with a choice of vector fields $\epsilon_1(k), \epsilon_2(k)$ which are orthonormal to each other and orthogonal to k^a . Such a choice can always be made and implies that the symplectic structure between $\hat{P}, \hat{A}$ is of canonical form, i.e. for $I, J \in \{1, 2\}$

$$\{\hat{P}_I(k), \hat{A}_J(k')\} = \delta_{IJ} \delta^{(3)}(k, k'). \quad (2.25)$$

On this subspace the Gauss constraint is trivially solved, and all gauge-degrees of freedom have been removed. Expressed in

these variables the continuum Hamiltonian of our model takes the form:

$$H = \int d^3k \sum_{I=1,2} \left(|\hat{P}_I(k)|^2 + \omega(k)^2 |\hat{A}_I(k)|^2 \right). \quad (2.26)$$

2.3. Scalar Field Renormalization With Multiple Field Species

In this subsection we discretize the model (2.26) with ω from (2.20) with the scalar field techniques introduced in (Lang et al., 2018a). Due to the form of the Hamiltonian we are close to the analysis in (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) to which we refer the reader for all details. Indeed, we can understand the Hamiltonian as two decoupled field species (P_I, A_I) labeled by $I = 1, 2$, where we use the Fourier inversion:

$$\begin{aligned} P_I(x) &:= \frac{1}{2\pi} \int d^3k e^{-ik \cdot x} \hat{P}_I(k), \\ A_I(x) &:= \frac{1}{2\pi} \int d^3k e^{-ik \cdot x} \hat{A}_I(k). \end{aligned} \quad (2.27)$$

We introduce a family of discretisations of the spatial manifold σ in terms of cubic cell complexes as described in the previous subsection such that $N_{M,2M}(m) = 2^D$ for all $m \in \mathbb{Z}_M^3 = \{0, 1, \dots, M-1\}^3$. With the evaluation maps E_M from (2.3) we discretize both field species:

$$\begin{aligned} P_{M,I}(m) &:= P_I(m \epsilon_M), \\ A_{M,I}(m) &:= E_M^3(E_M A_I)(m). \end{aligned} \quad (2.28)$$

We must also introduce a discretization of ω which is supposed to map from $L_M \rightarrow L_M$. Since we have two field species $I = 1, 2$ it could turn out that each supports its own covariance. To take this possibility into account, we will keep the discretisations $\omega_{M,I}$ dependent on the field species I in the following. However, as initial discretization we take them to be equal, that is:

$$\omega_{M,1} = \omega_{M,2} = \omega_M^{(0)} \equiv \frac{1}{p^{2(n-1)}} (-\Delta_M + p^2)^n, \quad (2.29)$$

with Δ_M some initial discretization such that $\lim_{M \rightarrow \infty} \omega_M^{(0)} = \omega$.

Since the Hamiltonian is essentially of free harmonic oscillator form for each I , it motivates to introduce the discrete annihilation and creation fields:

$$\mathbf{a}_{M,I}(m) := \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\omega_{M,I}/\epsilon_M^3} A_{M,I}(m) - i \sqrt{\epsilon_M^3/\omega_{M,I}} P_{M,I}(m) \right), \quad (2.30)$$

such that

$$H_M = \hbar \sum_{I=1,2} \sum_{m \in \mathbb{Z}_M^3} \overline{\mathbf{a}_{M,I}(m)} (\omega_{M,I} \mathbf{a}_{M,I})(m). \quad (2.31)$$

For any resolution M we define the corresponding Hilbert spaces $\mathcal{H}_{M,I}$ for specie I with Fock vacuum $\Omega_M^{(0)} \in \mathcal{H}_M := \otimes_I \mathcal{H}_{M,I}$ annihilated by the operators corresponding to (2.30), i.e.

$$\widehat{\mathbf{a}}_M[f_M] := \sum_I \sum_{m \in \mathbb{Z}_M^3} \widehat{\mathbf{a}}_{M,I}(m) f_{M,I}(m), \quad (2.32)$$

(with $f_{M,I}(m) \in L_M^2 := \ell_2(\mathbb{Z}_M^3)^2$). Thus, Ω_M is simultaneously annihilated by the quantization of (2.31). Denoting by $\langle \cdot, \cdot \rangle_{\mathcal{H}_M}$ the scalar product on $\mathcal{H}_M$ it follows

$$\langle \Omega_M, e^{i\widehat{\mathbf{a}}_M[f_M]} \Omega_M \rangle_{\mathcal{H}_M} = e^{-\frac{i}{2} \langle f_M, \omega_{M,1}^{-1} f_M \rangle} e^{-\frac{i}{2} \langle f_M, \omega_{M,2}^{-1} f_M \rangle} \quad (2.33)$$

Each $\mathcal{H}_M$ can be represented as Hilbert space $L_2(\mathbb{R}^{2M^3}, d\nu_M)$ where $\nu_M = \nu_{M,1} \nu_{M,2}$ is a Gaussian measure with covariance $c_M = \text{diag}(c_{M,1}, c_{M,2})$ and $c_{M,I} = (\hbar/2)\omega_{M,I}^{-1}$. Hence, we have at our disposal an initial family of Osterwalder-Schrader data $(\mathcal{H}_M, \widehat{H}_M, \nu_M)$ which under a renormalization step, does not change its general structure (Lang et al., 2018b) but leads to a new family of (Gaussian) covariances, i.e. $\{c_{M,I}^{(n)}\}_M$. Our goal is to find a family of measures that remains invariant under the coarse graining induced by the maps $I_M \rightarrow 2M$ defined in **Subsection 2.1**.

Indeed, the fact that our model is essentially two copies of a free scalar field allows making use of many tools developed in (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d). We recall from **Section 3.1** of (Lang et al., 2018d) that determination of the fixed points for any power n in (2.20) can be reduced to studying the renormalization group flow for $n = 2$ at the cost of an additional contour integral by a standard application of the residue theorem: Starting from the initial covariance:

$$c_M^{(0)} = \frac{p^{(n-1)}(\hbar/2)}{(-\Delta_M + p^2)^{n/2}} = \frac{p^{(n-1)}(\hbar/2)}{2\pi i} \oint_{\gamma} dk_o \frac{1}{k_o^{n/2}} \frac{1}{-\Delta_M + p^2 - k_o}, \quad (2.34)$$

with γ being a contour consisting of a part along $i\mathbb{R}$ (excluding the origin) and an arc closing at infinity on the positive half plane. For brevity, we relabel $\bar{p}^2 := p^2 - k_o$. Now, since the RG flow is linear and only changes Δ_M , determination of the fixed points boils down to the case $n = 2$ up to said contour integral along γ .

As we had already seen in (Lang et al., 2018b) that the RG flow is easiest studied in the Fourier transformed representation, we recall the discrete Fourier transform and its inverse on L_M for any D (with $k_M = 2\pi/M$)

$$f_M(m) = \sum_{l \in \mathbb{Z}_M^D} \widehat{f}_M(l) e^{ik_M l \cdot m}, \quad \widehat{f}_M(l) = M^{-D} \sum_{m \in \mathbb{Z}_M^D} f_M(m) e^{-ik_M l \cdot m}. \quad (2.35)$$

Going to the discrete Fourier picture and assuming translational invariance of the covariance, we know that the kernel of the covariance at the fixed point can be written as:

$$c_{M,I}^*(m) = \frac{p^{(n-1)}(\hbar/2)}{2\pi i} \oint_{\gamma} dk_o \frac{1}{k_o^{n/2}} \sum_{l \in \mathbb{Z}_M^3} e^{ik_M l \cdot m} \widehat{c}_{M,I}^*(l). \quad (2.36)$$

Further, it was observed in (Lang et al., 2018d) that the renormalization group flow decouples for each direction and

thus the covariance can be transformed via another application of the residue theorem into:

$$\widehat{c}_{M,I}^*(l) = \frac{1}{(2\pi i)^3} \left(\prod_{b=1}^3 \oint_{\gamma} dz_b \right) \frac{1}{\bar{p}^2 - \sum_b z_b} \prod_{b=1}^3 \widehat{c}_{M,I,b}^*(l_b; z_b). \quad (2.37)$$

For $I_M \rightarrow 2M$ from (2.7) with a discretization using regular cubes as (2.10) the fixed point obtained from the flow starting with the fraction in (2.34) has been already computed in (Lang et al., 2018b) and reads:

$$\widehat{c}_{M,I,b}^*(l, z) = \widehat{c}_M^*(l, q) : \\ = \frac{\varepsilon_M^2}{q^3} \frac{q \cosh(q) - \sinh(q) + (\sinh(q) - q) \cos(k_M l)}{\cosh(q) - \cos(k_M l)}, \quad (2.38)$$

where $q = \varepsilon_M \sqrt{z}$. Note that indeed $I_M \rightarrow 2M$ is the same in each direction b and the same for both field species I , hence we obtain the same fixed point for both I .

For the deleting kernel $I_M^{\text{Del}} \rightarrow 2M$ from (2.8) the fixed point can be computed to be³

$$\widehat{c}_{M,I,b}^*(l, z) = \widehat{c}_M^{\text{Del}}(l, q) := \frac{\varepsilon_M^2}{q} \frac{\sinh(q)}{\cosh(q) - \cos(k_M l)}. \quad (2.39)$$

Thus, we finished the analysis of the direct Hamiltonian Renormalization applied to our toy model for a gauge theory which has been reduced to the gauge-invariant subspace before quantization. Keep in mind that in **Section 3.2.2** of (Lang et al., 2018b) it was already explained that renormalization of the Hamiltonian leads to replacing in the discretization (2.31) the initial covariance with the fixed pointed one, that is $\omega_{M,I} \mapsto \omega_M^*$.

Also, since both field species behaved exactly the same, i.e. $\omega_{M,I} = \omega_M$, the same universality and continuum properties discussed in (Lang et al., 2018c; Lang et al., 2018d) apply to this case as well.

3. RENORMALIZATION WITH FORM FACTORS FOR FREE VECTOR BOSONS

In this section we turn toward those discretisations for which the fields are discretized with respect to the edges of some finite graph. This brings us closer to lattice gauge theories which are typically formulated in terms of holonomies, that is exponentials of the connection. For this purpose, **Subsection 3.1** introduces discretisations where the fields are integrated along one-dimensional curves and their canonical conjugated pairs against $D - 1$ faces, where D is the number of spatial dimensions. We can express the discretization in a language maximally close to (Lang et al., 2018c) and the previous section, if we smear both objects with *form factors* of curves and $D - 1$ faces respectively.

³Note that the earlier work (Lang et al., 2018c) contains a typo: While in **eqn (3.61)** (in (Lang et al., 2018c)) we quote obviously the initial covariance, we missed to explicitly write the fixed point given by (2.39) above.

Due to our earlier considerations we have an understanding how sensible injection maps on the quantum level can be chosen, which we do in **Subsection 3.2** calling them “deleting” and “filling” kernel respectively. These relate the quantities of some resolution M to those on a finer resolution pM , where $p \in \mathbb{N}$ can be any arbitrary factor. However, to keep the notation simple, we will use throughout this paper the choice $p = 2$.

Afterward, we want to investigate a toy model in order to test how the different coarse graining maps and their corresponding fixed pointed theories behave with respect to each other. As we want to study models which allow for the existence of holonomy operators in the Fock representation that supports the continuum Hamiltonian, we have to introduce a deformation of free Maxwell theory. This deformation is discussed in **Subsection 3.3**.

In **Subsections 3.4** and **3.5** we will again employ tools developed in (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) to determine the fixed pointed Hilbert spaces for the coarse graining maps defined by the deleting as well as the filling kernel. The task amounts to finding a suitable fixed pointed covariance defining a Gaussian measure on the Hilbert spaces of finite resolution, which we will derive in closed form for both maps. This demonstrates robustness of the continuum theory even under drastic changes of the coarse graining procedure.

3.1. Injection and Evaluation Maps

As in the previous section, we consider a $(D + 1)$ -dimensional manifold of the form $\mathbb{R} \times \sigma$ on which an infinite dimensional, conservative Hamiltonian system is defined. Via an IR cut-off we restrict to the compact submanifold σ_R , omitting the cut-off R in all subsequent formulas.

Let the phase space be coordinatized by vector fields P^a and covector fields A_a with $a = 1, \dots, D$ which read in terms of smearing against test functions $f, g \in L(\sigma)^D$

$$P[g] := \langle P, g \rangle := \int_{\sigma} d^D x P^a(x) g_a(x), \quad (3.1)$$

$$A[f] := \langle A, f \rangle := \int_{\sigma} d^D x A_a(x) f^a(x), \quad (3.2)$$

and which have elementary Poisson brackets:

$$\begin{aligned} \{P[g], P[g']\} &= \{A[f], A[f']\} = 0, \\ \{P[g], A[f]\} &= \kappa_o \langle g, f \rangle := \kappa_o \int_{\sigma} d^D x g_a(x) f^a(x), \end{aligned} \quad (3.3)$$

with κ_o being the coupling constant of the theory, which we set to one in the following: $\kappa_o = 1$.

We discretize the theory by introducing smearings of A_a along the 1-dimensional edges of some dual cell complex. For the case of $\sigma_R = [0, R]^D$, it suggests itself to consider regular lattices, where at each vertex there are $2D$ many edges incident. In the following, we will restrict to this choice, to keep the notation simple. Note that the edges of the lattice are understood to be *paths*, i.e. semianalytic curves. The set of all paths forms the groupoid $\mathcal{P}$, which is closed under concatenation of elements and features an inverse element for each path—however there is no natural identity element on $\mathcal{P}$. We understand an element $e \in \mathcal{P}$ as the embedding $e: [0, 1] \rightarrow \sigma_R$. Since we want to focus for the purpose of this article on regular lattices (e.g. cubic lattices for $D = 3$),

we are mostly interested in a subset of $\mathcal{P}$: Given a lattice γ_M , where M denotes the number of vertices in each direction, we denote the set of oriented edges in γ_M by $\mathcal{P}^M \subset \mathcal{P}$.

A smearing of the field A_a against an edge can be obtained by allowing in (3.2) not only test functions in $L(\sigma)$ but distributions such as *form factors* F_e for any edge $e \in \mathcal{P}^M$, i.e.:

$$(F_e)^a(x) = \int_e dy^a \delta^{(D)}(x, y). \quad (3.4)$$

Similarly, since we are interested in those lattices γ_M which stemmed from some dual cell complex, we can associate with each edge e a choice of some $(D - 1)$ -dimensional face $S(e)$, such that $S(e) \cap e' = \emptyset$ iff $e \neq e'$ and at the unique point $S(e) \cap e$ its normal points in the same direction as $\dot{e}$. Then, we can also introduce the dual form factors of the face S , e.g.:

$$(f^S)_a(x) = \int_S dy^b \wedge dy^c \epsilon^{abc} \delta^{(3)}(x, y) \text{ if } D = 3, \quad (3.5)$$

$$(f^S)_a(x) = \int_S dy^b \epsilon_{ab} \delta^{(2)}(x, y) \text{ if } D = 2. \quad (3.6)$$

Note that there is a natural non-distributional Poisson bracket between the form factors for curves and the dual form factors for faces:

$$\langle F_e, f^S(e') \rangle = \delta_{ee'}. \quad (3.7)$$

We can now restrict the set of our observables with respect to which the physical configuration (E^a, A_a) is probed. We want to keep only those observables that can be understood as restricting A_a to the edges of a lattice and E^a to its dual faces. This can be achieved by introducing injection and evaluation maps between test functions in $L(\sigma^D)$ and functions on the lattice $L_M \equiv L(\mathcal{P}^M, \mathbb{R})$:

$$\begin{aligned} I_M^{\text{Del}} : L_M &\rightarrow L(\sigma)^D \\ f_M &\mapsto (I_M^{\text{Del}} f_M)^a(x) := \sum_{e \in \mathcal{P}^M} (F_e)^a(x) f_M(e), \end{aligned} \quad (3.8)$$

$$\begin{aligned} E'_M : L(\sigma)^D &\rightarrow L_M \\ f^a &\mapsto (E'_M f)(e) := \langle f, f^S(e) \rangle. \end{aligned} \quad (3.9)$$

Using property (3.7) one easily verifies that $E'_M \circ I_M^{\text{Del}} = \text{id}_M$. Further, we can understand

$$A[I_M^{\text{Del}} f_M] = \int d^3 x A_a(x) (I_M^{\text{Del}} f_M)^a(x) = \sum_{e \in \mathcal{P}^M} A[F_e] f_M(e), \quad (3.10)$$

as A_a restricted to the lattice γ_M . We introduced a superscript on I_M^{Del} and call it in the following “deleting kernel” due to its similarity with (2.8).⁴ Yet, this construction is far from unique and in order to demonstrate this we introduce a second choice called “filling kernel.” In its spirit, this map is constructed to be similar to the standard choice employed for scalar fields, i.e. (2.3).

⁴Deleting kernels are favored in the literature on cylindrical consistency of gauge theories, see for example the projective spaces of the Ashtekar-Lewandowski Hilbert space in the context of Loop Quantum Gravity (Ashtekar et al., 1995; Thiemann, 2007; Rovelli, 2004). Note however, that the Ashtekar-Lewandowski Hilbert space for each edge is a Hilbert space over $\text{SU}(2)$ in contrast to the Fock space we consider in this manuscript.

Due to the multiple choices of cell complexes used to define (2.3), we have an ambiguity regarding the injection map for the “filling kernel.” We restrict us to the choices of parallelepipeds (2.11) since discretisations with regular cubes have been extensively studied in the papers (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d) and this new choice will demonstrate the robustness of the renormalization procedure under considerably drastic changes. For $i \in \{1, \dots, D\}$, we define

$$I_M^{\text{Fil},i} : L_M \rightarrow L(\sigma)^D, \quad (3.11)$$

$$f_M \mapsto (I_M^{\text{Fil},i} f_M)^a(x) = \frac{1}{2^{D-1}} \sum_{e \in \mathcal{P}_M} \sum_{e' \in \mathcal{P}^{2M}} (F_{e'})^a(x) f_M(e) r_{ee'}^{\text{Fil},i}, \quad (3.12)$$

with

$$r_{ee'}^{\text{Fil},i} = \begin{cases} \delta_{\hat{e}(0), \hat{e}'(0)} \delta_{\hat{e}(0), \hat{b}}, & \text{if } \text{Im}(e') \subset {}^{i,b}\chi(e(0)), \\ 0, & \text{else,} \end{cases} \quad (3.13)$$

where $\text{Im}(e_i)$ denotes the image of $e_i : [0, 1] \rightarrow \sigma$, ${}^{i,b}\chi$ is defined in (2.11), $\hat{b}$ is the normal vector of unit length pointing in direction b and $\delta_{\hat{e}(0), \hat{b}}$ denoting the Kronecker delta in the tangent space, i.e. it is non-vanishing only if $\hat{e}(0) = \hat{b}$. Note that the cases in (3.13) are meant to be checked for all possible $b = 1, \dots, D$ separately.

It is easy to check that for a suitable choice of faces $S(e)$ we get $E'_M \circ I_M^{\text{Fil},i} = \text{id}_M$ for all i . We recall that the parameter i of the filling kernel determines the choice of parallelepipeds from (2.11) and thus all derived quantities in the coarse graining procedure will depend on it. In what follows we fix $i \in \{1, 2, 3\}$ and check the coarse graining maps for all of them separately, thus not displaying the label i explicitly.

3.2. Coarse Graining for Deleting and Filling Kernel

In this subsection we concatenate injection and evaluation maps to *coarse graining maps* $I_{M \rightarrow 2M}$ both for deleting and filling kernel on the classical level and use $I_{M \rightarrow 2M}$ to build isometries between Fock quantized Hilbert spaces of different resolutions.

3.2.1. Classical Coarse Graining Maps

First, we introduce the coarse graining maps for the deleting kernel from $L_M \rightarrow L_{2^n M}$ via: ($n \in \mathbb{N}$)

$$I_{M \rightarrow 2^n M}^{\text{Del}} := \left(\frac{\varepsilon_{2^n M}}{\varepsilon_M} \right)^\alpha E_{2^n M} \circ I_M^{\text{Del}}, \quad (3.14)$$

with $\alpha \in \mathbb{R}$. They relate a set of test functions on coarse resolution M with a set of test functions at finer resolution $2^n M$. Their action on test functions can be written explicitly as:

$$f_{2^n M}(e') := (I_{M \rightarrow 2^n M}^{\text{Del}} f_M)(e') = \frac{1}{2^{n\alpha}} \sum_{e \in \mathcal{P}^M} r_{ee'}^{\text{Del}} f_M(e), \quad (3.15)$$

where $e' \in \mathcal{P}^{2^n M}$ and

$$r_{ee'}^{\text{Del}} = \begin{cases} 1, & \text{if } e' \subset e, \\ 0, & \text{else.} \end{cases} \quad (3.16)$$

The free parameter $\alpha \in \mathbb{R}$ can be chosen in such a way that the condition of *cylindrical consistency* is satisfied, that is for all A and $f_M \in L_M$:

$$A[I_{2^n M} \circ I_{M \rightarrow 2^n M} f_M] = A[I_M f_M]. \quad (3.17)$$

Using that $F_e = F_{e_1} + F_{e_2} + \dots + F_{e_{2^n}}$ if $e = e_1 \circ e_2 \circ \dots \circ e_{2^n}$ we find

$$\begin{aligned} A[I_{2^n M}^{\text{Del}} I_{M \rightarrow 2^n M}^{\text{Del}} f_M] &= \int d^D x \sum_{e' \in \mathcal{P}^{2^n M}} A_a(x) (F_{e'})^a(x) (2^{-an} \sum_{e \in \mathcal{P}^M} f_M(e) r_{ee'}^{\text{Del}}) \\ &= \int d^D x \sum_{e \in \mathcal{P}^M} A_a(x) f_M(e) (2^{-an} \sum_{e' \in \mathcal{P}^{2^n M}} (F_{e'})^a(x) r_{ee'}^{\text{Del}}) \\ &= \int d^D x \sum_{e \in \mathcal{P}^M} A_a(x) 2^{-an} (F_e)^a(x) f_M(e) = A[I_M^{\text{Del}} f_M] 2^{-an}. \end{aligned} \quad (3.18)$$

Hence, it must be $\alpha = 0$.

If we were to introduce a coarse graining map of the filling kernel as the analogue of (3.14), a calculation similar to (Subsection 3.2.1) demonstrates, that the latter is *not* cylindrical consistent unless A_a is constant over each ${}^{i,b}\chi$. However, requiring cylindrical consistency for the classical coarse graining map is not necessary per se, thus this finding does not rule out the filling kernel. The important property for the inductive limit construction is the compatibility condition between the quantum isometries, which follows from the weaker condition

$$I_{2^n M \rightarrow 2^{n'} M} I_{M \rightarrow 2^n M} = I_{M \rightarrow 2^{n'} M}, \quad (3.19)$$

with $n < n' \in \mathbb{N}$. Indeed, (3.19) can be achieved also for the filling kernel when defining $I_{M \rightarrow 2^n M}$ as the analogue of (3.15):

$$f_{2^n M}(e') := (I_{M \rightarrow 2^n M}^{\text{Fil}} f_M)(e') = \frac{1}{2^n} \sum_{e \in \mathcal{P}^M} r_{ee'}^{\text{Fil}} f_M(e), \quad (3.20)$$

with $e' \in \mathcal{P}^{2^n M}$ and $r_{ee'}^{\text{Fil}}$ from (3.13).

Lastly, it turns out—for both filling and deleting kernel—that demanding the map $I_{M \rightarrow 2^n M}$ to be an isometry, i.e.

$$\langle I_{M \rightarrow 2^n M} f_M, I_{M \rightarrow 2^n M} \tilde{f}_M \rangle_{2M} = \langle f_M, \tilde{f}_M \rangle_M, \quad \forall f_M, \tilde{f}_M, \quad (3.21)$$

can be used to fix an auxiliary scalar product on L_M :

$$\langle f_M, \tilde{f}_M \rangle_M := \varepsilon_M^D \sum_{e \in \mathcal{P}^M} f_M(e) \tilde{f}_M(e). \quad (3.22)$$

3.2.2. Isometric Injections on the Quantum Level

In this section we construct coarse graining maps between Hilbert spaces corresponding to different resolutions. These maps drive the renormalization group (RG) flow between the inner products on the Hilbert spaces $\mathcal{H}_M$. Once a fixed point family of Hilbert space measures is found, it can be used to obtain a continuum Hilbert space via the method of inductive limits (Kadison and Ringrose, 1986; Janas, 1988). To use the latter toolbox, certain requirements must be met for the coarse graining maps $I_{M \rightarrow 2M}$. It must be guaranteed that $I_{M \rightarrow 2M}$ are isometric injections, i.e.

$$I_{M \rightarrow 2M}^\dagger I_{M \rightarrow 2M} = \text{id}_{\mathcal{H}_M}, \quad (3.23)$$

and that they are subject to the compatibility condition, i.e. for each $n < n' \in \mathbb{N}$:

$$J_{2^n M \rightarrow 2^{n'} M} \circ J_{M \rightarrow 2^n M} = J_{M \rightarrow 2^{n'} M}. \quad (3.24)$$

These two properties were also imposed for scalar field models and indeed the same procedure of constructing the injections from (Lang et al., 2018a) can be used again. We utilize a Fock quantization of the discretized field $A_M = A \circ I_M$. Upon choosing the vacuum vector $\Omega_M \in \mathcal{H}_M$ of the discretized Hamiltonian, we consider the dense linear span of vectors of the form

$$\exp(i\hat{A}_M[f_M])\Omega_M, \quad (3.25)$$

where

$$\hat{A}_M[f_M] := \varepsilon_M \sum_{e \in \mathcal{P}^M} \hat{A}_M(e) f_M(e) = \varepsilon_M \sum_{m \in \mathbb{Z}_M^D} \sum_{a=1}^D \hat{A}_{M,a}(m) f_{m,a}(m), \quad (3.26)$$

and we denote the edge $e = e_{m,a}$ with initial vertex m and direction a .

In the same manner as in (Lang et al., 2018a), we define the injections between Fock spaces as:

$$J_{M \rightarrow 2M}(e^{i\hat{A}_M[f_M]}\Omega_M) := e^{i\hat{A}_{2M}[I_{M \rightarrow 2M}f_M]}\Omega_{2M}, \quad (3.27)$$

where $I_{M \rightarrow 2M}$ is the respective version of its action on test functions for deleting or filling kernel.

By construction, this map is maximally parallel to the case of scalar fields and therefore many properties can be transferred to this setting. We refer to (Lang et al., 2018a; Lang et al., 2018b) for further details.

3.3. Toy Model: Definition and Discretization for a Proca Like Theory

In this subsection, we define a toy model which allows for holonomy like operators in the continuum, i.e. $\exp(i\hat{A}[F_\alpha])$ has finite expectation values for α being some closed curve in σ . Then, we discretize this theory with respect to smearings along the curves of a lattice γ_M as discussed before.

3.3.1. Definition of the Continuum Model

In close analogy to the model of Section 2.3 we study a field theory with $D = 3$ spatial dimensions and Hamiltonian

$$H := \frac{\varepsilon_o}{2} \int d^3x (P_a(x)P^a(x) + c^2 A^a(x)(\omega^2 A)_a(x)), \quad (3.28)$$

where in the following we set $c = \varepsilon_o = 1$. In order to allow for the continuum QFT to support the exponentials of Wilson loops as operators, i.e. $\exp(i\hat{A}[F_\alpha])$ with some closed curve, α we chose

$$\omega^2 = \frac{1}{p^{2(n-1)}}(-\Delta + p^2)^n, \quad (3.29)$$

with some mass term $p > 0$ and $n \geq 4$ to ensure existence of the covariance following from (3.28) when evaluated on form factors F_α as in (3.4):

Lemma: Let $\alpha : [0, 1] \rightarrow \sigma$ be a (closed) curve. The continuum vacuum expectation value of the holonomy along α is finite if $n \geq 4$ and $p > 0$, i.e.:

$$\langle \Omega, e^{i\hat{A}[F_\alpha]}\Omega \rangle_{\mathcal{H}} = e^{-\hbar \langle F_\alpha, \omega^{-1} F_\alpha \rangle / 4} < \infty. \quad (3.30)$$

Proof: We consider only the case $n = 4$ as higher powers are automatically included due to positive definiteness of $-\Delta$ and $p > 0$. The vacuum expectation value will be finite if $\langle F_\alpha, \omega^{-1} F_\alpha \rangle$ remains finite with ω from (3.29). It suffices to check whether

$$\int \frac{d^3k}{(p^2 + k^2)^2} (\hat{F}_\alpha)^a(k) \overline{(\hat{F}_\alpha)_a(k)} < \infty, \quad (3.31)$$

where

$$\begin{aligned} (\hat{F}_\alpha)^a(k) &= \int d^3x e^{ik \cdot x} (\hat{F}_\alpha)^a(x) = \int_\alpha dy^a \int d^3x e^{ik \cdot x} \delta^{(3)}(x, y) \\ &= \int_\alpha dy^a e^{ik \cdot y} = \int_0^1 dt \dot{\alpha}^a(t) e^{ik \cdot \alpha(t)}. \end{aligned} \quad (3.32)$$

First, we give a bound from above for the absolute value of $\hat{F}_\alpha$

$$\|\hat{F}_\alpha(k)\| \leq \beta := \sup_{a=1,2,3; t \in [0,1]} |\dot{\alpha}^a(t)|. \quad (3.33)$$

Using this approximation and going to spherical coordinates $d^3k \rightarrow r^2 \sin(\theta) dr d\theta d\varphi$ we get:

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{d^3k}{(p^2 + k^2)^2} (\hat{F}_\alpha)^a(k) \overline{(\hat{F}_\alpha)_a(k)} &\leq \beta^2 \int_{\mathbb{R}^3} \frac{d^3k}{(k^2 + p^2)^2} \\ &= 4\pi\beta^2 \int_0^\infty \frac{r^2 dr}{(r^2 + p^2)^2} = 2\pi\beta^2 \int_{\mathbb{R}} \frac{r^2 dr}{(r^2 + p^2)^2} = \frac{2\pi\beta^2}{p} \int_{\mathbb{R}} \frac{x^2 dx}{(x^2 + 1)^2} \\ &= \frac{\pi^2\beta^2}{p} < \infty, \end{aligned} \quad (3.34)$$

where we used the residue theorem in the last step. Hence, the vacuum expectation value is well-defined.

Conversely, a similar calculation shows that for lower powers of n in ω the vacuum expectation value diverges (and due to (3.34) also if $p = 0$). One should therefore either change the test functions and not use form factors or study different theories. In principle, we could consider free Maxwell electrodynamics, the Proca action or even the free graviton theory and study their behavior under a renormalization group flow with the methods of (Lang et al., 2018a). But here we have altered the Hamiltonian H in order to ensure that the expectation values of holonomies with respect to the vacuum (which is annihilated by H) are well-defined. This happens by introducing a higher order polynomial in the Laplacian (3.29) which of course breaks Lorentz invariance. However, our model just serves to test theories with well-defined holonomy operators (but not well-defined electric flux operators) in the usual Fock space setting. Ultimately, we will be interested in coupling general relativity to gauge theories. In this case, theories such as Loop Quantum Gravity (Thiemann, 2007; Rovelli, 2004; Ashtekar and Lewandowski, 1995) indicate that insertion of such Lorentz invariance breaking higher polynomials is not necessary (Thiemann, 1998; Liegener and Thiemann, 2016).

3.3.2. Initial Discretization on Cubic Lattice

In order to test the coarse graining maps on the quantum level, we need to first introduce a discretization of the phase space of $(P^a(x), A_a(y))$ with $a = 1, 2, 3$ with symplectic structure (3.3) and a discretization of the Hamiltonian (3.28).

We work on a cubic lattice, with M vertices in each direction labeled by $m \in \mathbb{Z}_M^D = \{0, 1, \dots, M-1\}^D$ with $D = 3$. At each vertex m we have three in- and three outgoing edges. We use smearings against form factors to discretize the Hamiltonian. Denoting the edges on the lattice by $e_{m,a}$ (labeled by initial point m and a direction $a = 1, 2, 3$ and $\dot{e}(t) = \hat{a}$ for all $t \in [0, 1]$) we have

$$A_{M,a}(m) := A[F_{e_{m,a}}] = \int_{[0,1]} dt A_a(e_{m,a}(t)), \quad (3.35)$$

$$P_M^a(m) := P[f^S(e_{m,a})] = \int_{[0,1]^2} du dv P^a(S_{e_{m,a}}(u, v)), \quad (3.36)$$

Similar to Section 2.3 we interpret this structure as three different field species $a = 1, 2, 3$ (This is due to $D = 3$. To make the distinction between directions and field species clear, we will write in this section an arbitrary D for directions but keep $a = 1, 2, 3$ for the field species). Moreover, at each m the field specie a is supported only on edges along direction $\hat{a}$. In order to distinguish the a priori different species, we associate to each of them their own discretised $\omega_{M,a}$, while of course our initial discretization is such that

$$\omega_{M,1} = \omega_{M,2} = \omega_{M,3} = \omega_M^{(0)}, \quad (3.37)$$

with $\omega_M^{(0)}$ some discretization of (3.29), such that $\lim_{M \rightarrow \infty} \omega_M^{(0)} = \omega$.⁵

Since the Hamiltonian is of free harmonic oscillator form for each a , we can repeat the discussion from Section 2.3: We introduce the discrete annihilation and creation fields

$$\mathbf{a}_{M,a}(m) := \frac{1}{\sqrt{2\hbar}} (\sqrt{\omega_{M,a} \varepsilon_M} A_{M,a}(m) - i \sqrt{\omega_{M,a} \varepsilon_M}^{-1} P_M^a(m)), \quad (3.38)$$

such that

$$H_M = \hbar \sum_{a=1,2,3} \sum_{m \in \mathbb{Z}_M^D} \overline{\mathbf{a}_{M,a}(m)} (\omega_{M,a} \mathbf{a}_{M,a}(m)), \quad (3.39)$$

For each specie a , we define the corresponding Hilbert spaces $\mathcal{H}_{M,a}$ with Fock vacuum $\Omega_M^{(0)} \in \mathcal{H}_M := \otimes_a \mathcal{H}_{M,a}$ annihilated by each (3.38) and thus simultaneously by (3.39). Denoting by $\langle \cdot, \cdot \rangle_{\mathcal{H}_M}$ the scalar product on $\mathcal{H}_M$ it follows (with $f_{M,a}(m) \in L_M := L(\mathcal{P}^M, \mathbb{R}) \equiv \ell_2(\mathbb{Z}_M^D)^3$)

$$\langle \Omega_M, e^{\hat{A}_M} [f_M] \Omega_M \rangle_{\mathcal{H}_M} = \prod_{a=1,2,3} e^{-(\hbar/4) \langle f_{M,a}, \omega_{M,a}^{-1} f_{M,a} \rangle} = e^{-(f_M, {}^M f_M)/2}, \quad (3.40)$$

⁵Indeed, we will see in the next sections that the coarse graining induces different flows of $\omega_{M,a}$ for different a in case of the filling kernel, leading ultimately to different fixed pointed families $\{\omega_{M,a}^*\}_M$. However, this “direction dependence” is artificial in the sense that it is only present for finite M , while in the continuum limit $M \rightarrow \infty$ the covariances of all species a agree.

with covariance $c_M = \text{diag}(c_{M,1}, c_{M,2}, c_{M,3})$ and $c_{M,a} = (\hbar/2) \omega_{M,a}^{-1}$. As Gaussian measures do not change their structure under coarse graining (Lang et al., 2018b) the task boils down to find a fixed pointed family $\{c_M^*\}_M$ for the coarse graining maps $J_M \rightarrow J_{2M}$ of both the deleting as well as the filling kernel. Then, we can also use that the fixed pointed Hamiltonian is given by (3.39) when replacing $\omega_{M,a} \mapsto \omega_{M,a}^*$.

Also, we discussed already in Section 2.3 that the fixed point for choice $n \geq 4$ in (3.29) can be achieved by finding the fixed point of $n = 2$ due to the fact that both are related via the contour integral (2.34) and replacing $p^2 \mapsto \bar{p}^2 = p^2 - k_0$.

We end this section by choosing an explicit initial discretization of the covariance, i.e. $c_M^{(0)} = \text{diag}(c_{M,1}^{(0)}, c_{M,2}^{(0)}, c_{M,3}^{(0)})$, which acts on test functions $f_{M,a}(m) \in L_M^3$. We assume that every field specie has a translational invariant covariance, i.e. its kernel is for $m, n \in \mathbb{Z}_M^D$:

$$c_{M,a}(m, n) = c_{M,a}(m - n), \quad (3.41)$$

which holds true for the following initial discretization of the derivatives inside

$$\omega_M^2 = \left(- \sum_{b=1}^D \partial_b^M \partial_b^M + \bar{p}^2 \right), \quad (3.42)$$

with:

$$(\partial_b^M f_M)_a(m) := \frac{1}{\varepsilon_M} [f_{M,a}(m) - f_{M,a}(m - \hat{b})], \quad (3.43)$$

$$(\partial_b^M f_M)_a(m) := \frac{1}{\varepsilon_M} [f_{M,a}(m) - f_{M,a}(m + \hat{b})], \quad (3.44)$$

and $\hat{b}$ is the normal vector pointing in direction b . We see that c_M does not mix the different species a , therefore we can apply the discrete Fourier transform from (2.35) on each subspace of fixed a to get as initial starting point for the covariance (see (Lang et al., 2018b) for details):

$$\hat{c}_{M,a}^{(0)}(l) = \frac{\hbar/2}{\bar{p}^2 - \varepsilon_M^{-2} \sum_{d=1}^D (2 \cos(k_M l_d) - 2)}, \quad (3.45)$$

with $l = \{l_1, \dots, l_D\}$ and $k_M = 2\pi/M$. Note that the right-hand side of (3.45) is independent of a due to the initial choice (3.37). This will change once we study the RG flow of the filling kernel.

Lastly, let us recall from (Lang et al., 2018d) that an initial covariance of the form (3.45) can be transformed via the residue theorem into several integrals over a product of “one-dimensional” covariances, i.e. decouples in each direction:

$$\hat{c}_{M,a}^{(0)}(l) = \frac{\hbar/2}{(4\pi i)^3} \left(\prod_{b=1}^D \oint_{\Gamma} dz_b \right) \frac{1}{\bar{p}^2 - \sum_b z_b^2} c'_M(l_a, a; z_a) \prod_{b \neq a} c'_M(l_b, a; z_b), \quad (3.46)$$

where Γ is a contour surrounding the real axes (closing at $\pm \infty$ and thus including both poles) and $(l_i \in \mathbb{Z}_M)$

$$c'_M(l', a; z) := \frac{1}{z - \varepsilon_M^{-2} (2 \cos(k_M l') - 2)}. \quad (3.47)$$

Note that the way in which we split the integrals is purely conventional and does not affect the continuum limit $M \rightarrow \infty$.

Also, the initial covariance does not have a direction dependency, hence the label a does not appear on the right hand side of (3.47).

A factorization property like (3.46) becomes useful if it can be established that the covariance does not change this structure under a renormalization step. In such a case, each of the $\hat{c}'_M$ will drive into its respective fixed point (Lang et al., 2018d). Indeed, this will be case for both the deleting and the filling kernel as we discuss in the next two sections. There, we will study the different Hamiltonian RG flows in order to find the fixed pointed covariances $\hat{c}^*_{M,a} = (\hbar/2)(\omega^*_{M,a})^{-1}$ for each field specie a . They completely describe the Hilbert spaces and the Hamiltonians at finite resolution.

3.4. Toy Model: Fixed Points of the Deleting Kernel

From now on we set $D = 3$ explicitly in all formulae. We study the RG flow of the coarse graining for the deleting kernel from (3.27), i.e.

$$I_{M \rightarrow 2M}(e^{\hat{A}_M} [f_M] \Omega_M^{(n+1)}) := e^{\hat{A}_{2M}} [I_{M \rightarrow 2M}^{\text{Del}} f_M] \Omega_{2M}^{(n)}, \quad (3.48)$$

which is equivalent to the flow of the family of Hilbert space measures

$$\langle \Omega_M^{(n+1)}, e^{\hat{A}_M} [f_M] \Omega_M^{(n+1)} \rangle_{\mathcal{H}_M^{(n+1)}} = \langle \Omega_{2M}^{(n)}, e^{\hat{A}_{2M}} [I_{M \rightarrow 2M}^{\text{Del}} f_M] \Omega_{2M}^{(n)} \rangle_{\mathcal{H}_{2M}^{(n)}}, \quad (3.49)$$

that is (see (Lang et al., 2018b)):

$$\langle f_M, c_M^{(n+1)} g_M \rangle_M := \langle I_{M \rightarrow 2M}^{\text{Del}} f_M, c_{2M}^{(n)} I_{M \rightarrow 2M}^{\text{Del}} g_M \rangle_{2M}, \quad (3.50)$$

with $f_M, g_M \in L_M = \ell_2(\mathbb{Z}_M^3)$ and deleting kernel ($m' \in \mathbb{Z}_{2M}^3$)

$$(I_{M \rightarrow 2M}^{\text{Del}} f_M)_a(m') = \sum_{m \in \mathbb{Z}_M^3} f_{M,a}(m) (\delta_{m',2m} + \delta_{m',2m+a}). \quad (3.51)$$

We see that (3.51) does not mix the field species for different a with each other and does not distinguish between different a . Together with the fact that the initial covariance was written as diagonal matrix $c_M = \text{diag}(c_{M,1}, c_{M,2}, c_{M,3})$, this implies that the same holds at each iteration of the RG flow and thus also the fixed point measure will be a product of three times the same Gaussian measures for each a .

However, for each field specie a the different directions with respect to the lattices vertices $m = (m_1, m_2, m_3)$ behave differently as $\hat{a}$ enters the right-hand side of (3.51). Thus, in direction $\hat{a}$ the $I_{M \rightarrow 2M}^{\text{Del}}$ behaves as the one-dimensional blocking kernel studied in (Lang et al., 2018b), that is

$$(I_{M \rightarrow 2M}^{\text{Del}} f_M)_a(2m_a, \dots) = (I_{M \rightarrow 2M}^{\text{Del}} f_M)_a(2m_a + 1, \dots), \quad (3.52)$$

for $m_a \in \mathbb{Z}_M$ being the a^{th} component of $m \in \mathbb{Z}_M^3$. However, for $b \neq a$, the kernel behaves as the one-dimensional deleting kernel from (Lang et al., 2018c), that is for m_b being the b^{th} component of m :

$$(I_{M \rightarrow 2M}^{\text{Del}} f_M)_a(\dots, 2m_b + 1) = 0. \quad (3.53)$$

Thus, the flow of the coarse graining map from (3.51) introduces a “direction dependence” of the covariance at the quantum level for finite resolution M . This dependence only vanishes in the continuum limit $M \rightarrow \infty$.

Since the RG flow in (3.50) does not mix the different directions, for a decoupled covariance of the form (3.46) each “one-dimensional covariance” $\hat{c}'_M$ will flow into its respective fixed point. And since the RG flows for direction a and $b \neq a$ behave like the ones of the injections studied in (Lang et al., 2018b) and (Lang et al., 2018c) respectively, the fixed points are already known and read:

$$(\hat{c}'_M)^*(l', b; z) = \hat{c}_M^{\text{Del}}(l', z) \text{ for direction } b \neq a, \quad (3.54)$$

$$(\hat{c}'_M)^*(l', a; z) = \hat{c}_M(l', z) \text{ for direction } a, \quad (3.55)$$

with $l_i \in \mathbb{Z}_M, z \in \mathbb{C}$ and the definitions from (2.38) and (2.39).

It remains to plug the fixed points for each direction into (3.46) and to restore the correct n -dependence via (2.34). Thus, we know the complete fixed pointed covariance $\hat{c}^* = \text{diag}(\hat{c}^*_{M,1}, \hat{c}^*_{M,2}, \hat{c}^*_{M,3})$ with the following kernels for the Fourier transform of the covariances:

$$\hat{c}^*_{M,a}(l) = \frac{p^{(n-1)} \hbar}{(4\pi i)^4} \oint_{\gamma} dk_o \frac{1}{k_o^{n/2}} \left(\prod_{b=1}^3 \oint_{\Gamma} dz_b \right) \frac{1}{\bar{p}^2 - \sum_{b=1}^3 z_b} \hat{c}_M(l_a, z_a) \prod_{b \neq a} \hat{c}_M^{\text{Del}}(l_b, z_b), \quad (3.56)$$

where we remember that $\bar{p}^2 = p^2 - k_o$. For further details, see (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d).

3.5. Toy Model: Fixed Points of the Filling Kernel

We turn toward the second choice of coarse graining map that was motivated in this paper. While of course further coarse graining maps can be constructed, the analysis of this section presents already an indication of universality—as it will transpire that the continuum limit $M \rightarrow \infty$ of the fixed pointed theories for both kernels agree.

Again it is $D = 3$. The three different choices of filling kernels are labeled by $i \in \{1, 2, 3\}$ and their explicit action is obtained by using the form of the characteristic functions in (2.11):

$$\begin{aligned} (I_{M \rightarrow 2M}^{\text{Fill},i} f_M)_a(m') &= \sum_{m \in \mathbb{Z}_M^3} f_{M,a}(m) \begin{cases} \delta(m'_a - \sum_{b \neq a} \delta^{\text{odd}}(m'_b)) / 2, m'_a \neq 0, & \text{for } i = a, \\ \delta(m'_a - \delta^{\text{odd}}(m'_1) + \delta^{\text{odd}}(m'_2)) / 2, m'_a \neq 0, & \text{else,} \end{cases} \\ &\quad \delta(m'_1 + \delta^{\text{odd}}(m'_2)) / 2, m'_1 \neq 0, & (3.57) \end{aligned}$$

where $s = 1, 2, 3$ is distinct from both a, i , that is $a \neq s \neq i$ and $\delta^{\text{odd}}(n) = 0$ iff $n \in 2\mathbb{Z}_M$ and $= 1$ else. Like in the previous subsection, we see that different field species a will not talk to each other, therefore keeping the structure $c_M = \text{diag}(c_{M,1}, c_{M,2}, c_{M,3})$ intact during the whole RG flow.

However, a notable difference to the map $I_{M \rightarrow 2M}^{\text{Del}}$ is that the choice of i leads to different fixed pointed families for the field

specie labeled by a —since (3.57) singles out the case with $a = i$. On top of that, the directions of the lattice vertices $m \in \mathbb{Z}_M^3$ do not decouple in an obvious way. Thus, we need to carefully study how the matrix elements of a covariance transform under a renormalization step, which reads for fixed a and $f_M, g_M \in L_M$

$$\langle f_{M,a}, c_{M,a}^{(n+1)} g_{M,a} \rangle_M := \langle I_{M \rightarrow 2M}^{\text{Fil},i} f_{M,a}, c_{2M,a}^{(n)} I_{M \rightarrow 2M}^{\text{Fil},i} g_{M,a} \rangle_{2M} \quad (3.58)$$

Here, we only show the case $a = i = 1$ explicitly, all other choices work analogously. By writing explicitly $\langle \cdot, \cdot \rangle_M$ with (3.22) abbreviating $f_M(m) := f_{M,a=1}(m)$ and plugging in (3.57), we perform the following manipulations:

$$\begin{aligned} & \varepsilon_{2M}^{-2D} \langle f_M, c_{M,1}^{(n+1)} g_M \rangle_M \\ &= \sum_{m, \tilde{m} \in \mathbb{Z}_{2M}^3} (I_{M \rightarrow 2M}^{\text{Fil},1} f_M)(m) (I_{M \rightarrow 2M}^{\text{Fil},1} g_M)(\tilde{m}) c_{2M,1}^{(n)}(m, \tilde{m}), \quad (3.59) \\ &= \sum_{m'_1, m'_2, m'_3 \in \mathbb{Z}_{2M} \tilde{m}'_1, \tilde{m}'_2, \tilde{m}'_3 \in \mathbb{Z}_{2M}} c_{2M,1}^{(n)}(m'_1, m'_2, m'_3, \tilde{m}'_1, \tilde{m}'_2, \tilde{m}'_3), \quad (3.60) \\ & \times f_M \left(\left\lfloor \frac{m'_1 - \delta^{\text{odd}}(m'_2) - \delta^{\text{odd}}(m'_3)}{2} \right\rfloor, \left\lfloor \frac{m'_2}{2} \right\rfloor, \left\lfloor \frac{m'_3}{2} \right\rfloor \right) g_M \\ & \left(\left\lfloor \frac{\tilde{m}'_1 - \delta^{\text{odd}}(\tilde{m}'_2) - \delta^{\text{odd}}(\tilde{m}'_3)}{2} \right\rfloor, \left\lfloor \frac{\tilde{m}'_2}{2} \right\rfloor, \left\lfloor \frac{\tilde{m}'_3}{2} \right\rfloor \right) \\ &= \sum_{m, \tilde{m} \in \mathbb{Z}_M^3, \delta, \tilde{\delta} \in \{0,1\}^3} c_{2M,1}^{(n)}(2m_1 + \delta_1, 2m_2 + \delta_2, 2m_3 + \delta_3, 2\tilde{m}_1 \\ & + \delta_1, 2\tilde{m}_2 + \delta_2, 2\tilde{m}_3 + \delta_3) \times f_M(m_1 + (\delta_1 - \delta_2 - \delta_3)/2, m_2, m_3) g_M \\ & (\tilde{m}_1 + (\tilde{\delta}_1 - \tilde{\delta}_2 - \tilde{\delta}_3)/2, \tilde{m}_2, \tilde{m}_3), \quad (3.61) \end{aligned}$$

where in the last step we expressed $m'_i = 2m_i + \delta_i$ with $m, \tilde{m} \in \mathbb{Z}_M^3$ and $\delta, \tilde{\delta} \in \{0,1\}^3$. We can now shift the summation parameter $m_1 \mapsto m_1 - (\delta_1 - \delta_2 - \delta_3)/2$ using that $f_M(0) = f_M(M)$ and $c_{2M,1}^{(n)}(0, \dots) = c_{2M,1}^{(n)}(2M, \dots)$:

$$\begin{aligned} & \varepsilon_{2M}^{-2D} \langle f_M, c_{M,1}^{(n+1)} g_M \rangle_M = \sum_{m, \tilde{m} \in \mathbb{Z}_M^3} f_M(m) g_M(m) \\ & \times \sum_{\delta, \tilde{\delta} \in \mathbb{Z}_M^D} c_{2M,1}^{(n)} \left(2m_1 + \delta_1 - 2 \frac{\delta_1 - \delta_2 - \delta_3}{2}, \dots, 2\tilde{m}_1 + \tilde{\delta}_1 \right. \\ & \left. - 2 \frac{\tilde{\delta}_1 - \tilde{\delta}_2 - \tilde{\delta}_3}{2}, 2\tilde{m}_2 + \tilde{\delta}_2, \dots \right). \quad (3.62) \\ & \times \sum_{\delta, \tilde{\delta} \in \mathbb{Z}_M^D} c_{2M,1}^{(n)} \left(2m_1 + \delta_1 - 2 \frac{\delta_1 - \delta_2 - \delta_3}{2}, \dots, 2\tilde{m}_1 + \tilde{\delta}_1 \right. \\ & \left. - 2 \frac{\tilde{\delta}_1 - \tilde{\delta}_2 - \tilde{\delta}_3}{2}, 2\tilde{m}_2 + \tilde{\delta}_2, \dots \right) \end{aligned}$$

As this equation is for arbitrary f_M, g_M , it must hold component wise and gives us the following recursion relation for the RG flow:

$$\begin{aligned} c_{M,1}^{(n+1)}(m, \tilde{m}) &= 2^{-2D} \sum_{\delta, \tilde{\delta} \in \mathbb{Z}_M^3} c_{2M,1}^{(n)} \left(2m_1 + \sum_i \delta_i, 2m_2 + \delta_2, 2m_3 \right. \\ & \left. + \delta_3, 2\tilde{m}_1 + \sum_i \tilde{\delta}_i, 2\tilde{m}_2 + \tilde{\delta}_2, 2\tilde{m}_3 + \tilde{\delta}_3 \right), \end{aligned}$$

where we realized that $\sum_i \delta_i$ can be obtained by interchanging the summation parameter $\delta_i \mapsto \delta_i + 1$ in the cases where $\delta_2 + \delta_3 = 1$.

In order to proceed, we employ the assumption of the covariance to be translational invariant, i.e. $c_{2M,1}(m, \tilde{m}) = c_{2M,1}(m - \tilde{m})$, and go into Fourier space, where the recursion relation reads:

$$\begin{aligned} \hat{c}_{M,1}^{(n+1)}(l) &= 2^{-2D} \sum_{\delta, \tilde{\delta}, \eta \in \{0,1\}^3} \hat{c}_{2M,1}^{(n)}(l + \eta M) \\ & \exp \left(ik_{2M}(l + \eta M) \cdot \left[\begin{pmatrix} \sum_i \delta_i \\ \delta_2 \\ \delta_3 \end{pmatrix} - \begin{pmatrix} \sum_i \tilde{\delta}_i \\ \tilde{\delta}_2 \\ \tilde{\delta}_3 \end{pmatrix} \right] \right) \\ &= 2^{-2D} \sum_{\eta \in \{0,1\}^3} \hat{c}_{2M,1}^{(n)}(l + \eta M) [1 + e^{ik_{2M}(l_1 + \eta_1 M)}] [1 \\ & + e^{ik_{2M}(l_1 + l_2 + (\eta_1 + \eta_2)M)}] [1 + e^{ik_{2M}(l_1 + l_3 + (\eta_1 + \eta_3)M)}] \times [1 \\ & + e^{-ik_{2M}(l_1 + \eta_1 M)}] [1 + e^{-ik_{2M}(l_1 + l_2 + (\eta_1 + \eta_2)M)}] [1 \\ & + e^{-ik_{2M}(l_1 + l_3 + (\eta_1 + \eta_3)M)}] \\ &= 2^{-2D} \sum_{\eta \in \{0,1\}^3} \hat{c}_{2M,1}^{(n)}(l + \eta M) 2^D [1 + \cos(k_{2M}(l_1 + \eta_1 M))] \times [1 \\ & + \cos(k_{2M}(l_1 + l_2 + (\eta_1 + \eta_2)M))] [1 + \cos(k_{2M}(l_1 + l_3 \\ & + (\eta_1 + \eta_3)M))] \\ &= 2^{-D} \sum_{\delta \in \{0,1\}^3} \hat{c}_{2M,1}^{(n)}(l_1 + \delta_1 M, (l_1 + l_2) + \delta_2 M, (l_1 + l_3) + \delta_3 M) \\ & \times [1 + (-)^{\delta_1} \cos(l_1)] [1 + (-)^{\delta_2} \cos(k_{2M}(l_1 + l_2))] [1 \\ & + (-)^{\delta_3} \cos(k_{2M}(l_1 + l_3))] \quad (3.63) \end{aligned}$$

where in the last step we introduced $\hat{c}_{2M,1}(l_1, l_1 + l_2, l_1 + l_3) \equiv \hat{c}_{2M,1}(l_1, (l_1 + l_2) - l_1, (l_1 + l_3) - l_1)$ and used the periodicity $\hat{c}_{2M,a}(l + 2M) = \hat{c}_{2M,a}(l)$ to relabel $\eta \rightarrow \delta$ and $\cos(l + k_{2M}M) = -\cos(l)$ (due to $k_{2M}M = \pi$).

We observe that if the initial covariance could be written as a product of the form $c_l(l_1) c_l(l_1 + l_2) c_l(l_1 + l_3)$ then every element of the RG flow would have this property (similar to (Lang et al., 2018d)). Thus, we aim at splitting $\hat{c}_M^{(0)}$ via another application of (3.46). For this purpose, note the following identity for $\hat{c}_M^{(0)}$ from (3.45):

$$\hat{c}_{M,a}^{(0)}(l) = \frac{\hbar}{2} \Xi_a^{-1} [z_1, z_2, z_3] \Big|_{z_a = \epsilon^{-2} (2 \cos(k_M l_a) - 2), z_b \neq a = \epsilon^{-2} (2 \cos(k_M (l_a + l_b)) - 2)} \quad (3.64)$$

with

$$\Xi_a[\{z\}] := \bar{p}^2 - z_a - 2\varepsilon_M^{-2}$$

$$\sum_{b \neq a} \left[\left(\frac{\varepsilon_M^2}{2} z_a + 1 \right) \left(\frac{\varepsilon_M^2}{2} z_b + 1 \right) + \sqrt{1 - \left(\frac{\varepsilon_M^2}{2} z_a + 1 \right)^2} \sqrt{1 - \left(\frac{\varepsilon_M^2}{2} z_b + 1 \right)^2} - 1 \right].$$

Hence, with (3.46) and c'_M from (3.47) we find the desired splitting:

$$\hat{c}_{M,a}^{(0)}(l) = \frac{\hbar/2}{(4\pi i)^3} \left(\prod_{b=1}^3 \oint_{\Gamma} dz_b \right) \Xi^{-1}[\{z\}] c'_M(l_a, a; z_a) \prod_{b \neq a} c'_M(l_a + l_b, a; z_b). \quad (3.65)$$

Moreover, the recursion with $[1 \pm \cos(\dots)]$ is the same as in (Lang et al., 2018b) and thus is known to lead to the fixed point $\hat{c}_M$ from (2.38). In other words, we know to which fixed point family the flow induced by recursion (3.63) drives to. Lastly, we again restore the contour integral to take $n \geq 4$ into account and obtain the final result:

$$\hat{c}_{M,a=i}^*(l) = \frac{p^{(n-1)} \hbar}{(4\pi i)^3} \oint_{\gamma} \frac{dk_o}{k_o^{n/2}} \left(\prod_{b=1}^3 \oint_{\Gamma} dz_b \right) \Xi_a[\{z\}] \hat{c}_M(z_a, l_a) \prod_{b \neq a} \hat{c}_M(z_b, l_a + l_b). \quad (3.66)$$

Analogously, iterating the same steps for $a \neq i$ we get: (with $a \neq s \neq i$)

$$\hat{c}_{M,a \neq i}^*(l) = \frac{p^{(n-1)} \hbar}{(4\pi i)^3} \oint_{\gamma} \frac{dk_o}{k_o^{n/2}} \left(\prod_{b=1}^3 \oint_{\Gamma} dz_b \right) \Xi_a[\{z\}] \hat{c}_M(z_a, l_a) \hat{c}_M(z_i, l_a + l_i) \hat{c}_M(z_s, l_i + l_s - l_a),$$

where Ξ'_a can be obtained from a similar splitting as in (3.64).

If one performs the continuum limit $M \rightarrow \infty$, one sees that the artificial direction dependence as well as the difference between the field species a will be lost and the continuum theory agrees thus with the continuum limit from (3.56), i.e. the fixed point of the deleting kernel. In other words, the projections of same continuum theory with different coarse graining projections carrying the same label M differ. Yet, the difference is merely due to the fact that the coarse graining maps are different. The continuum theory is in both cases the same and

thus displays universality with respect to this change of the coarse graining map.⁶

4. CONCLUSIONS

In this paper we performed preliminary steps to extend the Hamiltonian Renormalization Group to Abelian gauge theories. This serves as a further step toward the construction of interacting QFTs for those systems which are subject to constraints.

When constraints are present, a possible strategy is to perform a symplectic reduction and go to the reduced phase space on which the constraints have been implemented. In general, the geometry (i.e. the symplectic structure) of the reduced phase space may be very complicated, but at least for the Gauss constraint of Abelian gauge theories the procedure is well understood: one can split the phase space in transversal and longitudinal modes and then gauge-fix the unphysical longitudinal modes. This allows to proceed with canonical quantization and renormalization along the methods for scalar fields from (Lang et al., 2018a; Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d). In a class of models that includes free Maxwell theory we performed a reduced phase space quantization obtaining a family of Fock Hilbert spaces $\mathcal{H}_M$, one for each resolution M . For this class, we could test different injections $J_{M \rightarrow 2M} : \mathcal{H}_M \rightarrow \mathcal{H}_{2M}$. It transpired that the resulting models can be understood as two decoupled field species, both of them running into their fixed point, which we knew analytically due to previous studies in (Lang et al., 2018b; Lang et al., 2018c).

The reduced phase space approach results in a renormalization flow which is very close to that of scalar fields. In order to test renormalization flows that take the vector field structure into account we considered a second class of models without Gauss constraint which includes free Proca theory. The motivation for considering generalisations of free Maxwell and Proca theory is that some of these models allow for well defined holonomy operators in the corresponding Fock representations at the price of losing Poincaré invariance. We consider these models as mere toy models for quantum gravity theories (Thiemann, 1998; Liegener and Thiemann, 2016) that are based on Hilbert space representations with both well defined holonomy operators and Hamiltonians without breaking symmetries. In particular we are thinking about discretisations of the Hamiltonian operators studied in this paper using holonomies themselves which would simulate the proposal of (Thiemann, 1998; Liegener and Thiemann, 2016). In a future publication (Liegener and Thiemann, 2020) we will also aim at

⁶Of course, this does not guarantee universality under *any* changes of coarse graining map—a property which cannot be true in general. However, it is possible to show that for the Hamiltonian RG formulation all coarse graining maps are unitary equivalent, albeit the initial discretisations may change under said unitary map, see (Bahr and Liegener, 2020) for all details.

imposing the Gauss constraint after quantization. The idea of introducing a “smoothing” operator into the Hamiltonian in order to allow for holonomy operators in the corresponding Fock representation is in some sense dual to the idea of using smoothed form factors studied in (Varadarajan, 2000). Note also that we could have made our deformation of Proca or Maxwell theory phenomenologically more interesting by changing $\omega^2 \mapsto -\Delta + p^2 ((-\Delta + p^2)/\mu^2)^n$ with p arbitrarily small but finite and μ arbitrarily large but finite so that the Lorentz violation will only manifest itself at energies above μ . Even in this case holonomies are still well-defined operators and the presented strategy to determine the fixed point remains the same.

We chose two different coarse graining maps in order to understand how stable the fixed points of the theory are under changes of the injection maps. Both maps—deleting and filling kernel—are mathematical well-defined, but the level of experience that we have for them differs: the deleting kernel has already been actively studied in the literature and found application in the non-Abelian case of Loop Quantum Gravity where it enabled the construction of an inductive limit Hilbert space. Spin networks (a possible basis of said Hilbert space) carry distributional excitations such that a smooth quantum geometry can only be obtained by distributions on the Hilbert space. Conceptually, reobtaining smooth geometry could be easier when working with the filling kernel, as it excites all edges as the resolution increases. However, extensive studies on the latter kernel have not been performed as of today.

Both maps employ discretisations of the spatial manifold where the fields are smeared along edges of a cuboidal lattice. Choosing such cubic lattices might at first glance look like a restriction of the theory since it gives rise to the so-called “staircase problem” (Sahlmann et al., 2001): albeit square lattices suffice to separate the points in phase space as M gets large, one does not have access to “45” degree line observables at any finite resolution. Yet, the continuum theory does allow considering holonomy operators along such curves which are not straight. This stresses the point that the lattice just serves to construct the continuum theories, all other investigations have to start from there.

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We demonstrated for our model classes that the relevant fixed points can be found for the filling as well as for the deleting kernel. Due to the fact that the discretisations were expressed in terms of smearings with form factors, the investigation exploited many of the findings from previous applications of the Hamiltonian Renormalization Group in (Lang et al., 2018b; Lang et al., 2018c; Lang et al., 2018d). Finally, we found analytically closed formulas for the respective fixed points and saw that the Hamiltonian renormalization leads to reliable results.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The contribution among the authors is 50-50. TT contributed among others the proposal for the toy models and the proof of them being well-defined. KL provided explicit computations of the fixed point analysis for the various coarse graining maps.

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Quantum Gravity: A Fluctuating Point of View

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In this contribution, we discuss the asymptotic safety scenario for quantum gravity with a functional renormalization group approach that disentangles dynamical metric fluctuations from the background metric. We review the state of the art in pure gravity and general gravity–matter systems. This includes the discussion of results on the existence and properties of the asymptotically safe ultraviolet fixed point, full ultraviolet-infrared trajectories with classical gravity in the infrared, and the curvature dependence of couplings also in gravity–matter systems. The results in gravity–matter systems concern the ultraviolet stability of the fixed point and the dominance of gravity fluctuations in minimally coupled gravity–matter systems. Furthermore, we discuss important physics properties such as locality of the theory, diffeomorphism invariance, background independence, unitarity, and access to observables, as well as open challenges.

Keywords: quantum gravity, asymptotic safety, background independence, diffeomorphism invariance, ward identities

1 INTRODUCTION

One of the major challenges in theoretical physics is the unification of the standard model of particle physics (SM) with quantum gravity. Based on the classical Einstein–Hilbert action, gravity is perturbatively nonrenormalizable and hence cannot be expanded about a vanishing gravitational coupling, the Newton coupling. A very promising way out has been proposed by Weinberg [1], the *asymptotic safety scenario*. It draws from the theory of critical phenomena developed for investigating the phase structure of condensed matter and statistical systems. In the language of critical phenomena, the standard perturbation theory about a vanishing Newton coupling is an expansion about the free, Gaussian fixed point of the theory and fails since this fixed point is ultraviolet (UV)-repulsive in the relevant couplings. In turn, the asymptotic safety scenario builds upon the conjecture that quantum gravity also exhibits a nontrivial UV fixed point, the Reuter fixed point. This asymptotically safe fixed point should exhibit a finite-dimensional critical hypersurface, which renders the theory finite and predictive even beyond the Planck scale.

The method of choice for respective investigations is the renormalization group. Most investigations of asymptotically safe gravity have been performed with the functional renormalization group (fRG) in its form for the effective action [2]. The fRG approach to quantum gravity has been initiated by the seminal work [3], where the UV fixed point has been studied in the Einstein–Hilbert truncation. In this approximation, one retains only two couplings, the Newton coupling G_N and the cosmological constant Λ . Already this basic truncation exhibits a UV fixed point in four dimensions [3, 4]. This exciting finding has triggered a plethora of works for

asymptotically safe gravity with and without matter. (We refer the reader to the textbooks [5, 6] and reviews [7–15]. For very recent accounts of the challenges for asymptotically safe gravity, see 16, 17. For generic reviews on the fRG, we refer to 18–27.)

The fRG approach to gravity centers around the quantum effective action of the theory $\Gamma[\bar{g}_{\mu\nu}, h_{\mu\nu}]$, the quantum analog of the classical action. Here, $\bar{g}_{\mu\nu}$ is a generic metric background and the graviton field $h_{\mu\nu}$ accounts for quantum fluctuations about this background. The computation of the effective action $\Gamma[\bar{g}_{\mu\nu}, h_{\mu\nu}]$ is tantamount to that of the path integral: the n -point correlation functions of the dynamical fluctuation field h are given by n derivatives of the effective action with respect to the correlation field, evaluated on the equations of motion, $h = 0$ and $\bar{g} = \bar{g}_{\text{EoM}}$, that is, on-shell. These correlation functions are nothing but the moments of the path integral and carry the dynamics of the quantum theory.

This seemingly introduces a background dependence of the approach. However, the approach has inherent on-shell background independence, also related to physical diffeomorphism invariance. Indeed, the background effective action $\Gamma[g_{\mu\nu}] = \Gamma[g_{\mu\nu}, 0]$ is diffeomorphism-invariant. The latter properties are the backbones of any quantum gravity approach, and their realization even within approximations is chiefly important.

The present review outlines the properties and results of the fRG approach to asymptotically safe quantum gravity in terms of *background* and *fluctuation* correlation functions of gravitons, shortly baptized the *fluctuation approach* to gravity. This approach is based on the observation that the dynamics of quantum gravity is encoded in the correlation functions of the fluctuation field h . Reliable computations of observables can only be done from these correlation functions. This situation calls for a systematic improvement of the standard background-field approximation. In this approximation, the correlation functions of the background metric and the fluctuation field are identified. We refrain from going into more details here, the underlying assumptions and challenges are discussed in **Sections 5 and 6**.

The fluctuation approach resolves these differences, and by now, it has matured enough to host a large number of results: this includes investigations of the Reuter fixed point in pure gravity in a rather elaborate truncation within a vertex expansion with momentum-dependent two-, three-, and four-point functions; the computation of the background-effective action for backgrounds with constant curvature; investigations of the stability of general gravity–matter systems; investigation of convergence properties of the expansion (*apparent convergence*); and a potential close perturbativeness of the asymptotically safe UV regime (*effective universality*). (We refer the reader to **Section 8** for an explanation of the terminology and respective results.)

In **Section 2**, we discuss the general quantum field theory setting of quantum gravity, which we use for the fluctuation approach. This includes a discussion of the necessary gauge fixing and background independence of the approach. In **Section 3**, we discuss general parametrizations of the full metric in terms of a metric background and a fluctuation field. The preparation in

Sections 2 and 3 allows us to introduce the fRG approach to quantum gravity in **Section 4** as well as discussing the standard approximation used in the field, the background field approximation, in **Section 5**. The symmetry identities that relate the dynamical correlation functions of the fluctuation field and those of the background metric are discussed in **Section 6**. These symmetry identities imply the necessity to go beyond the background field approximation, and thus, we detail the fluctuation approach in **Section 7**. With the preparation of the sections before, we discuss the results of the fluctuation approach in **Section 8** and close with a short conclusion and outlook in **Section 9**.

2 QUANTUM FIELD THEORY APPROACH TO QUANTUM GRAVITY

The present contribution discusses the advances and open problems of a quantum field theory approach to quantum gravity that is based on the computation of metric correlation functions or, more generally, correlation functions of operators in quantum gravity. Formally, such an approach is based on the existence of a path integral for quantum gravity, for example, defined by the integration over the space of all metrics $\{g_{\mu\nu}\}$ with a specific *classical* action for gravity S_{grav} , a standard choice being the Einstein–Hilbert action,

$$S_{\text{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^4x \sqrt{g} (2\Lambda - R), \quad (1)$$

with the abbreviation $g = \det g_{\mu\nu}$. In **1**, we have introduced the Newton coupling G and the cosmological constant Λ . R stands for the Ricci scalar. In most works in the fRG approach, the theory is considered in its Euclidean version, which is indicated here by the missing minus sign in the square root of the determinant. The expectation value of a diffeomorphism invariant operator $\mathcal{O}[g_{\mu\nu}]$ is formally given by

$$\langle \mathcal{O}[\hat{g}_{\mu\nu}] \rangle = \frac{\int \mathcal{D}\hat{g}_{\mu\nu} \mathcal{O}[\hat{g}_{\mu\nu}] e^{-S_{\text{grav}}[\hat{g}_{\mu\nu}]}{\int \mathcal{D}\hat{g}_{\mu\nu} e^{-S_{\text{grav}}[\hat{g}_{\mu\nu}]}}, \quad (2)$$

Here and in the following, $\hat{}$ indicates the fields that are integrated over. The formal definition **2** faces several problems. Some of them are standard problems of the quantization of gauge theories, and some of them are specific to quantum gravity. The latter problems include, for example, the lack of perturbative renormalizability of gravity for S_{EH} [28–31], the apparent unitarity problems for higher derivative gravity a la Stelle [32–34], and the question whether the integration measure $\mathcal{D}\hat{g}$ includes a sum over all topologies [35]. The latter question is also an eminent one in lattice gravity (see, e.g., 36–42). Note in this context that a general measure $\mathcal{D}\mu(\hat{g})$ can always be absorbed with a change of the gravity action in **2**,

$$\mathcal{D}\mu(\hat{g}) = \mathcal{D}\hat{g}_{\mu\nu} e^{-\Delta S_{\text{grav}}[\hat{g}_{\mu\nu}]}, \quad (3)$$

with a potentially nonlocal action ΔS_{grav} . In the fRG approach, the task of a finite definition of **2** and its computation is turned into

the task of solving a flow equation for the quantum effective action $\Gamma[\bar{g}, \phi]$. Here, $\bar{g}_{\mu\nu}$ is the background or reference metric, and ϕ are fluctuation fields, the expectation values of the fluctuation field operators $\hat{\phi}$. The latter includes the fluctuation field $\hat{h}_{\mu\nu}$ of the metric, $g_{\mu\nu} = g_{\mu\nu}(\bar{g}, h)$, and potential matter fields ϕ_{mat} and auxiliary fields such as the ghosts c_μ of the gauge fixing in gravity,

$$\phi = \langle \hat{\phi} \rangle, \quad \phi = (h_{\mu\nu}, c_\mu, \bar{c}_\mu, \phi_{\text{mat}}). \quad (4)$$

In the case of further gauge fields, one may also use background fields for the gauge fields, which are suppressed here for the sake of convenience. A reparameterization $g_{\mu\nu}(\bar{g}, h)$ seemingly introduces a background-metric dependence of the formulation. This is common to many approaches to quantum gravity due to the necessity of defining metric fluctuations. Accordingly, the question of background independence of the present approach is an eminent one and is discussed later. Here, we only want to mention the most common split between the background metric and the fluctuation field, the *linear* split,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (5)$$

This split also underlies most of the results discussed in **Section 8**. Note that from now on the lowering and raising of indices is done with the background metric $\bar{g}$, if not specified otherwise. **Equation 5** emphasizes one specific problem with the background field approach in quantum gravity: while $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are metrics, their difference $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ is not. Indeed $h_{\mu\nu}$ has no geometrical meaning at all. This is discussed in more detail in **Section 3**.

2.1 Gauge Fixing

In gauge theories such as gravity with the diffeomorphism (gauge) group or the simpler case of non-abelian gauge theories, the practical computation of observables **2** faces the gauge group redundancy in the path integral measure. While this redundancy is a finite-dimensional one within discrete lattice formulations, it is an infinite-dimensional one in functional approaches based on graviton correlation functions. In particular, it prohibits the straightforward definition of the propagator, which is key in most functional approaches.

Therefore, most of the latter approaches require a gauge fixing. (For a brief discussion of gauge-invariant functional approaches, see **Section 6.3**.) Put differently, we have to choose a parametrization of the theory. Typically, this is done with a linear gauge fixing for the fluctuation field $h_{\mu\nu}$ that carries the metric degrees of freedom,

$$S_{\text{gf}}[\bar{g}, h] = \frac{1}{2\alpha} \int d^4x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu. \quad (6)$$

A common gauge fixing condition F_μ is given by

$$F_\mu[\bar{g}, h] = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\beta}{4} \bar{\nabla}_\mu h^\nu_\nu, \quad (7)$$

where $\bar{\nabla}$ is the covariant derivative with the background metric $\bar{g}_{\mu\nu}$. The gauge fixing **7** is introduced in the path integral with the Faddeev–Popov trick and the Jacobi determinant of the respective reparameterization. The Faddeev–Popov determinant Δ_{FP} can be rewritten in terms of a fermionic path integral with the ghost fields c_μ and $\bar{c}_\mu$. The ghost action related to **7** reads

$$S_{\text{gh}}[\bar{g}, \phi] = \int d^4x \sqrt{\bar{g}} \bar{c}^\mu M_{\mu\nu} c^\nu, \quad (8)$$

with the Faddeev–Popov operator

$$M_{\mu\nu} = \bar{\nabla}^\rho (g_{\mu\nu} \nabla_\rho + g_{\rho\nu} \nabla_\mu) - \frac{1+\beta}{2} \bar{g}^{\sigma\rho} \bar{\nabla}_\mu g_{\nu\sigma} \nabla_\rho. \quad (9)$$

Again, Δ_{FP} is the covariant derivative with the background metric $\bar{g}_{\mu\nu}$, while ∇ is that with the full metric $g_{\mu\nu}$. Note that $M_{\mu\nu}$ is linear in the fluctuation field h . The background metric $\bar{g}_{\mu\nu}$ cannot be avoided, and both gauge fixing and ghost action depend on it. This implies that also the quantum effective action depends on both metrics, the background metric $\bar{g}_{\mu\nu}$ and the full metric $g_{\mu\nu}(\bar{g}, h)$, as we shall see later. Note however that the correlation functions of diffeomorphism-invariant operators and the solutions to the quantum equations of motion do not depend on the gauge fixing. Hence, they are background-independent as explained below.

2.2 Background Independence

Background independence of the construction is more than a formal property to aim for. We briefly recollect the standard arguments for background independence in the background field approach to quantum gauge field theories. We first restrict ourselves to pure gravity. Seemingly, background dependence of the path integral is introduced by gauge fixing such as **6** and the respective Faddeev–Popov determinant Δ_{FP} . The latter is the Jacobian of the reparameterization of the path integral in terms of gauge-fixed fields. We emphasize that the gauge fixing should be rather understood as a specific choice of field coordinates in the configuration space that facilitates the integration. The Faddeev–Popov trick is nothing but a convenient way to introduce these coordinates. In any case, it leads us to the expectation values of diffeomorphism-invariant operators defined in **2** for pure gravity with a path integral with the gauge-fixed action,

$$\langle \mathcal{O}[\hat{g}] \rangle = \int \mathcal{D}\hat{\phi} e^{\int d^4x \sqrt{\hat{g}} \left(-S_{\text{gf}}[\hat{g}, h] - S_{\text{gh}}[\hat{g}, \phi] - S_{\text{grav}}[\hat{g}] \right)} \mathcal{O}[\hat{g}] e^{-S_{\text{grav}}[\hat{g}]} \quad (10)$$

Note that integration over the diffeomorphism group (from the Faddeev–Popov trick) has been factored out in the numerator and denominator. This relies on the diffeomorphism-invariance of S_{grav} , $\mathcal{O}[\hat{g}]$, and $\mathcal{D}\hat{g}$. The full integration measure $\mathcal{D}\hat{\phi}$ in **10** now also includes the ghost fields, $\hat{\phi} = (\hat{h}_{\mu\nu}, \hat{c}_\mu, \hat{\bar{c}}_\mu)$. Naturally, the right-hand side in **10** is independent of the background field as the left-hand side trivially is (see **2**). This background-metric independence is captured in the Nielsen or split Ward identity derived from taking a $\bar{g}_{\mu\nu}$ derivative of **10**. Typically, one also

subtracts the Dyson–Schwinger equation for $\langle \mathcal{O} \rangle$, which reads schematically

$$\int \mathcal{D}\hat{h} \frac{\delta}{\delta \hat{h}} [e^{-S_{\text{gf}} - S_{\text{gh}}} \mathcal{O} e^{-S_{\text{grav}}}] = 0. \quad (11)$$

This leads us to the Nielsen identity for general diffeomorphism-invariant operators $\mathcal{O}[g]$ with

$$\left\langle \mathcal{O}[\hat{g}] \left(\frac{\delta}{\delta \hat{g}} - \frac{\delta}{\delta \hat{h}} \right) (S_{\text{gh}} + S_{\text{gf}}) [\hat{g}, \hat{h}] \right\rangle = 0. \quad (12)$$

If solving the path integral within approximations, the check of the Nielsen identity 12 is crucial as it carries the physical background independence.

The identity 12 constitutes infinitely many relations for diffeomorphism-invariant correlation functions and can be rephrased in terms of derivatives of the effective action. Correlation functions are conveniently derived from the generating functional $Z[\bar{g}, J]$ obtained by adding source terms for the fluctuation fields to the exponent in the path integral,

$$Z[\bar{g}, J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\hat{\phi} e^{-S - S_{\text{gf}} - S_{\text{gh}} + \int d^4x \sqrt{\bar{g}} J^a \hat{\phi}_a}, \quad (13)$$

where $J = (J_{h_{\mu\nu}}, J_{c_\mu}, J_{\bar{c}_\mu}, J_{\text{mat}})$ and the normalization $\mathcal{N}$ is the denominator in 10. Lowering and rising the field indices are done with the metric γ^{ab} in field space. (For details, see Supplementary Material.)

In 13, the action $S = S[\bar{g}, \hat{\phi}]$ is the “classical” action of the gravity–matter system under consideration. The gauge fixing action S_{gf} and the ghost action S_{gh} of the full gravity–matter system may include further gauge fixings of gauge fields. Note that for gravity–matter systems, the “classical” action may not be based on the Einstein–Hilbert action of general relativity as discussed before. More generally also, the matter part may not simply be that of a standard renormalizable QFT in the presence of a dynamical metric background.

The generating functional $Z[\bar{g}, J]$, or rather the Schwinger functional $\log Z[\bar{g}, J]$, generates connected n -point correlation functions of the fluctuation field with

$$\frac{\delta^n \log Z[\bar{g}, J]}{\delta J^{a_1} \dots \delta J^{a_n}} = \langle \hat{\phi}_{a_1} \dots \hat{\phi}_{a_n} \rangle_{\text{con}}, \quad (14)$$

where the indices a_i stand for Lorentz and internal indices as well as species of fields. The subscript con in 14 indicates the connected part of the correlation function. We have included a factor of $1/\sqrt{\bar{g}}$ in the definition of the functional derivative (see Supplementary Material). This cancels the $\sqrt{\bar{g}}$ factor in the space-time integral in the source term of 13. If instead, we had used $\sqrt{\bar{g}}$ in the source term, derivatives with respect to the current J would generate infinite-order correlation functions.

Note that the generating functional 13 can be expressed with the right-hand side of 10 with the operator $\mathcal{O} = e^{\int d^4x \sqrt{\bar{g}} J^a \hat{\phi}_a}$. However, this operator is neither diffeomorphism-invariant nor background-independent. For that reason, it cannot be mapped into a manifestly background-independent form such as 10. For $J = 0$, we have $\mathcal{O} = 1$ and $Z = 1$, which is trivially

background-independent. Accordingly, for $J \neq 0$, the gauge-fixed generating functional $Z[\bar{g}, J]$ is background-dependent, as is the effective action $\Gamma[\bar{g}, \phi]$,

$$e^{-\Gamma[\bar{g}, \phi]} = \frac{1}{\mathcal{N}} \int \mathcal{D}\hat{\phi} e^{-(S + S_{\text{gf}} + S_{\text{gh}}) + \int d^4x \sqrt{\bar{g}} (\hat{\phi}_a - \phi_a) \frac{\delta \Gamma}{\delta \phi_a}}. \quad (15)$$

For the relation 15, we have used that the effective action Γ is the Legendre transformation of the Schwinger functional $W[\bar{g}, J] = \log Z[\bar{g}, J]$. This leads to

$$J^a[\bar{g}, \phi] = (-1)^{s_a} \frac{\delta \Gamma[\bar{g}, \phi]}{\delta \phi_a}, \quad (16)$$

with the fermion number $s_a = 1$ for fermions and $s_a = 0$ for bosons. Then, background-independence is achieved on the fluctuation field equations of motion (EoM) for $J[\bar{g}, \phi] = 0$. The on-shell vanishing of the currents entails that all diffeomorphism-invariant quantities are background-independent on-shell, and this independence is carried by 12.

An important consequence of background-independence is the equivalence of the solutions $\bar{g}_{\text{EoM}}[\phi]$ to the fluctuation field EoM and the background field EoM,

$$\frac{\delta \Gamma[\bar{g}_{\text{EoM}}^{\text{fluc}}, \phi]}{\delta h_{\mu\nu}} = 0 \quad \leftrightarrow \quad \frac{\delta \Gamma[\bar{g}_{\text{EoM}}^{\text{back}}, \phi]}{\delta \bar{g}_{\mu\nu}} = 0, \quad (17)$$

with

$$\bar{g}_{\text{EoM}}^{\text{fluc}} = \bar{g}_{\text{EoM}}^{\text{back}} = \bar{g}_{\text{EoM}}. \quad (18)$$

If the fluctuation EoM holds, the current J is vanishing, and hence, the background EoM is nothing but the Nielsen identity 12. In turn, if the background EoM holds, the current J necessarily vanishes.

3 FIELD PARAMETRIZATIONS

So far, we have not specified the relation between the background metric and the full metric $g(\bar{g}, h)$, which defines the role of the fluctuation field h . While most of the computations are done within the linear split 5, it is worth discussing the general case. This not only allows us to achieve a better understanding of the linear split but also allows us to discuss the challenges for manifestly diffeomorphism-invariant formulations.

The importance of the different splits for the path integral has been already mentioned in the context of the path integral measure (see the introduction of Section 2 around 2). In the flow equation approach to quantum gravity detailed in the next section (Section 4), the discussion of the path integral measure translates into that of the ordering of fluctuations: the fRG approach to quantum gravity is based on a Wilsonian successive integrating out of quantum fluctuations. In its form of a flow equation for the quantum effective action, $\Gamma[\bar{g}, \phi]$ has a simple form in terms of the full field-dependent fluctuation field propagator $G[\bar{g}, \phi]$ of the theory (see 30). This is the connected part of the two-point function of the fluctuation field,

$$G[\bar{g}, \phi](x, y) = \langle \hat{\phi}(x) \hat{\phi}(y) \rangle - \phi(x) \phi(y). \quad (19)$$

The definition 19 requires a gauge fixing (or reparameterization), as discussed in the previous section. Moreover, the Wilsonian cutoff regularizes the spectrum of the propagator. Consequently, the fRG approach crucially depends on the split of the full metric g into the background metric $\bar{g}$ and the fluctuation field h for two reasons:

- Ordering of fluctuations:** The quantum fluctuations of the fluctuation field h are successively integrated out and are ordered in terms of the background covariant Laplacian. Therefore, the meaning of this ordering depends on the chosen split.
- Relevance of higher order correlations:** The physics included with higher order correlation functions crucially depends on the chosen split. Thus, a different split orders quantum fluctuations differently. This leads to potentially qualitative differences for the convergence of a given approximation scheme.

In this section, we briefly introduce and discuss the different splits considered so far in the fRG approach to asymptotically safe quantum gravity.

3.1 Linear Split

We begin with the standard and simplest split, the linear split (see also 5). It is given by

$$g = \bar{g} + h, \quad \text{with} \quad \mathcal{D}\bar{g} = \mathcal{D}h. \quad (20)$$

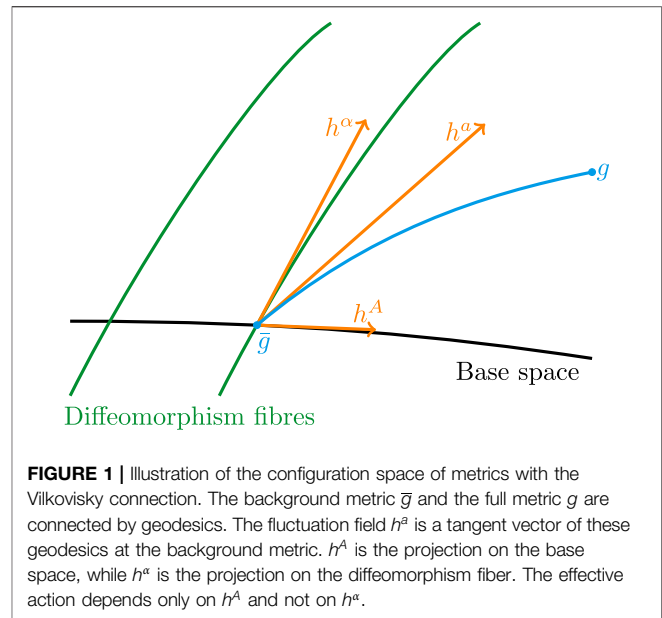
The Jacobian of this transformation is unity, and the path integral measures agree. As mentioned before, with such a definition, the fluctuation field $h = g - \bar{g}$ is not a metric and has no geometrical meaning in the configuration space of metrics. Still, it is the natural choice as it facilitates explicit computations and the implementation of the quantization of the theory for a given classical action S_{grav} on the space of metrics. Still, its lack of a geometrical interpretation makes it difficult to discuss the reparameterization invariance of the theory and the consequences of background independence. (For more details, see Section 6.) These intricacies have led to more elaborated splits based on the fiber bundle structure of the configuration space of metrics.

3.2 Exponential Split

In recent years, the exponential split has attracted some attention [43–59]. It is given by

$$g = \bar{g} \exp h, \quad \text{with} \quad \mathcal{D}\bar{g} = J_{\text{exp}} \mathcal{D}h. \quad (21)$$

The full metric is proportional to the exponential of the fluctuation field h indicating a Lie algebra nature of the fluctuation field h . Note that the parametrization 21 restricts the metric g , and in particular it does not allow for signature changes. Therefore, it is potentially not a reparameterization of the path integral in terms of integration over all metrics but a definition of another candidate for quantum gravity. Moreover,



the assumption may change the integration. In summary, it is unclear whether a path integral with the exponential split and the measure $\mathcal{D}h$ describes the same quantum theory as that with the measure $\mathcal{D}\bar{g}$. This parametrization is also linked to unimodular gravity (see, e.g., [60–66]).

3.3 Geometrical Split

We briefly describe the geometrical approach to quantum gravity pioneered by Vilkovisky and DeWitt (see, e.g., [67–70]). In the fRG approach to gravity, it has been discussed in 21, 71–75. It is a general framework, and all parametrizations used in the literature can be understood as different choices for the geometrical structure of the configuration space of metrics $g_{\mu\nu}$. This also allows for a better understanding of the Wilsonian integrating out of quantum fluctuations underlying the different splits.

In the linear split, as discussed in Section 3.1, the fluctuation field h neither is a metric nor does it have a geometrical interpretation in the configuration space Φ . In turn, in the geometrical approach, the fluctuation field is constructed such that it has a geometrical meaning. The background metric and the full metric are linked by geodesics with respect to a given connection in the configuration space. The Vilkovisky connection Γ_V is a specifically useful one: it is constructed with the demand of maximal orthogonality between the diffeomorphism fiber in the configuration space and the base space. If such disentanglement is achieved, the path integral and the effective action only depend on the propagating degrees of freedom and the gauge redundancies are completely removed. This leads to the following conditions,

$$\Gamma_{VBC}^A = \Gamma_{gBC}^A, \quad \Gamma_{VB\gamma}^A = 0, \quad \Gamma_{V\beta\gamma}^A = 0, \quad (22)$$

where

$$\Gamma_{gBC}^A = \frac{1}{2} g^{AD} (g_{DB,C} + g_{DC,B} + g_{BC,D}), \quad (23)$$

is the Riemannian metric on the quotient space $\Phi/\mathcal{G}$, where $\mathcal{G}$ is the group of diffeomorphisms. This quotient space is labeled with capital Latin letters $A, B, C, \dots$, while the diffeomorphism fiber is labeled with Greek letters $\alpha, \beta, \gamma, \dots$. The full space is labeled with small Latin letters $a, b, c, \dots$ (For further details on the notation and the setup in the context of RG gravity, see e.g., 72.)

The background metric $\bar{g}$ and the full metric g are connected by a geodesic. With the Vilkovisky connection, the fluctuation field is a tangent vector on this geodesic at the background metric (Gaussian or geodesic normal coordinates). This is illustrated in **Figure 1** and leads to

$$g = \bar{g} + h - \frac{1}{2}\Gamma_V h^2 + \mathcal{O}(h^3), \quad \text{with} \quad \mathcal{D}\bar{g} \simeq \mathcal{D}h. \quad (24)$$

The relation between g and $\bar{g}$ is nonpolynomial. Still, the Jacobian does not depend on the fluctuation field, and we have dropped it in 24. In this setting, it can be shown that the effective action Γ only depends on the projection h^A of the tangent vector h^a onto the base space of the fiber bundle: $\Gamma = \Gamma[\bar{g}, h^A]$. In turn, the projection of h^a onto the diffeomorphism fiber, h^α , drops out. Hence, the effective action is diffeomorphism-invariant as h^A is a diffeomorphism scalar. Trivially, an infrared (IR) regularization of the h^A -path integral is diffeomorphism-invariant.

We close this section with some remarks on the implications of such a geometrical setup for “physical” gauge fixings, linear and exponential splits, and locality. The geometrical construction comes as close as possible to the definition of the configuration space of a gauge theory in terms of “physical” gauge-invariant fields and correlation functions. Such a parameterization is tantamount to a specific gauge fixing as already mentioned in **Section 2.2**. We may call such a gauge fixing “physical,” having in mind that it removes most of the redundancies related to the gauge group, in gravity that related to the diffeomorphism group. Note however that the terminology “physical gauge fixing” is not well-defined and also used differently in other contexts. In non-abelian gauge theories, the projection is unique and singles out the Landau–DeWitt gauge as the “physical” one. In gravity, one is left with a one-parameter family of gauges with the gauge fixing parameter β (see 7).

It is worth emphasizing that a gauge fixing condition for the geometrical field (or Gaussian normal field) h^a is different from that for the fluctuation field h in the linear split. Only for specific choices of the latter, the maximal disentanglement of the geometrical construction is manifestly obtained. We also remark that the linear split is obtained by using a vanishing connection, hence entirely ignoring the geometrical structure of the configuration space. The exponential split simply uses the Riemannian part Γ_g of the configuration space, hence ignoring the diffeomorphism group.

Finally, the geometrical construction with the Vilkovisky connection is highly nonlocal in configuration space, one of the ensuing problems being caustics and Gribov copies. This also raises the question of locality in the configuration space and that of momentum locality of the correlation functions of the geometrical fluctuation field h . The latter is discussed in detail in

Section 7.4. Both locality issues highlight the challenges for manifest gauge- or diffeomorphism-invariant functional approaches to quantum gravity.

4 FLOW EQUATION FOR GRAVITY

With the quantum field theory approach to quantum gravity outlined in the last sections, we are now in the position to discuss the flow equation approach to gravity. (For reviews, see 7–17, and for generic fRG reviews, see 18–27.) As already mentioned in the introduction of **Section 3**, the fRG approach to gravity is based on a successive integrating out of quantum fluctuations. Typically, this is done with an ordering of quantum fluctuations in momentum space: the regulator introduces a suppression of low-momentum fluctuations below an IR cutoff scale $p^2 \lesssim k^2$, and one RG step with $k \rightarrow k - \Delta k$ relates to the integration of momentum modes $p^2 \approx k^2$. In gravity, the implementation of such a momentum cutoff necessitates the choice of a background metric $\bar{g}$, and the (covariant) momenta are those related to the covariant Laplacian in the background metric, $\Delta_{\bar{g}}$, with the spectral values $p_{\bar{g}}^2$.

Remarkably, the flow equation is insensitive to field reparameterizations of quantum gravity discussed in the last section or even physically different formulations: For the derivation, let us assume that a finite generating functional for correlation functions of the fluctuation field is given. In terms of a path integral, this is given by 13 with an assumed diffeomorphism-invariant regularization and renormalization procedure. More generally, such a *finite* generating functional is given by its defining property 14 under the assumption that these correlation functions are *finite*. Then, the flow equation can be readily derived without the necessity of referring to a specific representation of $Z[\bar{g}, J]$ such as the path integral. (For detailed discussion, see 76.) The correlation functions of h depend on $\bar{g}$, as does the generating functional for $J \neq 0$ via the gauge fixing (see **Section 2**).

The flow equation for the effective action is derived from the IR regularized generating functional,

$$Z_k[\bar{g}, J] = \exp\left(-\int d^4x \sqrt{\bar{g}} \frac{\delta}{\delta J^a} R_k^{ab} \frac{\delta}{\delta J^b}\right) Z[\bar{g}, J], \quad (25)$$

with a $\bar{g}$ -dependent IR regulator R_k . Typically, the background dependence enters the regulator via a background Laplacian and background covariant derivatives. In flat space, the eigenvalues of the Laplacian are just momentum squared, p^2 . As already discussed above, the regulator suppresses then IR momentum modes with $p^2 \lesssim k^2$. In turn, UV momentum modes with $p^2 \gtrsim k^2$ propagate freely, and the generating functional includes all quantum contributions generated by these modes.

It is convenient to write the regulator R_k in terms of the classical or quantum dispersion of the field at hand,

$$R_k^{ab}(p) = T_k^{ab}(p) r_k(x), \quad \text{with} \quad x = \frac{p^2}{k^2}, \quad (26)$$

where momentum-squared is counted in cutoff units. In these units, the IR regime is given by $x \lesssim 1$ and the UV regime by $x \gtrsim 1$. The tensor part T_k^{ab} of the regulator is proportional to the classical or quantum dispersion of the field. Classically, it is the second derivative of the action with respect to the fields ϕ^a and ϕ^b , that is, $(S^{(2)})^{ab}(p)$. It carries the kinetic information about the field whose propagation is regularized. In turn, the dimensionless *shape function* r_k specifies how the propagation is regularized. In most cases, the latter part is chosen such that the physical cutoff scales agree for all fields. This is typically achieved with identical (e.g., for several scalar or bosonic fields) or related shape functions (e.g., for scalars and Dirac fermions). It can be shown that such a choice also improves the convergence of generic expansion schemes [21, 77]. Moreover, r_k has to be chosen such that the IR suppression of momentum modes and the UV decay of the regularization are guaranteed. These properties lead to the following asymptotics of the regulator shape function,

$$\lim_{x \rightarrow 0} r_k(x) \rightarrow \infty, \quad (27a)$$

$$\lim_{x \rightarrow \infty} r_k(x) = 0. \quad (27b)$$

The first limit, 27a, guarantees the IR suppression of momentum modes. For example, for a scalar field in d dimensions with a quadratic dispersion $\propto p^2$, a regulator shape function $r_k(x \rightarrow 0) = 1/x$ introduces a low momentum mass k^2 for this field. Indeed this is the common choice for the IR limit, but more singular choices work as well. Eq. 27a also entails that for $k \rightarrow \infty$, all momentum modes are suppressed and the theory approaches the UV-scaling regime. For asymptotically free theories, this is the classical theory, and for asymptotically safe theories, this is the nontrivial quantum UV theory.

The second limit, 27b, guarantees that the UV behavior of the theory is unchanged by the IR regularization. We shall see below that the limit in 27b has to be approached sufficiently fast. In our example of a scalar field in d dimensions, the regulator shape function has to decay with at least $\min((1/x)^{d/2}, 1/x)$ for rendering the IR flows finite. This is also discussed later in more detail below 39. Note that the latter limit is that of a mass or Callan–Symanzik cutoff. Then, changing k changes a relevant parameter of the theory and hence changes the theory at all scales. Accordingly, the Callan–Symanzik cutoff is not a *local* momentum cutoff. The limit 27b also has another implication: for $k \rightarrow 0$, the limit 27b holds for all momenta and the cutoff is removed from the theory. We remark that it is precisely this property, which is at stake for the Callan–Symanzik cutoff and similar ones.

Subject to the existence of a finite full generating functional $Z[\bar{g}, J]$, the regularized generating functional $Z_k[\bar{g}, J]$ is also finite (and smaller than $Z[\bar{g}, J]$). The flow equation for the Schwinger functional $W_k[\bar{g}, J] = \log Z_k[\bar{g}, J]$ is derived by taking the logarithmic k -derivative of 25. Schematically, this leads us to

$$\partial_t W_k[\bar{g}, J] = -\frac{1}{2} \text{Tr} \left(\frac{\delta^2 W_k}{\delta J^2} + \frac{\delta W_k}{\delta J} \frac{\delta W_k}{\delta J} \right) \partial_t R_k, \quad (28)$$

where the RG-“time” t is defined with $t = \ln k/\Lambda$, and Λ is some reference scale Λ . The trace sums over position space, Lorentz and internal indices, and species of fields. For the sake of a concise

presentation, we have suppressed all space-time and internal indices including species of fields. We emphasize that for the explicit form of 28, the order of derivatives is important as J contains fermionic currents.

The term in parenthesis in 28 is nothing but the full two-point correlation functions of the theory: the first term is the connected part, that is, the scale-dependent propagator $G_k[\bar{g}, \phi]$ of the theory (see 19). The second term is simply ϕ^2 , the disconnected part. The scale-dependent effective action $\Gamma_k[\bar{g}, \phi]$ is defined as the modified Legendre transformation of the Schwinger functional,

$$\Gamma_k[\bar{g}, \phi] = \int d^4x \sqrt{\bar{g}} J^a \phi_a - W_k[\bar{g}, J] - \frac{1}{2} \int d^4x \sqrt{\bar{g}} \phi_a R_k^{ab} \phi_b, \quad (29)$$

where $J = J[\bar{g}, \phi]$ is given by 16. The source term in (IV) depends on $\sqrt{\bar{g}}$, just as the source term in 13. Otherwise, the Legendre transform would not be linear in the mean field ϕ . Note also that the classical action of gravity may be unbounded, for example, in the case of the Einstein–Hilbert action. Then, the Legendre transformation is defined on a saddle point.

The flow equation for the effective action [2, 78, 79] follows straightforwardly from 28. The part proportional to ϕ^2 is canceled by the flow of the last term in (IV), and the flow of $\Gamma_k[\bar{g}, \phi]$ is given by

$$\partial_t \Gamma_k[\bar{g}, \phi] = \frac{1}{2} \text{Tr} G_k[\bar{g}, \phi] \partial_t R_k, \quad (30)$$

where $G_k[\bar{g}, \phi]$ is the full field-dependent propagator $\delta^2 W_k/\delta J^2$, and the trace has been defined below 28. It now contains a relative minus sign for Grassmann-valued fields. With the definition of the Legendre transformation in (IV), the full propagator is given by

$$G_k[\bar{g}, \phi] = \frac{1}{\Gamma_k^{(0,2)}[\bar{g}, \phi] + R_k}. \quad (31)$$

The flow equation for the effective action depends on the second derivative of the effective action with respect to the fluctuation fields, $\Gamma_k^{(0,2)}[\bar{g}, \phi]$. The flow of the latter is derived from 30, with two derivatives with respect to the fluctuation field ϕ . This flow depends on itself and the vertices $\Gamma_k^{(0,3)}[\bar{g}, \phi]$ and $\Gamma_k^{(0,4)}[\bar{g}, \phi]$. This leads to a tower of coupled differential equations for the n -point vertices $\Gamma_k^{(n,m)}[\bar{g}, \phi]$, which is discussed in more detail in Section 7.1. We use the following notation for derivatives,

$$\Gamma_k^{(n,m)}[\bar{g}, \phi] = \frac{\delta^{n+m} \Gamma_k[\bar{g}, \phi]}{\delta \bar{g}^n \delta \phi^m}, \quad (32)$$

for general functionals of $\bar{g}$ and ϕ . The functional derivative in 32 includes a factor of $1/\sqrt{\bar{g}}$ (see Supplementary Material).

The different parameterizations of the metric field, discussed in Section 3, do not influence the flow equation for the effective action 31, and they only differ by their corresponding expansion schemes induced by the relations between metric and fluctuations (20, 21, 24). Still, from the viewpoint of diffeomorphism-invariance, the different parameterizations differ qualitatively. While the geometrical approach with the fluctuation field 24 by construction leads to a diffeomorphism-invariant effective action

at all cutoff scales, diffeomorphism-invariance is broken in the linear split (20) and the exponential split 21 at a finite cutoff scale.

For all field parameterizations, a diffeomorphism-invariant effective action with one metric g is obtained at vanishing fluctuation graviton field $h = 0$,

$$\Gamma_k[g, \varphi] = \Gamma_k[g, \varphi]|_{h=0}, \quad \text{with} \quad \varphi = \phi(h=0), \quad (33)$$

the *background effective action*. Its flow equation is given by 30, evaluated at vanishing fluctuation field $h = 0$,

$$\partial_t \Gamma_k[g, \varphi] = \frac{1}{2} \text{Tr} G_k[g, \varphi] \partial_t R_k. \quad (34)$$

Importantly 34 is not closed: the right-hand side depends on $\Gamma_k^{(0,2)}$, the two-point function of the fluctuation fields including the fluctuation graviton field h , while the left-hand side knows nothing about h . Hence, the information about $\delta^2 \Gamma_k / \delta h^2$ has to be obtained separately.

5 BACKGROUND FIELD APPROXIMATION

The background field approximation, introduced in [3, 80] for YangMills theory and gravity, respectively, is the most commonly used approximation in the fRG approach to quantum gravity (see the reviews 7–16). It elevates the diffeomorphism-invariance of the background effective action to that of the full effective action. To that end, we write the full effective action in an expansion about the background effective action in 33,

$$\Gamma_k[\bar{g}, \phi] = \Gamma_k[g, \varphi] + S_{\text{gf}}[\bar{g}, h] + \Delta \Gamma_k[\bar{g}, \phi]. \quad (35a)$$

The gauge fixing term S_{gf} is defined in 6 and $\Delta \Gamma_k[\bar{g}, \phi = 0] = 0$. In the background field approximation, the last term in 35a is assumed to be negligible,

$$\Delta \Gamma_k[\bar{g}, \phi] \approx 0. \quad (35b)$$

The underlying assumption is that the dynamics of a gauge theory is carried by gauge-invariant fluctuations, while $\Delta \Gamma_k$ carries quantum deformations of the gauge fixing procedure and should not drive the dynamics. Then, derivatives with respect to $\bar{g}$ and h agree in the linear split and are related in a simple way in the other parameterizations via 21 and 24.

In the approximation 35 and with the linear split 20, the second derivatives of the effective action with respect to the background metric and the fluctuation field agree at $\phi = 0$ up to the gauge fixing term:

$$\Gamma_k^{(0,2)}[\bar{g}, \varphi] = \Gamma_k^{(2,0)}[\bar{g}, \varphi] + S_{\text{gf}}^{(0,2)}[\bar{g}, 0]. \quad (36)$$

Inserting 35 into 30 leads us to a closed and diffeomorphism-invariant flow for the background effective action $\Gamma_k[g, \varphi]$.

5.1 Properties of the Background Approximation

It is the simple relation 36 and the manifest diffeomorphism-invariance of the approximation at all cutoff scales that make the

background field approximation so attractive. A large amount of the results in asymptotically safe quantum gravity has been obtained in this approximation, and it is still the commonly used approximation in the field. This asks for independent checks of these results and its embedding in systematic expansion schemes that go beyond it. In the present work, we review the *fluctuation approach* (see Section 7), which includes the correlation functions of the fluctuation graviton field h . The results in the background field approximation are qualitatively in line with the results in the fluctuation approach discussed in Section 8. This confirms—in most cases—the underlying assumption 35b. Nonetheless, some words of caution are needed.

Despite its seeming manifest diffeomorphism-invariance, the background field approximation is at odds with diffeomorphism-invariance and background independence. To understand this counterintuitive remark, we recall some features of the background field formalism to standard quantum field theories, for example, the SM and QCD. The introduction of the background field to the gauge fixing allows defining a gauge-invariant background effective action. It is evident from its introduction that it is an auxiliary symmetry. The background field can even generate gauge-invariant background effective actions in theories that explicitly break gauge-invariance. This is clear from the construction of diffeomorphism-invariant background effective actions in gravity in the presence of a background-covariant momentum regulator. In a gauge-invariant theory without a cutoff, it can be shown that the physical gauge-invariance of the theory is carried by the fluctuation field in terms of nontrivial Ward- or Slavnov-Taylor identities. The underlying transformations are called quantum gauge/diffeomorphism transformations. This physical symmetry carries over to the auxiliary background gauge invariance via nontrivial Nielsen or split Ward identities. The latter encodes background independence of the theory and is introduced in Section 2.2. The Slavnov-Taylor and Nielsen identities for gravity are discussed in detail in Section 6.

In summary, *only* if the fluctuation correlation functions satisfy the nontrivial symmetry relations *and* the Nielsen identities, the auxiliary background gauge-invariance is physical. Then, it carries the underlying symmetry, and we have background independence.

5.2 Regulator Dependence of the Background Effective Action

In this section, we first argue that regulator choices within the general class defined with 26 and 27 can be used within the background field approximation to even change the (non-) existence or the nature of an asymptotically safe UV fixed point. This seems to casts some doubts on the reliability of results obtained in the background field approximation. However, we then show that the comparison with fluctuation results and the proper use of Nielsen identities (see Section 6) suffices to further restrict the general class of regulators such that it is adapted to the background field approximation.

The regulator term is the origin of the reliability problems of a naive use of the background field approximation within the fRG

approach: it generates additional terms in $\Delta\Gamma_k[\bar{g}, h]$ in **35a** via the background-metric dependence of the regulator. In the background field approximation **35b**, this background-metric dependence is elevated to a dynamical one: in the approximation **36**, the fluctuation two-point function $\Gamma_k^{(0,2)}$ is computed from background-metric derivatives of the (integrated) flow with the exception of the gauge fixing term. These derivatives also hit the regulator. Accordingly, we have added dynamics via the choice of the regulator, and it remains to be proven in each application that this does not change the results qualitatively.

This has been discussed early on at the example of scalar theories and Yang–Mills theory in [81, 82]. In particular, it has been shown that the one-loop β -function in Yang–Mills theory can be changed from its universal result with regulator choices in the background field approximation. More precisely, it has been shown that for regulators, $R_k(\bar{\Delta}_s)$ with spin $s = 1$ and spin $s = 0$ covariant Laplacians $\bar{\Delta}_s = \Delta_s(\bar{A})$ the coefficient Z_F of the $\text{tr}F^2$ -term in the effective action runs at one loop as

$$\left. \frac{\partial_t Z_F}{Z_F} \right|_{1\text{-loop}} = n \beta_{\alpha_s, 1\text{-loop}}, \text{ for } R_k(x \rightarrow 0) \propto \frac{1}{x^{n-1}}. \quad (37)$$

(For details, refer to 82.) This spoils the universality of the one-loop β -function in the Yang–Mills theory. If one does not resort to the background field approximation, the correct one-loop β -function is obtained.

We now discuss the origin of this peculiar behavior. We follow the argument in [83] and for the general case including gravity (refer to [21, 72, 73, 83]). Simply put, we would like to show that the background effective action at a finite cutoff scale k and in particular in the limit $k \rightarrow \infty$ carries no physics without further restrictions of the regulator. We parameterize the regulator with

$$R_k = \Gamma_k^{(0,2)} r_k(\bar{\nabla}) \quad (38)$$

(see also 26). Note that in 38, we have introduced a $\bar{\nabla}$ -dependent shape function, which is more general than the $x = \bar{\Delta}/k^2$ -dependent one (defined in 26). Still we use x in a slight abuse of notation for identifying the UV and IR limits. As already explained around 26, the shape function r_k is a free function of the covariant derivative with the limits 27. In particular it has to decay in UV. With the parameterization 38, the flow equation 34 for the background effective action with $\bar{g} = g$ reads

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{1+r_k} \partial_t r_k + \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(0,2)}} \partial_t \Gamma_k^{(0,2)} \frac{1}{1+r_k} r_k. \quad (39)$$

From the first term on the right-hand side of the flow 39, we deduce that the UV limit of the shape function is constrained: $r_k(x \rightarrow \infty) \leq 1/x^{d+e}$ with $x = \Delta/k^2$, as discussed below 27. In turn, the IR limit $x \rightarrow 0$ of r_k can be singular without spoiling the finiteness of 39. In order to obtain a general background effective action, we simply demand that r_k solves the differential equation,

$$\partial_t r_k = -r_k \frac{1}{\Gamma_k^{(0,2)}} \partial_t \Gamma_k^{(0,2)} + 2 \partial_t Y_k. \quad (40)$$

This is a simple differential equation that admits a solution at least locally (in the flow time t). Note that the UV decay of r_k also constrains the UV limit of Y_k with $\partial_t Y_k(x \rightarrow \infty) \leq 1/x^{d+e}$. Inserting a shape function r_k of 40 into 39, we arrive at

$$\partial_t \Gamma_k = \text{Tr} \partial_t Y_k(\nabla). \quad (41)$$

Equation 41 constrains the IR limit of the function Y_k : its flow $\partial_t Y_k$ has to be trace class for rendering the flow of the background effective action finite. If we also assume the trace-class property for Y_k , the order of t -derivative and trace can be swapped.

Apart from these trivial constraints, the choice of $Y_k(\nabla)$ is at our disposal. Integrating the flow 41 from some scale $\Lambda < k$, and taking the UV limit with $k \rightarrow \infty$ we arrive at

$$\lim_{k \rightarrow \infty} \Gamma_k[g, \varphi] = [\Gamma_\Lambda - \text{Tr} Y_\Lambda] + \text{Tr} Y_k(\nabla) \rightarrow \text{Tr} Y_k(\nabla). \quad (42)$$

The term $\Gamma_\Lambda - \text{Tr} Y_\Lambda$ is k - and Λ -independent, and the latter property follows from RG-consistency: $\partial_\Lambda \Gamma_k \equiv 0$ for $k \neq \Lambda$ (see, e.g., 84). In the last relation in 42, we have assumed that the effective action is dominated by the UV term $\text{Tr} Y_k$. This assumption underlies most fixed-point analyses.

We emphasize that the result 42 is exact and no approximation has been applied. **Equation 42** implies that without suitable restrictions on the regulator function r_k , the flow of the background effective action $\Gamma_k[g, \varphi]$ (for large cutoff scales) has no physics content at all. Even at one- and two-loop order in perturbatively renormalizable theories, it does not reproduce universal results without further restrictions on the regulator.

The IR limit with $r_{k \rightarrow 0} = 0$ puts a severe restriction onto r_k , which constrains the integrated flow together with the RG consistency at the initial cutoff scale Λ , $\partial_\Lambda \Gamma_{k=0} = 0$. However, in the UV limit, the restriction

$$r_{k \rightarrow \infty} \rightarrow \infty, \quad (43)$$

does effectively not restrict the UV scaling. The latter is dominated by the UV-relevant operators that satisfy 43 by definition. Note that so far, we have discussed the flow of the background effective action $\Gamma_k[g, \varphi]$ without resorting to approximations.

The above issues are already present for the full flow and emphasize the auxiliary nature of the background effective action at $k \neq 0$. In particular, no conclusion can be drawn from its regularity or singular behavior in the UV limit with $k \rightarrow \infty$. This situation is further complicated by the background field approximation. Then, the field dependence that originates from the regulator term is fed back into the flow equation as dynamical contributions. As we have discussed above, these contributions are ambiguous in particular in the UV limit. In conclusion, the background field approximation, while having the appeal of simplicity and seeming diffeomorphism-invariance, has to be applied with great care. To that end, we split the problems discussed above in their physics origin:

- (1) Physical diffeomorphism-invariance and background independence are carried by nontrivial Slavnov–Taylor and Nielsen identities of the fluctuation field.

- (2) The background field dependence of the regulator term is potentially dangerous in the UV and has to be separated.

A first step in the resolution of the issues of the background field dependence is to monitor the field-dependence that originates in the regulator. The related equation and discussion in the Yang–Mills theory and gravity can be found in 21, 72, 73, 82, 83, 85, 86. (For applications to gravity, see also 57, 87–90.) The equation that monitors this dependence is given by

$$\text{Tr} \left[\frac{\delta \sqrt{\bar{g}} R_k}{\delta \bar{g}_{\mu\nu}} \frac{\delta}{\delta R_k} \right] \Gamma_k[\bar{g}, \phi] = \frac{1}{2} \text{Tr} \frac{\delta \sqrt{\bar{g}} R_k}{\delta \bar{g}_{\mu\nu}} G_k[\bar{g}, \phi]. \quad (44)$$

Equation 44 allows to disentangle the background-metric dependence stemming from the regulator from the rest. In the Yang–Mills example from 37, it can be shown that the regulator-field dependence is responsible for a contribution $(1-n)\beta_{\alpha,1-\text{loop}}$. Subtracting the contribution from the regulator-field dependence, the universal result is obtained. Indeed, even without an explicit computation, we can already infer from 44 that the universal 1-loop β -function of the dimensionless Yang–Mills coupling is achieved for IR regular regulators: the projection of the right-hand side of 44 on the dimensionless term proportional to $\text{tr} F_{\mu\nu}^2$ can only depend on the cutoff scale k in the presence of an additional scale. For IR-regular regulators, such a scale is absent and the k -derivative of 44 vanishes. In turn, IR-singular regulators implicitly introduce a further IR scale, and the k -derivative of 44 does not vanish. This explains the structure of the result in 37. We emphasize that the modification of the dynamics in the background field approximation via the regulator term is not restricted to IR-singular regulators. The latter fact is a peculiarity of the *universal* one-loop running of the *dimensionless* Yang–Mills coupling. In particular, we emphasize that for nonuniversal couplings and theories with dimensionful couplings such as gravity, the flow of 44 does not vanish for IR-regular regulators.

Based on this analysis it has been suggested in 81, 82, that within the background field approximation, the corresponding field-dependence should be subtracted before applying the approximation $\Gamma_k^{(0,2)} \simeq \Gamma_k^{(2,0)}$ for the right-hand side of the flow. This idea has been picked up by 91–93 for scalar theories, $f(R)$ gravity and gravity matter systems. (For more details, see **Section 6**.) These works are based on the relation 44, where one derivative with respect to the background is taken. To fully resolve $\Delta\Gamma_k$ in 35a, a further field derivative of 44 is needed. Furthermore 44 does not comprise the full difference between h and $\bar{g}$ derivatives. While the background field correlation functions are diffeomorphism-covariant due to background diffeomorphism invariance, the fluctuation correlation functions satisfy difficult Slavnov–Taylor identities. This is well-known and well-studied (though not fully conclusively) in non-abelian gauge theories where one also has access to respective lattice results, for a recent review and related references [27]. In turn, the related analysis, while in high demand, is less advanced in quantum gravity (see also 16, 27). This is detailed in the next section.

6 SYMMETRY IDENTITIES

Physical observables are diffeomorphism-invariant and background-independent. The underlying symmetry is dynamical and is solely carried by the dynamical fluctuation fields. It is called *quantum* diffeomorphism invariance and reads

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \mathcal{L}_\omega(\bar{g}_{\mu\nu} + h_{\mu\nu}), \quad \bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}. \quad (45)$$

The background metric triggers an a priori *auxiliary* symmetry, the *background* diffeomorphism invariance. It is given by the transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \mathcal{L}_\omega h_{\mu\nu}, \quad \bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + \mathcal{L}_\omega \bar{g}_{\mu\nu}. \quad (46)$$

Here, $\mathcal{L}_\omega$ is the Lie derivative with respect to some vector field ω_μ , which reads for a rank-two tensor

$$\mathcal{L}_\omega T_{\mu\nu} = \omega_\rho \bar{\nabla}^\rho T_{\mu\nu} + T_{\mu\rho} \bar{\nabla}^\rho \omega_\nu + T_{\nu\rho} \bar{\nabla}^\rho \omega_\mu. \quad (47)$$

Both transformations, 45 and 46, generated diffeomorphism transformations on the full metric $g_{\mu\nu}$, so they do not differ on the functional of $g_{\mu\nu}$. Moreover, while 46 is an auxiliary symmetry, it still comprises the information of the dynamical quantum diffeomorphism symmetry 45 via the Nielsen identities. The latter carry the background independence of the theory.

Any fRG computation needs to introduce a gauge fixing and a regularization, which both apparently break diffeomorphism invariance and (on-shell) background independence. Thus, it is an important issue in the fRG approach to quantum gravity to discuss how these properties can be preserved in a nonperturbative computation. For each symmetry broken by the cutoff term, we can formulate a nontrivial modified symmetry identity, which captures the cutoff deformation of the underlying symmetry and smoothly approaches the unbroken symmetry identity at vanishing cutoff scale, $k = 0$. We now first discuss how the Nielsen identities take care of background independence and afterward discuss quantum diffeomorphism invariance due to the Slavnov–Taylor identities. Note that also in discrete gravity models, the Ward identities play a crucial role (see 94) for a review of tensor models.

6.1 Background Independence

As discussed in **Section 2.1**, we always need to split the full metric into a background metric $\bar{g}$ and a fluctuation field h . This split introduces an additional symmetry given by all transformations of the background metric and of the fluctuation field that leave the full metric invariant.

$$g(\bar{g}, h) \rightarrow g(\bar{g} + \delta\bar{g}, h + \delta h) = g(\bar{g}, h). \quad (48)$$

For example, in the linear split 20, we have $\delta\bar{g} = -\delta h$. This symmetry is guaranteeing background independence since we can always find a transformation that changes the background according to our choice. This symmetry is broken off-shell by the gauge fixing and ghost action and further broken by the regularization on- and off-shell. The breaking of the symmetry is described by the Nielsen (or split Ward) identities [95, 96]. They encode the background independence of the physical

observables and allow us to restore the symmetry at vanishing cutoff.

Let us first discuss the Nielsen identities without the regulator. The Ward identity for the effective action for any symmetry transformation $\mathcal{G}$ is given by

$$\mathcal{W} = \mathcal{G}\Gamma - \left\langle \mathcal{G}(S_{\text{gf}} + S_{\text{gh}}) \right\rangle = 0, \quad (49)$$

where S_{gf} and S_{gh} are defined as in 6 and 8. We apply this to the transformation of the metric split 48 and obtain the Nielsen identity $\text{NI} = 0$, with

$$\text{NI} = \frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} - \int \left\langle \frac{\delta\hat{h}}{\delta\bar{g}_{\mu\nu}} \right\rangle \cdot \frac{\delta\Gamma}{\delta h} - \left\langle \left[\frac{\delta}{\delta\bar{g}_{\mu\nu}} - \int \frac{\delta\hat{h}}{\delta\bar{g}_{\mu\nu}} \cdot \frac{\delta}{\delta h} \right] (S_{\text{gf}} + S_{\text{gh}}) \right\rangle, \quad (50)$$

where $h_{\mu\nu} = \langle \hat{h}_{\mu\nu} \rangle$, and the fluctuation field is understood as function of the full metric and the background metric $\hat{h}(g, \bar{g})$. For the linear split 20, we have $\delta\hat{h}_{\rho\sigma}(x)/\delta\bar{g}_{\mu\nu}(y) = 1/\sqrt{\bar{g}}\delta(x-y)1/2(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho)$ (see Supplementary Material), and thus,

$$\text{NI}_{\text{lin}} = \frac{\delta\Gamma_{\text{lin}}}{\delta\bar{g}_{\mu\nu}} - \frac{\delta\Gamma_{\text{lin}}}{\delta h_{\mu\nu}} - \left\langle \left[\frac{\delta}{\delta\bar{g}_{\mu\nu}} - \frac{\delta}{\delta h_{\mu\nu}} \right] (S_{\text{gf}} + S_{\text{gh}}) \right\rangle. \quad (51)$$

The Nielsen identity for the exponential split 21 resembles 51: there is a nontrivial difference between the background-metric and fluctuation-field derivatives due to the gauge fixing and ghost terms. In 17, we have pointed out that at $k = 0$, a solution of the background EoM is also a solution of the quantum EoM and vice versa. This implies together with 51 that the expectation value $\langle [\delta_{\bar{g}} - \delta_{\hat{h}}](S_{\text{gf}} + S_{\text{gh}}) \rangle$ needs to vanish on-shell. This is indeed nontrivial and does not happen off-shell.

In comparison, for the fully diffeomorphism-invariant Vilkovisky–DeWitt or geometrical effective action with the split given by 22, the dependence on the gauge fixing action and the ghost action is vanishing, and thus, the Nielsen identity reads

$$\text{NI}_{\text{geo}} = \frac{\delta\Gamma_{\text{geo}}}{\delta\bar{g}_{\mu\nu}} - \int \left\langle \frac{\delta\hat{h}}{\delta\bar{g}_{\mu\nu}} \right\rangle \cdot \frac{\delta\Gamma_{\text{geo}}}{\delta h}. \quad (52)$$

In contradistinction to the linear and exponential split, the $\bar{g}$ and h derivatives are directly related.

The Nielsen identities entail that for all metric splits, the effective action is not a function of the full metric g but depends separately on the background metric $\bar{g}$ and the fluctuation field h . Consequently, the effective action has no simple expansion in terms of diffeomorphism-invariant quantities in $g_{\mu\nu}$. Still, the Nielsen identities relate $\bar{g}$ and h derivatives such that on the solution of the Nielsen identities, the effective action carries background independence and only depends on one field.

So far, the analysis has been performed in the absence of the cutoff term, that is, at $k = 0$. At finite k , the regulator term introduces a further breaking of the split symmetry 48. The Nielsen identities turn into modified Nielsen identities, $\text{mNI} = 0$, that read for a general split

$$\text{mNI} = \text{NI} - \frac{1}{2} \text{Tr} \frac{\delta\sqrt{\bar{g}}R_k}{\sqrt{\bar{g}}\delta\bar{g}_{\mu\nu}} G_k - \text{Tr} R_k G_k \left[\frac{\delta}{\delta\phi} \left\langle \frac{\delta\hat{\phi}}{\delta\bar{g}_{\mu\nu}} \right\rangle \right]. \quad (53)$$

Note that in the last term in 53, only the metric fluctuation h contributes as the other fluctuation fields do not depend on the background metric. Furthermore, in the linear split, the last term is vanishing, and consequently, the mNI simplifies to

$$\text{mNI}_{\text{lin}} = \text{NI}_{\text{lin}} - \frac{1}{2} \text{Tr} \frac{\delta\sqrt{\bar{g}}R_k}{\sqrt{\bar{g}}\delta\bar{g}_{\mu\nu}} G_k. \quad (54)$$

While some of the properties and consequences of the mNI are theory-dependent, most of them are generic, and much can be learned about applications in gravity from investigations in general theories: mNIs have been discussed in detail gravity, gauge theories, in scalar theories [3, 15, 21, 57, 72, 73, 81, 82, 87–93, 98–106].

There is an important qualitative difference between the breaking of the metric split symmetry 48 at finite k and at $k = 0$. We have already discussed in Section 2.2 that the Nielsen identity at vanishing cutoff scale, $k = 0$, encodes background independence, manifested in the equivalence of the solutions of the background and fluctuation EoMs, 17. At finite cutoff scale, $k \neq 0$, we necessarily have background dependence, as the quantum fluctuations have to be ordered in a specific background. This is also manifest in the missing equivalence of the background and fluctuation EoMs, the respective solutions do not agree,

$$\frac{\delta\Gamma_k[\bar{g}_{\text{EoM}}^{\text{fluc}}, 0]}{\delta h_{\mu\nu}} = 0 = \frac{\delta\Gamma_k[\bar{g}_{\text{EoM}}^{\text{back}}, 0]}{\delta\bar{g}_{\mu\nu}}, \quad \bar{g}_{\text{EoM}}^{\text{fluc}} \neq \bar{g}_{\text{EoM}}^{\text{back}}. \quad (55)$$

(For a detailed discussion, see 97, 106, 107). The difference between the solutions can be parameterized by a term proportional to the regulator R_k , which is most easily seen in the modified Nielsen identity in the geometric approach, 52 and 53.

The difference between $\bar{g}_{\text{EoM}}^{\text{fluc}}$ and $\bar{g}_{\text{EoM}}^{\text{back}}$ was explicitly computed in 97, 107 for backgrounds with constant curvature. The ansatz for the background effective action is

$$\Gamma_k[\bar{g}] = \int d^4x \sqrt{\bar{g}} k^4 f(r) = V \tilde{f}(r), \quad (56)$$

where V is the space-time volume and $r = \bar{R}/k^2$ is the dimensionless background curvature. Thus, the background EoM becomes

$$\Gamma_k^{(\bar{g})}[\bar{g}, 0] \sim r f'(r) - 2f(r) = 0, \quad (57)$$

which is displayed in the right panel of Figure 2 at the UV fixed point for different numbers of scalar fields N_s . The ansatz for the fluctuation one-point function reads

$$\Gamma_k^{(h_{\text{tr}})}[\bar{g}, 0] = \int d^4x \sqrt{\bar{g}} k^3 f_1(r) = \frac{V}{k} f_1(r), \quad (58)$$

and thus, the quantum EoM is simply

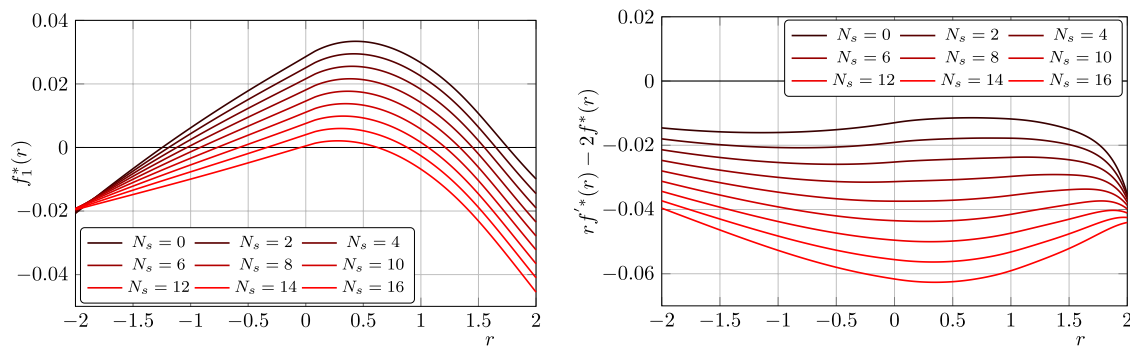


FIGURE 2 | Displayed are the potential of the one-point function and the derivate of the background potential for different numbers of scalar fields at the fixed point, as defined in 57 and 59. A zero in these functions indicates a solution to the quantum and background EoM, respectively. While the former always has two solutions, a minimum at negative curvature and a maximum at positive curvature, the latter shows no solution at all. The figures are taken from 97.

$$\Gamma_k^{(h_{\mu\nu})}[\bar{g}, 0] \sim f_1(r) = 0. \quad (59)$$

This is shown in the left panel of **Figure 2** at the UV fixed point for different numbers of scalar fields N_s .

The background EoM does not display a solution in the whole investigated region, while the quantum EoM has two solutions, a minimum at negative curvature and a maximum at positive curvature. For a larger number of scalar fields, these two solutions merge. However, in this regime, the approximation lacks reliability due to large values of the graviton anomalous dimension. Importantly, **Figure 2** manifests in explicit computation the difference between the background and quantum EoM, 55. The background EoM was also extensively investigated in the background field approximation with different choices of regulator and parameterization. For example, in 108, the linear split was used and a solution at large negative curvature was found. However, in 109, 110, two further solutions at positive curvature were found due to a different choice of the regulator. A solution at positive curvature was also found in 111 and with the exponential parameterization in 51.

In 106, a modification of the fRG equation was proposed. There, the effective action was defined as the Legendre transform of a normalized Schwinger functional, $\hat{W}_k[\bar{g}, J] = \log(Z_k[\bar{g}, J]/Z_k[\bar{g}, 0])$. This modification implies that the solutions to the quantum and background EoMs agree even at a finite cutoff scale. This does not imply that the modified effective action is background-independent at finite k since there are differences in the higher order correlation function. However, it allows for constructing improved background field approximations, which might allow resolving some tensions between background and fluctuation results.

6.2 From BRST to Diffeomorphism Invariance

While the auxiliary background diffeomorphism invariance 46 remains unbroken, the physical quantum diffeomorphism

invariance 45 turns into a BRST symmetry due to the gauge fixing, which is then further broken by the regulator. The related symmetry identities are called (modified) Slavnov–Taylor identities [(m)STI] [112, 113]. They encode physical diffeomorphism invariance. We sketch the main ideas of the derivation and apply them to gravity.

In case of the linear gauge fixing condition 7, the generator of BRST transformation (or BRST operator) denoted by $\mathfrak{s}$, including the Nakanishi–Lautrup field λ_μ , is given by

$$\mathfrak{s}(\bar{g}_{\mu\nu}, h_{\mu\nu}, c_\mu, \bar{c}_\mu, \lambda_\mu, \phi_{\text{mat}}) = (0, \mathcal{L}_c(\bar{g}_{\mu\nu} + h_{\mu\nu}), c_\rho \bar{\nabla}^\rho c_\mu, \lambda_\mu, 0, \mathfrak{s}\phi_{\text{mat}}). \quad (60)$$

In (B), the vector field ω_μ in the Lie derivative 47 is given by the ghost field, $\omega_\mu = c_\mu$. (For more details on the setup and the condensed notation used below, see 21). The Nakanishi–Lautrup field λ_μ transforms trivially under the BRST transformation, $\mathfrak{s}\lambda_\mu = 0$. The classical gauge-fixed action including the gauge fixing and the ghost action is invariant under this transformation, $\mathfrak{s}(S_{\text{grav}} + S_{\text{gf}} + S_{\text{gh}}) = 0$. Furthermore, $\mathfrak{s}$ is a nilpotent operator with $\mathfrak{s}^2 = 0$.

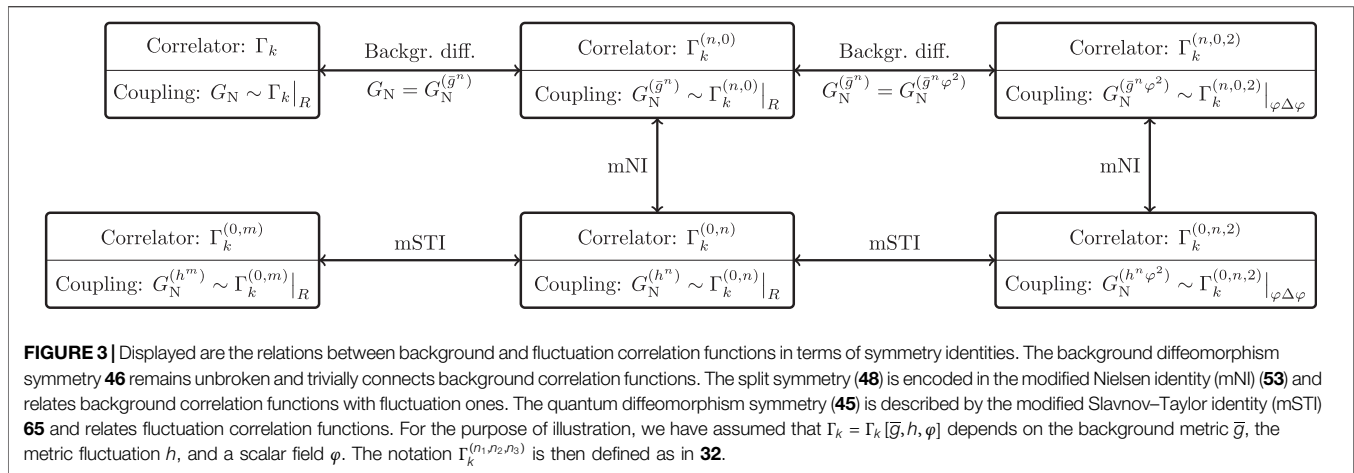
For the derivation of the STI, we include a source term $Q^a \mathfrak{s}\hat{\phi}_a$ for the BRST variations of the fields in the generating functional. The Schwinger functional now reads

$$e^W[\bar{g}, \phi, J, Q] = \int \mathcal{D}\hat{\phi} e^{-S_{\text{tot}} + \int d^4x \sqrt{\bar{g}} (J^a \hat{\phi}_a + Q^a \mathfrak{s}\hat{\phi}_a)}, \quad (61)$$

where $S_{\text{tot}} = S_{\text{grav}} + S_{\text{gf}} + S_{\text{gh}}$. The STI follows from the BRST-invariance of generating functional,

$$\int \mathfrak{s} \left(\mathcal{D}\hat{\phi} e^{-S_{\text{tot}} + \int d^4x \sqrt{\bar{g}} (J^a \hat{\phi}_a + Q^a \mathfrak{s}\hat{\phi}_a)} \right) = 0. \quad (62)$$

The source term $J^a \hat{\phi}_a$ is the only BRST-variant term. The BRST operator $\mathfrak{s}$ commutes with bosonic sources and anti-commutes with fermionic sources. This leads us to $\mathfrak{s}J^a \hat{\phi}_a = J^a \gamma^b_a \mathfrak{s}\hat{\phi}_b$, where the metric γ^b_a carries the minus sign for the fermionic terms (see Supplementary Material).



With these properties, we obtain the STI for the Schwinger functional,

$$\int \mathcal{D}\hat{\phi} \int d^4x \sqrt{\bar{g}} J^a \gamma_a^b (\hat{\phi}_b) e^{-S_{\text{tot}} + \int d^4x \sqrt{\bar{g}} (J^a \hat{\phi}_a + Q^a \hat{\phi}_a)} = \int d^4x \sqrt{\bar{g}} J^a \gamma_a^b \frac{\delta}{\delta Q^b} e^W[\bar{g}, \phi, Q] = 0. \quad (63)$$

This identity can be re-expressed in terms of the effective action. (See 21 for details.) Here, we just state the result for the STI in the absence of the cutoff term,

$$\text{STI} = \int d^4x \sqrt{\bar{g}} \frac{\delta \Gamma}{\delta \phi_a} \frac{\delta \Gamma}{\delta Q^a} = 0. \quad (64)$$

This equation is known as the quantum-master equation. The BRST variation of the effective action is given by $\delta \Gamma / \delta Q^a = \langle \hat{\phi}_a \rangle$. These variations can be interpreted as generalized vertices of the theory.

Equation 64 encodes diffeomorphism invariance at $k=0$ where the regulator vanishes. At finite cutoff scale, an additional regulator contribution has to be taken into account, and we are led to the mSTI,

$$\text{mSTI} = \text{STI} - \text{Tr} R_k \frac{\delta^2 \Gamma_k}{\delta Q \delta \phi} G_k = 0. \quad (65)$$

Some of the properties of the mSTI are theory-dependent, but most of them are generic: mSTIs in the presence and absence of background fields in gravity and gauge theories have been discussed in detail in [3, 15, 21, 22, 57, 72, 73, 81, 82, 87–93, 98–106, 114–132].

In summary, we have three symmetries:

- (1) The auxiliary background diffeomorphism invariance **46**, which remains unbroken.
- (2) The quantum diffeomorphism invariance **45**, which describes physical diffeomorphism invariance. It is broken and encoded in the mSTI **65**.
- (3) The split symmetry **48**, which guarantees background independence. It is broken as well and encoded in the mNI **53**.

The relations between background and fluctuation correlation functions are summarized in **Figure 3**. The relation between two fluctuation correlation functions can be expressed either with an mSTI or with a combination of mNI and background diffeomorphism invariance. However, it should be noted that in a truncated nonperturbative computation these two possibilities of relating fluctuation correlation function do not agree with each other. Nonetheless, it can be used to check the error of the truncation. (See **Section 7.1** for more details.)

Last but not least, the flow of mNI and the mSTI is proportional to itself, respectively. This is conveniently expressed in terms of the flow equation for composite operators, derived in [21, 128, 133]. Schematically, it reads

$$\partial_t \mathcal{O}_k[\bar{g}, \phi] = \frac{1}{2} \text{Tr} G_k \partial_t R_k G_k \mathcal{O}_k^{(0,2)}[\bar{g}, \phi]. \quad (66)$$

The operator $\mathcal{O}_k^{(0,2)}$ is contracted with $G_k \partial_t R_k G_k$ in the trace. The set of composite operators $\mathcal{O}_k[\bar{g}, \phi]$ with the flow **66** includes general correlation functions $Z^{(0,n)}[\bar{g}, J[\bar{g}, \phi]]$ with their disconnected parts as well as more general functions of the field-dependent source such as $J[\bar{g}, \phi] = \Gamma_k^{(0,1)}[\bar{g}, \phi]$. In the most general case of a functional with an explicit cutoff dependence, further terms enter **66** (see 21). An educative example is $\Gamma_k^{(0,1)}$: inserting it into **66** leads to the fluctuation field derivative of the flow **equation 30**. An instructive example for the case of general correlation functions, and the necessity of including the disconnected terms is the full two-point function $G_{\phi_1 \phi_2} + \phi_1 \phi_2$. **Equation 66** has been used in Yang–Mills theories for the traced Polyakov loop observables [134] and in gravity for the study of the renormalization and scaling of composite operators [135–138]. Importantly, the set of composite operators $\{\mathcal{O}_k\}$ includes modified symmetry identities, that is, $\text{Sym}_k = \text{mSTI}, \text{mNI}, \dots$ (see 21 and also 82, 118, 119, 139). Hence, the flow of the symmetry identities reads schematically

$$\partial_t \text{Sym}_k[\bar{g}, \phi] = \frac{1}{2} \text{Tr} G_k \partial_t R_k G_k \text{Sym}_k^{(0,2)}[\bar{g}, \phi]. \quad (67)$$

Equation 67 implies that once we have solved these identities at a scale k , then the identities are satisfied at all scales. However,

this only holds for untruncated flows or truncations that are compatible with 67. (More details can be found in Section 7.1.)

6.3 Challenges for Diffeomorphism-Invariant Flows

Gauge-invariant approaches to quantum field theories have received much attention over the decades both in perturbation theory and beyond. Such formulations also have met considerable challenges, except for lattice gauge theories that are based on link variables formulated in the gauge group. In turn, perturbation theory and nonperturbative functional approaches are based on correlation functions and in particular on the propagator of the algebra-valued gauge field. (For reviews on lattice approaches to quantum gravity see, e.g., 140–144).

Gauge-invariant functional formulations are based either on gauge-invariant or gauge-covariant variables such as the geometrical formulation, the field strength formulation, or Wilson line formulations similar to lattice gauge theories. Implementations in the flow equation approach range from generalized Polchinski equations with gauge-covariant kernels for the Wilson effective action [145–156] and its recent manifestations [157–159], over the geometrical or Vilkovisky–DeWitt flows for the effective action [21, 71–74], to a recent suggestion for a gauge-invariant flow for the effective action [160–164].

Most of these approaches rely explicitly or implicitly on the definition of projection operators on the subspace of the dynamical degrees of freedom. Typically, this is achieved by a gauge fixing, but the notation of a projection is far more versatile. The appropriate definition of this projection and the respective geometrical structure of the configuration space is at the root of the geometrical construction. This has been discussed in Sections 3.3, 6.2, and 6.3, and we refer to the discussions there. The notable nonlocality of the projections both in field space as well as momentum space is an *inherent* property of the construction of gauge-invariant subspaces. Consequently, it should be considered an *inherent* feature of such a construction. This inherent nonlocality may be buried in functional self-consistency relations, but it is present explicitly or implicitly without any doubt.

In any case, the situation calls for self-consistency checks of the final formulations of gauge-invariant or diffeomorphism-invariant flows. This necessity has been discussed already in 165: there the terminology of *complete* and *consistent* flows was introduced. The former flows generate all quantum fluctuations from a given classical action, while the latter flows generate a well-defined subset of quantum fluctuations from a given—partial—effective action. A well-known example for the latter is thermal flows, which only generate thermal fluctuations from the full quantum effective action at vanishing temperature. In 165, 166, an important and simple consistency check for flow equations has been suggested: any *complete* flow equation must generate the complete perturbation theory upon iteration from the given classical action. While one-loop perturbation theory in the fluctuation field is trivially achieved within one-loop exact flow equations, two-loop perturbation theory provides a nontrivial necessary, while not sufficient, consistency check.

These checks for diffeomorphism-invariant fRG approaches have been passed for the Wilsonian approach [145–156] or are

trivial for the geometrical effective action approach [21, 71–74]. It is a highly relevant and interesting question how the more recent proposals [158–164] fare in such a self-consistency check. Respective investigations either confirm the *completeness* of the approaches or may show their *consistency*, that is, they may integrate out a well-defined subset of quantum fluctuations. Finally, for potentially *consistent* flows, such an investigation may enable the construction of nontrivial two-loop consistent extensions. We emphasize that such an extension does not simply pass a two-loop test but more importantly allows for two-loop resummed nonperturbative approximations. The latter set of approximations certainly live up to the self-consistency of other state-of-the-art computations in asymptotically safe quantum gravity, while having the benefit of inherent diffeomorphism invariance.

7 FLUCTUATION APPROACH

In the last sections, we have detailed the need for an fRG approach to quantum gravity that goes beyond the background field approximation and that allows satisfying the nontrivial symmetry identities, the mSTI 65 and the mNI 53. For general metrics $g_{\mu\nu}$, this requires to solve the flow equation 30 for the two-field action $\Gamma_k[\bar{g}, h]$. It is already a formidable task for the one-field flow in the background field approximation discussed in Section 5. Indeed, already in scalar theories, one has to resort to approximations such as the derivative expansion or the vertex expansion, and this is no different in gravity. As already discussed, while the quantum dynamics of asymptotically safe gravity is generated and carried by the fluctuation correlation functions, it is the diffeomorphism-invariant background effective action $\Gamma[g]$ that allows for a more direct physics interpretation. The latter is extracted from the flow 34 that solely depends on the fluctuation two-point function $\Gamma^{(0,2)}[g, 0]$. The flow of the latter depends on higher order fluctuation correlation functions (see Section 7.1).

This suggests the expansion of the effective action $\Gamma_k[\bar{g}, h]$ in a vertex expansion of the fluctuation field h . Importantly, the vertex expansion in the fluctuation approach is a systematic approximation scheme, the strength and convergence of which have been shown in many nonperturbative approaches, and most notably in the fRG approach to QCD [167–171]. In the spirit of “toy” theories that can teach us something about technical properties and convergence, we consider non-abelian gauge theories as one of those standard quantum field theories that are as close as it gets to gravity. The vertex expansions fully disentangles the contributions from the background metric $\bar{g}$ and the fluctuation field h and reads for the effective action,

$$\Gamma_k[\bar{g}, \phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \Gamma_k^{(0, \phi_{a_1} \dots \phi_{a_n})}[\bar{g}, 0] \cdot \phi_{a_1} \dots \phi_{a_n}. \quad (68)$$

Evidently, if the expansion coefficients $\Gamma_k^{(0, \phi_{a_1} \dots \phi_{a_n})}$ are evaluated for general $\bar{g}$, we have a simple access to the full effective action. For example, if we choose $\bar{g} = \bar{g}_{\text{EoM}}^{\text{fluc}}$, the solution of the fluctuation EoM in 55, we have chosen an on-

shell expansion point. Accordingly, if we are interested in on-shell physics, only small fluctuations h should be relevant. In turn, if we choose another expansion point, for example, for technical reasons, it is very important to assess whether on-shell physics is in the radius of convergence of the expansion. This will be discussed in more detail in **Section 7.2**.

7.1 Hierarchy of Flow Equations

The background field approach leads to an extended hierarchy of flow equations. We first note that the background flow equation $\partial_t \Gamma_k[\bar{g}]$ (34) depends on the fluctuation two-point function $\Gamma_k^{(0,2)}[\bar{g}, 0]$ in a general background. The knowledge of the latter allows us to determine $\Gamma[\bar{g}]$ and is tantamount to the determination of the full propagator of the theory in a general background. However, the flow of the two-point function $\partial_t \Gamma_k^{(0,2)}[\bar{g}, 0]$ depends on $\Gamma_k^{(0,m)}[\bar{g}, 0]$ with $m = 2, 3, 4$. This continues for higher n -point functions and leads to an infinite tower of coupled equations,

$$\Gamma_k^{(0,m)}[\bar{g}, 0] = \text{fRG}_{0,m} \left[\bar{g}, \left\{ \Gamma_k^{(0,2 \leq j \leq m+2)}[\bar{g}, 0] \right\} \right]. \quad (69)$$

In other words, we need $\Gamma_k^{(0,2)}[\bar{g}, \phi]$ for general fluctuation fields for solving the flow equation of the background effective action. For most interacting quantum field theories, the task of resolving the full field dependence of the effective action is beyond reach. Already in scalar theories, one typically resorts to the computation of the full effective potential as well as additional vertices or momentum dependencies. In gravity, the full potential of the background curvature R has been investigated: $f(R)$ as well as potentials of tensor invariants [49, 51, 54, 59, 97, 107–111, 172–183]. Apart from this, as in other theories, explicitly or implicitly, a vertex expansion has been used. This entails a further expansion of 69 in powers of the background field and leads us to the hierarchy

$$\Gamma_k^{(n,m)}[\bar{g}, 0] = \text{fRG}_{n,m} \left[\bar{g}, \left\{ \Gamma_k^{(i \leq n, 2 \leq j \leq m+2)}[\bar{g}, 0] \right\} \right]. \quad (70)$$

Equation 70 is the full hierarchy of *integrated* flow equations to solve for quantum gravity. While its solution in terms of the vertex expansion has been baptized the *fluctuation approach*, it simply is the full problem at hand.

Apparently, 70 constitutes a system of equations for a two-field effective action. However, as discussed in **Section 6**, background independence at vanishing cutoff, $k = 0$, encoded in the Nielsen identities and carried over to the mNIs at finite cutoff scale k turns the effective action into a one-field effective action. In terms of the vertex expansion, this information is given by the mNI (53) for (n, m) -point functions,

$$\Gamma_k^{(n,m)}[\bar{g}, h] = \Gamma_k^{(n-1, m+1)}[\bar{g}, h] + \text{mNI}_{n,m} \left[\bar{g}, \left\{ \Gamma_k^{(i \leq n-1, j \leq m+1)}[\bar{g}, h] \right\} \right]. \quad (71)$$

This leaves us with two towers of functional relations. While the first one (70) describes the full set of correlation functions, the second one (71) can be used to iteratively solve the tower of mixed fluctuation background correlations on the basis of the

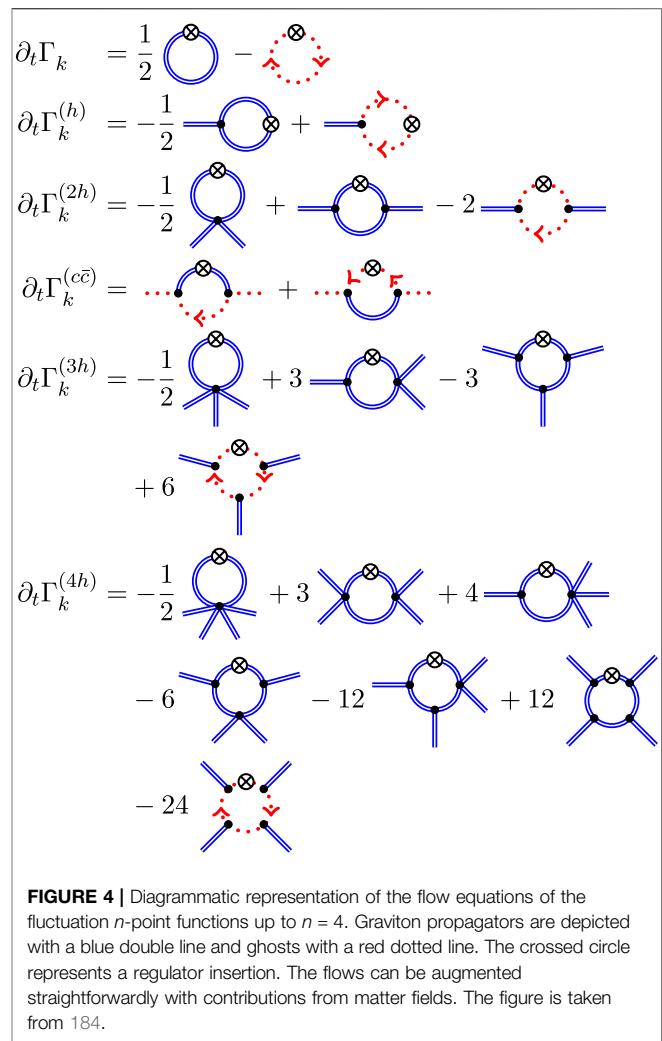


FIGURE 4 | Diagrammatic representation of the flow equations of the fluctuation n -point functions up to $n = 4$. Graviton propagators are depicted with a blue double line and ghosts with a red dotted line. The crossed circle represents a regulator insertion. The flows can be augmented straightforwardly with contributions from matter fields. The figure is taken from 184.

fluctuating correlation functions $\{\Gamma^{(0,m)}\}$. In both cases, we can solve the system for the higher order correlations of the background on the basis of the lower order correlations. If we use 71 with an iteration starting with the results from the flow equation for $\{\Gamma^{(0,m)}[\bar{g}_{\text{sp}}, h]\}$ for a specific background $\bar{g}_{\text{sp}}$, this closure of the system automatically satisfies the NI. Accordingly, *any* set of fluctuation correlation functions $\{\Gamma^{(0,m)}[\bar{g}_{\text{sp}}, h]\}$ can be iteratively extended to a full set of fluctuation background correlation functions in an iterative procedure.

An important feature of the fRG equations is that in the Landau limit of the gauge parameter $\alpha \rightarrow 0$ in 6, the flow equations for the transverse vertices $\Gamma_{k,\perp}^{(0,n)}$ are closed: the external legs of the vertices in the flow are transverse due to the transverse projection of the flow, the internal legs are transverse as they are contracted with the transverse propagator. Schematically, this reads for the integrated flows 70,

$$\Gamma_{k,\perp}^{(0,n)} = \text{fRG}_{\perp}^{(0,n)} \left[\left\{ \Gamma_{k,\perp}^{(0,m)} \right\} \right]. \quad (72)$$

In other words, the flow equation system of transverse fluctuation correlation functions is closed and determines the

dynamics of the system. In the fluctuation approach, the transverse system of graviton correlation function has been solved up to the four-graviton vertex [184]. A diagrammatic depiction of the system of flow equations is given in **Figure 4**, and a description of the respective results can be found in **Section 8**.

In turn, the flow equation system for longitudinal fluctuation correlation functions is not closed, and the transverse correlation functions $\Gamma_{k,\perp}^{(0,n)}$ feed into it,

$$\Gamma_{k,L}^{(0,n)} = \text{fRG}_L^{(0,n)} \left[\left\{ \Gamma_{k,\perp}^{(0,m)} \right\}, \left\{ \Gamma_{k,L}^{(0,m)} \right\} \right]. \quad (73)$$

Note that $\{\Gamma_{k,L}^{(0,n)}\}$ is the complement of the set of purely transverse correlation functions, so it consists of correlation functions with at least one longitudinal leg. On the other hand, the mSTIs are also nontrivial relations for the longitudinal correlation functions in terms of transverse vertices and longitudinal ones. This leads us to the schematic relation,

$$\Gamma_{k,L}^{(0,n)} = \text{mSTI}^{(0,n)} \left[\left\{ \Gamma_{k,\perp}^{(0,m)} \right\}, \left\{ \Gamma_{k,L}^{(0,m)} \right\} \right]. \quad (74)$$

(See 185 for non-abelian gauge theories.) In consequence, the mSTIs provide no direct information about the transverse correlation functions without further constraint. In the perturbative regime at large momenta, this additional constraint is given by the uniformity of the vertices. In turn, in strongly correlated regimes such a general constraint is absent. Indeed, one can show that the confinement property in a Yang–Mills theory in a covariant gauge necessitates the absence of uniformity of the vertices at low momenta. (For a detailed discussion in non-abelian gauge theories, see 168.)

Instead, we can simply use 74 for a given set of transverse correlation functions for constructing a BRST-invariant solution, which signals diffeomorphism invariance. For a given finite set of transverse correlation functions, generically such a solution can be found by integrating the flow (67). However, it may be *nonlocal*. The existence of BRST-invariant solutions for a general transverse input emphasizes the fact that the derivation of diffeomorphism-consistent solutions is not necessarily the hallmark of a good truncation. However, the comparison of 74 and 73 is a further nontrivial constraint on longitudinal correlation functions. Its evaluation is complicated by the fact that the solutions of two different functional relations for the same set of correlation functions do not agree in general in nontrivial truncations. Furthermore, it is very difficult to provide a measure for the closeness of the solutions. (For a related discussion in non-abelian gauge theories, see the recent review [27] and references therein.)

In summary, the evaluation of diffeomorphism invariance and self-consistency constitutes an intricate challenge. One has to utilize all the properties and relations discussed above. This holds for all fRG approaches to quantum gravity and not only to the fluctuation approach: only *local* BRST-invariant solutions should be considered physical, and the evaluation of locality and BRST invariance or their absence is intricate.

7.2 Flat Expansion Is a Curvature Expansion

As briefly mentioned in the introduction of this section, the choice of the background metric is important for the convergence of the

vertex expansion. However, an evaluation of the flow equations for $\Gamma_k^{(0,n)}[\bar{g}, 0]$ for generic metrics is yet an unresolved technical challenge. Even flows for spherically symmetric backgrounds already pose a formidable technical challenge that has only been solved recently within further approximations that hold for small curvatures [97, 107]. Therefore in most applications, one resorts to a curvature expansion in powers of the curvature. Such an expansion is tantamount to an expansion about the flat background with vanishing curvature,

$$\bar{g} = 1, \quad (75)$$

the Euclidean analog of the Minkowski metric. This has been baptized the *flat expansion*. With the flat background (75), Fourier transformations can be performed, and we are led to correlation functions $\Gamma_k^{(n,m)}(p_1, \dots, p_n, p_{n+1}, \dots, p_{n+m})$ in momentum space. This gives access to the powerful techniques of standard quantum field theory that allows solving the flow equations for general vertex functions in momentum space.

This expansion encompasses the standard curvature expansion with the additional benefit that generic covariant momentum dependences are systematically accessible. (For a respective brief discussion, see 184.) To understand this statement, we sketch the curvature expansion of the background field approximation with standard heat kernel techniques and the flat expansion in momentum space. We shall see that both lead to the same flow equations for the expansion coefficients of diffeomorphism-invariant operators. We expand the full one-field effective action in local curvature invariants and covariant derivatives

$$\Gamma_k[g_{\mu\nu}] = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} (2\Lambda_k - R) + O(R_{\mu\nu\rho\sigma}^2, \nabla^2). \quad (76)$$

In 76, the first term on the right-hand side is the Einstein–Hilbert action with a scale-dependent cosmological constant and Newton coupling. The second term includes all other curvature invariants starting with $R^2, \dots$. Covariant-derivative terms, schematically given by $\int \sqrt{g} R \nabla^2 R$ and terms with higher-orders in covariant derivatives ∇ , kick in at the next order and beyond. Note that the scale-dependent Einstein–Hilbert action without higher order terms is still a common approximation for the pure gravity sector in particular in many applications to gravity–matter systems. (For gravity–matter systems beyond the Einstein–Hilbert truncation see, e.g., 58, 65, 186–188.)

Similarly to 76, the flow of the background effective action can also be expanded in terms of *local* curvature invariants and covariant derivatives. This leads us to

$$\frac{1}{2} \text{Tr} G_k \partial_t R_k [g_{\mu\nu}] = \frac{1}{16\pi} \int d^4x \sqrt{g} (2a_{1,k} - a_{R,k} R) + O(R_{\mu\nu\rho\sigma}^2, \nabla^2), \quad (77)$$

with expansion coefficients $a_{\mathcal{O},k}$ of a given operator $\mathcal{O}$. In particular, we have $a_{1,k} = a_{R^0,k}$. By comparing 76 and 77, we arrive at the flow equations

$$\partial_t \frac{1}{G_k} = a_{R,k}, \quad \partial_t \frac{\Lambda_k}{G_k} = a_{1,k}. \quad (78)$$

Evidently, any *complete* projection procedure produces the *complete* set of flow equations of all expansion coefficients $a_{\mathcal{O},k}$. We emphasize that if the operator basis is overlapping, the flow of the effective action is unique, while the flow of the set of $a_{\mathcal{O},k}$ is not.

The standard procedure for projecting onto the flow of the cosmological constant and the Newton coupling, as well as that of higher order invariants, is by heat kernel techniques or explicit summation over the spectrum of the covariant Laplacians, in conjunction with the Euler–Maclaurin formula (see the reviews [7–16]). As no other *local* diffeomorphism-invariant operators are present at this order, the flow of G_k and Λ_k depends only on the given approximation of $\Gamma_k^{(0,2)}$ on the right-hand side of the flow. As already indicated, at higher orders of the curvature expansion, more and more invariants are present, and the projections on one single invariant only give unambiguous results if a complete basis of invariants is chosen. In fermionic systems, this is the well-known Fierz ambiguity. (See the review [26] for an extended discussion.) Consequently, at higher orders of the expansion, one typically has to deal with two truncation artifacts: first, we always have to deal with the truncation of $\Gamma_k^{(0,2)}$, and second, we have to deal with incomplete bases. We note that only very recently the second order has been mapped out (see 189). This emphasizes that we have to deal with an intricate technical challenge.

Now, we derive the flows in 78 within the flat expansion scheme. To that end, we note that the only *local* diffeomorphism-invariant term with no derivatives is the volume term $\mathcal{V} = \int \sqrt{g}$. Moreover, the only *local* diffeomorphism-invariant term with two derivatives is the curvature scalar term. This implies that we have a unique projection at the flat expansion point 75, schematically written as

$$\begin{aligned} -\frac{8\pi}{\mathcal{V}} \partial_t \Gamma_k[g] \Big|_{g \rightarrow 1} &= \partial_t \frac{\Lambda_k}{G_k} = a_{1,k}, \\ 16\pi \partial_{p^2} \partial_t \Gamma_{k,\text{TT}}^{(2)}[g] \Big|_{g \rightarrow 1} &= \partial_t \frac{1}{G_k} = a_{R,k}, \end{aligned} \quad (79)$$

where the subscript TT refers to the projection and normalization on the traceless-transverse part. (More details can be found, e.g., in 184.) **Equation 79** simply is **78**, as the flat expansion scheme is a consistent projection scheme.

This procedure can be extended beyond the set of *local* diffeomorphism-invariant operators:

- (1) Take general derivatives with respect to $h_{\mu\nu}(p)$.
- (2) Contract with all possible Lorentz tensor structures.
- (3) Take derivatives with respect to momenta.

In particular, apart from all *local* diffeomorphism-invariant term, the expansion captures general covariant momentum dependences including potential IR-singular terms and topological terms. A diffeomorphism-invariant example for the former is the Polyakov action in two dimensions,

$$-\frac{1}{96\pi} \int d^2x \sqrt{g} R \frac{1}{\Delta} R \quad (80)$$

(see, e.g., 190, 191). IR-singular terms are naturally covered by taking into account full momentum dependences of full vertices or momentum channels of specific tensor structures. This has been used extensively in gauge theories such as QCD not only within the fRG approach but also in other functional approaches based on Dyson–Schwinger equations or n -particle irreducible hierarchies.

A relevant example for a topological term in gravity is the Gauß–Bonnet term with the density

$$E[g] = \frac{1}{32\pi^2} (R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}). \quad (81)$$

Metric variations of the local density $E[g]$ are nonvanishing to all order of metric derivatives. In turn, its space-time integral $\chi[g] = \int d^4x \sqrt{g} E[g]$ is the Euler characteristic of the manifold M with $\chi[g] \in \mathbb{Z}$. Consequently, smooth metric variations of $\chi[g]$ (no change of the geometry) are vanishing. Note however that functional derivatives are distributional and do not fall into the class of smooth variations. Moreover, only the combination of the different curvature-squared invariants in **81** add up to the Euler characteristic $\chi[g]$. The single terms have a generic metric dependence, and with appropriate projections, we can capture their running coefficients. This is the manifestation of a more generic feature, which is already used in the extraction of anomalies in perturbation theory and anomalous as well as topological terms beyond perturbation theory (see, e.g., [192, 193]).

Below, we outline a cautious approach guided by the works 192, 193 in gauge theories. There, a simple example for a topological invariant is the Pontryagin index in a $U(1)$ theory with the density $1/(32\pi^2) F_{\mu\nu} \tilde{F}^{\mu\nu}$, where $\tilde{F}^{\mu\nu}$ is the dual field strength. This density is quadratic in the field and is discussed in Supplementary Material. Analogously to this example, we introduce the Gauß–Bonnet term with a local auxiliary field $\theta(x)$,

$$\chi[g, \theta] = \int d^4x \sqrt{g} \theta(x) E[g], \quad \chi[g, 1] \in \mathbb{Z}. \quad (82)$$

The auxiliary field $\theta(x) = \theta_{\text{top}} + \Delta\theta(x)$ can be seen as the local coupling of the Gauß–Bonnet density. Its constant part θ_{top} with $\nabla\theta_{\text{top}} = 0$ is the topological coupling, while its space-time-dependent part $\Delta\theta(x)$ is part of the couplings of the local diffeomorphism-invariants quadratic in the curvature. Applying two derivatives with respect to the metric field in momentum space leads us to

$$\frac{\delta^2 \chi[g, \theta]}{\delta g_{\mu\nu}(p) \delta g_{\rho\sigma}(q)} \Big|_{g=\delta} = \frac{1}{16\pi^2} \mathcal{T}^{\mu\nu\rho\sigma}(p, q) \theta(l) \delta(l+p+q). \quad (83)$$

The tensor structure $\mathcal{T}$ is given by

$$\begin{aligned} \mathcal{T}^{\mu\nu\rho\sigma} &= \Pi_0 \left(\delta^{\mu\nu} \delta^{\rho\sigma} - \delta^{\mu(\sigma} \delta^{\rho)} \right) + \Pi_2^{\mu\nu\rho\sigma} \\ &+ \delta^{\mu\nu} \Pi_1^{\rho\sigma} + \delta^{\rho\sigma} \Pi_1^{\mu\nu} - \delta^{\mu(\rho} \Pi_1^{\sigma)\nu} - \delta^{\nu(\rho} \Pi_1^{\sigma)\mu}, \end{aligned} \quad (84)$$

where we have defined

$$\begin{aligned}\Pi_0 &= p^2 q^2 - (p \cdot q)^2, \\ \Pi_1^{\alpha\beta} &= 2(p \cdot q) p^{(\alpha} q^{\beta)} - p^\alpha p^\beta q^2 - q^\alpha q^\beta p^2, \\ \Pi_2^{\alpha\beta\gamma\delta} &= p^\alpha p^\beta q^\gamma q^\delta + p^\gamma p^\delta q^\alpha q^\beta - p^\alpha p^{(\gamma} q^{\delta)} q^\beta - p^\beta p^{(\gamma} q^{\delta)} q^\alpha.\end{aligned}\quad (85)$$

The (local) total derivative property of the Gauß–Bonnet density is reflected in the fact that all Π_i are vanishing for $l = 0$ when momentum conservation implies $p = -q$. Accordingly, with $\theta(x) = \theta_{\text{top}}$ and $\theta(l) = \theta_{\text{top}} (2\pi)^4 \delta(l)$, the right-hand side of 83 vanishes. However, by collecting the θ terms on the left-hand and right-hand side of the flow, one can simply project the flow on the running of the coefficient of the topological term. We emphasize that the vanishing of the flow for constant θ is analogous to the vanishing of the flow $p^2 \partial_t Z_\phi$ for $p^2 = 0$. In conclusion, the present expansion scheme is well-capable and well-suited for describing IR divergent and topological terms.

In summary, the flat expansion allows projecting the flow equation on the flow of all coefficients $a_{\mathcal{O},k}$ for diffeomorphism-invariant operators of the form

$$\mathcal{O} = \int_x \sqrt{g} f_{\mu_1 \dots \mu_n} (\nabla_1, \dots, \nabla_n) \prod_{i=1}^n R_{\mu_i \dots \mu_i}. \quad (86)$$

Here, ∇_i acts only on the i th Riemann tensor. In the case of the fluctuation correlation functions $\Gamma_k^{(0,n)}[\bar{g}]$, no expansion in curvature invariants is possible, but an expansion in covariant tensor structures is possible, though being even more intricate. In case of the flat expansion, this is done with considering all tensor structures of $\Gamma_k^{(0,n)}(p_1, \dots, p_n)$. How this can be done has been worked out in QCD (see, e.g., 167–171), and respective computational tools are provided e.g., by [169, 194, 195] or are in preparation.

The findings of the present section can be summarized as follows:

- (1) The flat expansion encompassed the curvature expansion. There is no conceptual difference, and both expansions are expansions about the flat background $\bar{g} = 1$.
- (2) The expansion point of the curvature or flat expansion is not the solution of the EoM, $\bar{g}_{\text{EoM}}^{\text{fluc}}$, and checks of the convergence of the expansion are in high demand.
- (3) The fluctuation approach within the flat vertex expansion resolves the difference between fluctuation and background field. As such it simply improves upon the background field approximation within the curvature expansion without introducing other approximations: fluctuation approach results benchmark that in the background field approximation and provide nontrivial reliability checks for the latter.

There are an increasing number of computations that do not rely on the curvature expansion, for example, 49, 51, 54, 109–111, 175–182 in the background field approximation and 97, 107 in the fluctuation approach. This concludes our discussion of the formal properties of the fluctuation approach.

7.3 Tensor Structure and Momentum Dependence of Vertices

In the flat expansion, the vertices $\Gamma^{(n)} = \Gamma^{(0,n)}$ are typically rescaled with the wave function renormalizations Z_{ϕ_a} to obtain the RG-invariant vertices $\bar{\Gamma}^{(n)} = \bar{\Gamma}^{(0,n)}$

$$\Gamma_k^{(\phi_{a_1} \dots \phi_{a_n})}(\mathbf{p}) = \left(\prod_{i=1}^n Z_{\phi_{a_i}}^{\frac{1}{2}}(p_i^2) \right) \bar{\Gamma}_k^{(\phi_{a_1} \dots \phi_{a_n})}(\mathbf{p}), \quad (87)$$

where, $\mathbf{p} = (p_1, \dots, p_n)$. The wave function renormalizations can be fully absorbed by a redefinition of the fields $\bar{\phi}_a = \sqrt{Z_{\phi_a}} \phi_a$. The wave function renormalization enter the flow equations only via the anomalous dimensions η_{ϕ_a} defined by

$$\eta_{\phi_a}(p^2) = -\partial_t \ln Z_{\phi_a}(p^2), \quad (88)$$

which describes the running of the rescaled fields $\partial_t \bar{\phi}_a \propto \eta_{\phi_a} \bar{\phi}_a$. The RG-invariant vertices $\bar{\Gamma}_k^{(n)}$ are then parameterized with a complete set of tensor structures $\mathcal{T}_j$ and respective RG-invariant dressings $\mathcal{A}_{k,j}$

$$\bar{\Gamma}_k^{(\phi_{a_1} \dots \phi_{a_n})}(\mathbf{p}) = \mathcal{A}_{k,j}^{(\phi_{a_1} \dots \phi_{a_n})}(\mathbf{p}) \mathcal{T}_j^{(\phi_{a_1} \dots \phi_{a_n})}(\mathbf{p}; \text{couplings}), \quad (89)$$

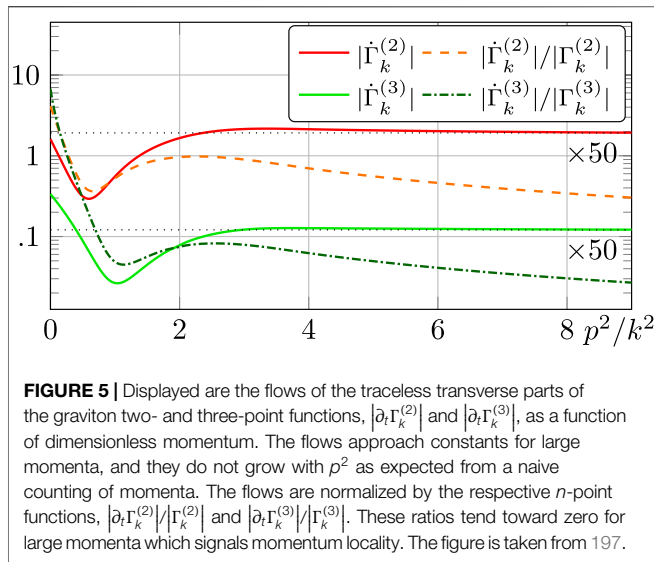
where the sum over j is implied. The size of the complete set of tensor structures increases rapidly for higher order vertices. The cutoff-dependent dressings $\mathcal{A}_{k,j}$ capture the overall coupling strength of the respective tensor structure and its momentum dependence.

In most applications to gravity, only the Einstein–Hilbert tensor structures deduced from the curvature scalar and the volume term are taken into account. This leads us to

$$\begin{aligned}\mathcal{A}_k^{(n)}(\mathbf{p}) &= G_n^{n-1}(\bar{p}^2), \\ \mathcal{T}^{(\phi_{a_1} \dots \phi_{a_n})} &= G_N S_{\text{EH}}^{(\phi_{a_1} \dots \phi_{a_n})}(\mathbf{p}; \Lambda_n),\end{aligned}\quad (90)$$

with the Einstein–Hilbert action 1 and the momentum-dependent global dressing $\mathcal{A}_k^{(n)}$ of the Einstein–Hilbert tensor structure. The prefactor G_N in the definition of the tensor structure leaves the latter independent of G_N . The couplings G_n and Λ_n resemble the Newton coupling and the cosmological constant, respectively, for each n -point function. They are called *avatars* of the respective coupling. In C, we have already simplified the momentum dependence of the couplings G_n : they only depend on the average momentum $\bar{p}^2 = (p_1^2 + \dots + p_n^2)/n$. The couplings G_n are extracted from the flow of the n -point functions at a momentum symmetric point. This definition mimics the definition of momentum-dependent couplings in gauge theories. The dimensionless counterparts of G_n and Λ_n are denoted by $g_n = G_n k^2$ and $\lambda_n = \Lambda_n/k^2$.

For $n = 0, 1$, we have $\Gamma^{(0,n)} = 0$ for a flat background. For $n = 2$, we have $G_n^{n/2-1} = G_2^0 = 1$, and hence, there is no Newton coupling G_2 for the two-point function. Instead, $\Gamma_k^{(0,2)}$ depends on the graviton mass parameter $\mu = -2\lambda_2$ and the dimensionless wave function renormalization $Z_h(p)$ of the fluctuation graviton. We emphasize that the graviton mass parameter μ should not be understood as a physical mass. Moreover, the graviton is not directly related to an asymptotic state. (For a recent discussion,



see 196 and also the review 16.) All dimensionless couplings are shown in **Figure 7** as a function of the cutoff scale on one exemplary UV-IR trajectory.

7.4 Momentum Locality

An important property of a physical coarse-graining procedure is momentum locality: it ensures that a coarse-graining step at a given cutoff scale k does not influence the physics at *momentum* scales $p \gg k$. In 197, it was defined by

$$\lim_{\frac{p^2}{k^2} \rightarrow \infty} \frac{|\partial_t \Gamma_k^{(n,m)}(\mathbf{p})|}{|\Gamma_k^{(n,m)}(\mathbf{p})|} = 0, \quad \text{with } \mathbf{p} = (p_1, \dots, p_{n+m}), \quad (91)$$

In this definition, all momenta p_i of the correlation function $\Gamma_k^{(n,m)}(\mathbf{p})$ need to be sent to infinity such that there are no trivial cancellations for the momenta of internal propagators. This is, for example, achieved with a symmetric momentum configuration. The norm of the n -point function refers to a normalized tensor projection.

The condition 91 is satisfied by all perturbatively renormalizable local quantum field theories of scalars, fermions, and vector fields (including gauge fields in linear gauges with linear momentum dependences) by trivial counting of the momenta. In turn, nonrenormalizable theories with nontrivial momentum dependences of vertices are easily nonlocal. For example, the scalar field theory with an interaction term of the type $\int \phi^2 (\partial\phi)^2$ does not fulfill 91. Note that this theory has the power counting of Einstein–Hilbert gravity.

Thus, a naïve momentum counting in gravity leads to the conclusion that the coarse graining is not momentum local, neither in Einstein–Hilbert gravity nor in a higher derivative theory of gravity. One needs nontrivial cancellations between diagrams. In 198, such a cancellation was observed for the first time in the transverse traceless part of the graviton two-point function with Einstein–Hilbert vertices. In 197, this was extended

to the transverse traceless part of the graviton three-point function. Both cases are displayed in **Figure 5**. There are three diagrams (plus one ghost diagram) contributing to the flow of the graviton three-point function (**Figure 4**). The cancellation takes place between all diagrams and holds for all gauge fixing parameters and all momentum configurations of the three-point function, as long as all external and internal momenta are sent to infinity.

We close this section with the remark that the results in 197, while highly nontrivial, should be considered to be the first step in a fully conclusive analysis. Most notably, the observed locality does not hold for all tensor structures of the n -point functions considered there. In our opinion, this may hint at persistent nonlocalities introduced by the gauge fixing. If this can be solidified in further investigations, this should allow for selecting gauge fixings that make the coarse graining procedure for a given regularization procedure momentum local. Note that while momentum locality of a coarse graining procedure is not a necessary property, it certainly improves the convergence of standard approximation schemes which are typically momentum local. Moreover, if no momentum local coarse graining procedure can be found for a given theory, this casts serious doubts on the interpretation of such a theory as a local quantum field theory.

8 STATE OF THE ART

We are now ready to review the state of the art of asymptotically safe quantum gravity within the fluctuation approach. To facilitate accessing the relevance of the different results for the self-consistency of the approach, we start with a brief overview:

UV Fixed Point (Section 8.1): The existence of a UV fixed point with a finite-dimensional critical hypersurface ensures the UV finiteness and predictivity of the theory. With the fluctuation approach, this has been investigated for pure gravity in 73, 93, 97, 107, 184, 197–206. The UV fixed point is comparable with results in the background field approximation and thus consolidates these results. Three UV-attractive directions are found associated with $\sqrt{g}$, $\sqrt{g}R$, and $\sqrt{g}R^2$. First signs for apparent convergence within the vertex expansion were found [184].

UV-IR Trajectory (Section 8.2): A UV-IR trajectory allows us to connect to a classical GR regime and IR-SM physics if matter couplings are included. Classical GR regimes are accessed for $\mu \rightarrow 0$ (Gaussian fixed point), $\mu \rightarrow \infty$, and $\mu \rightarrow -1$, where $\mu = -2\lambda_2$ is introduced below (Section 8.3). The case $\mu \rightarrow -1$ was investigated in 73, 184, 198, 199. In the classical regime, the modified STIs and modified NIs reduce to standard STIs and NIs, which can be solved for small k .

Momentum Dependence and Unitarity (Section 8.3): The full momentum dependence, in particular of the propagator, opens a path toward a first investigation of unitarity via spectral reconstructions. The truncations already include the momentum dependence of the graviton two-, three-, and four-point functions at the momentum symmetric point [184, 197, 198] as well as the momentum dependence of the graviton-matter three-point vertices [93, 202–204]. The momentum dependence

has been also used to show the absence of IR divergences in the IR regime [198, 199] and to show the absence of $R_{\mu\nu}^2$ contributions at the UV fixed point [184].

Curvature Dependence (Section 8.4): The curvature dependence of the correlation functions allows extending the results from a flat background to a generic background. The full curvature dependence of the fluctuation correlation functions contains the information of the diffeomorphism-invariant effective action (Section 7.1). The first steps in this direction in pure gravity and scalar gravity systems have been done in 97, 107, 206. In 97, 107, the difference of the background and quantum EoM due to the mNI was explicitly computed (see Section 6.1 and Figure 2).

Gravity–Matter Systems (Section 8.5): The aim is to incorporate the SM degrees of freedom in asymptotically safe quantum gravity and eventually to retrodict SM parameters and to constrain beyond the SM physics [207–218]. Minimally and nonminimally coupled gravity–matter systems have been investigated with (partial) fluctuation approach techniques in 53, 83, 93, 97, 187, 202, 204, 219–224. A particularly interesting question is for which matter content the UV fixed point exists. First bounds were computed in 219; however, a qualitative difference between the results in the background field approximation and the fluctuation approach was found [220]. It was shown in 202 that higher order curvature terms are needed to fully address this question. (For gravity–matter systems with higher derivative gravity in the background field approximation, see 58, 183, 186.) The investigation in 97 is a first step toward the computational confirmation of the existence of an asymptotically safe fixed point for general gravity matter in the minimally coupled approximation. This opens a path toward reliable stability investigations of fully coupled gravity–matter systems.

Effective Universality (Section 8.6): Last, we discuss the potential *close perturbativeness* of the UV fixed-point regime of asymptotically safe gravity. This leads us to the concept of *effective universality*: the so-called *avatars* of the Newton coupling extracted from different correlation functions may agree up to differences that can be inferred from the modified STIs that relate these couplings [93, 203]. If present, effective universality may have a dynamical origin. The analysis of this intriguing property is also intricate due to truncation artifacts and RG scheme dependences. We close this overview by commenting on the related bimetric approach and hybrids of the background field approximation and the fluctuation approach.

Hybrid approaches: In hybrid approaches, one substitutes part of the fluctuation flow equations with background flow equations [66, 219, 225–232]. In most cases, this concerns the notoriously difficult pure gravity couplings: the derivation of fluctuation flows of pure gravity vertices such as the three- and four-point functions requires a significant computer algebraic effort. In advanced truncations, this is accompanied with numerical loop integrations in every flow step and interpolations of dressing functions with potentially several momentum and angular dependences. In turn, using the background field approximation for these vertices reduces this task to computing the flow of a single background coupling, whose flow equation is known analytically. This considerable

reduction makes it chiefly important to construct reliable background field approximation schemes as discussed in Section 5.2

An alternative to the use of the background field approximation for the pure gravity couplings is their identification with matter–gravity couplings. Such an identification implicitly relies on the concept of *effective universality* discussed in more detail in Section 8.6. There it is discussed that while the full system shows effective universality, it is only maintained if using the pure gravity couplings for the matter–gravity couplings. In turn, effective universality, as well as the compatibility with the full system, is lost if using the matter–gravity couplings as pure gravity ones. This hints at a surprisingly complicated interaction structure in gravity–matter systems whose origin is yet to be understood.

Bimetric approach: The bimetric approach, developed in 100–102, 233, is tantamount to the fluctuation approach reviewed here, as it rests on the distinction between the background metric and the fluctuation field. Technically, fluctuation and background correlation functions are defined in terms of an expansion of the full metric $g_{\mu\nu} = (1 + \epsilon)\bar{g}_{\mu\nu}$ with the fluctuation field $h_{\mu\nu} = \epsilon\bar{g}_{\mu\nu}$. This allows one to order the flow and the effective action in powers of ϵ . The power ϵ^n of the effective action is simply the fluctuation n -point function. This reads schematically

$$\Gamma_k[\bar{g}, h] = \sum_n \frac{\epsilon^n}{n!} \int \Gamma_k^{(0, h_{\mu_1\nu_1} \dots h_{\mu_n\nu_n})}[\bar{g}, 0] \cdot \bar{g}_{\mu_1\nu_1} \dots \bar{g}_{\mu_n\nu_n}, \quad (92)$$

in analogy to 68. The $\Gamma_k^{(0, n)}[\bar{g}, 0]$ have been baptized *level- n vertices* comprising the respective *level- n couplings*. The last and most important step concerns the extraction of the correlation function $\Gamma_k^{(0, n)}[\bar{g}, 0]$ from $\int \Gamma_k^{(0, n)}[\bar{g}, 0] \cdot \bar{g}^n$, as the computation of the flow requires the knowledge of the correlation function and not their contractions with metrics. This computation is either done by i) considering an expansion about a specific background such as the flat background, ii) computing the flow of the effective action for a generic metric $\bar{g}$, or iii) assuming a global form of the effective action and simply computing the flow in this closed form. Option i) is the fluctuation approach reviewed here. It is not built on the metric split with ϵ . Option ii) asks for advanced computational heat kernel techniques even within restrictions. These techniques have seen rapid development in the past decade, which may open a path toward their use in ii). Option iii) has been considered so far for *level-one* couplings. The *level-two* correlation functions that are required for the right-hand side of the flow equation then have been obtained within a further background field approximation. In summary, the bimetric approach or rather the computational options ii) and iii) offer an alternative approach to compute fluctuation correlation functions that may provide important cross-checks for the results discussed here.

8.1 Ultraviolet Fixed Point

The UV fixed point in the fluctuation approach has been discussed in 73, 93, 97, 107, 184, 197–206. This includes work

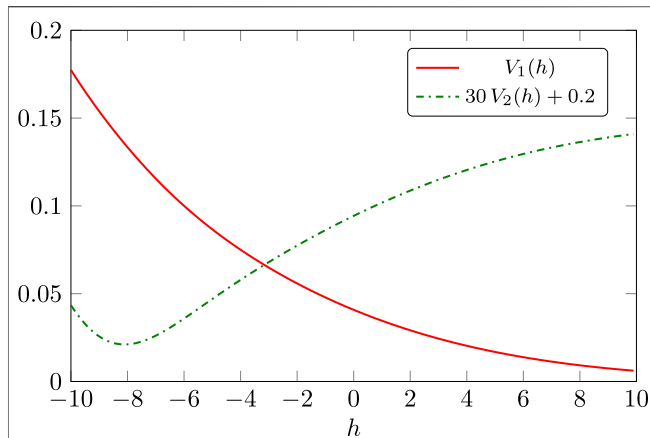


FIGURE 6 | Dimensionless fixed-point fluctuation potentials defined via $V = V_1(h) + \text{Tr}(h_{\text{TL}}^2)V_2(h)$, where h is the trace part and h_{TL} is the traceless part of the fluctuation graviton. Note that we rescaled and shifted V_2 , that is, V_2 is small compared to V_1 and always negative. The results are taken from 205.

in the vertex expansion about the flat background in pure gravity [184, 197–200] and gravity–matter systems [93, 201–204] as well as work including curvature dependence [97, 107, 206], a fluctuation potential [205] and in the geometrical approach [73]. In 184, the tower of fluctuation correlation functions was implemented until the graviton four-point function. All n -point functions were evaluated at the momentum-symmetric point, with external transverse traceless projections. A UV fixed point was found at

$$(\mu^*, \lambda_3^*, \lambda_4^*, g_3^*, g_4^*) = (-0.45, 0.12, 0.028, 0.83, 0.57), \quad (93)$$

where g_n and λ_n are the dimensionless Newton coupling and the momentum independent part of the graviton n -point function, respectively. (For more details, see 184.) The graviton mass parameter $\mu = -2\lambda_2$ is the momentum-independent part of the graviton two-point function. The critical exponents of the fixed point are given by

$$\theta_i = (4.7, 2.0 \pm 3.1i, -2.9, -8.0), \quad (94)$$

where a positive sign corresponds to a UV-attractive direction. The three UV-attractive directions were associated with the operators $\sqrt{g}$, $\sqrt{g}R$, and $\sqrt{g}R^2$. In contrast, the operator $\sqrt{g}R_{\mu\nu}^2$ is not generated in the present approximation. The latter property was inferred from the momentum dependence of the graviton three- and four-point function (see Section 8.3). Importantly, the first signs of apparent convergence were found in 184.

In Section 7.2, we have shown that the fluctuation approach in the flat expansion improves upon the background field approximation in the curvature expansion (see in particular the discussion at the summary at the end of Section 7.2). Accordingly, the fluctuation results for the UV fixed point detailed above extend and corroborate previous findings in the background field approximation within the curvature expansion.

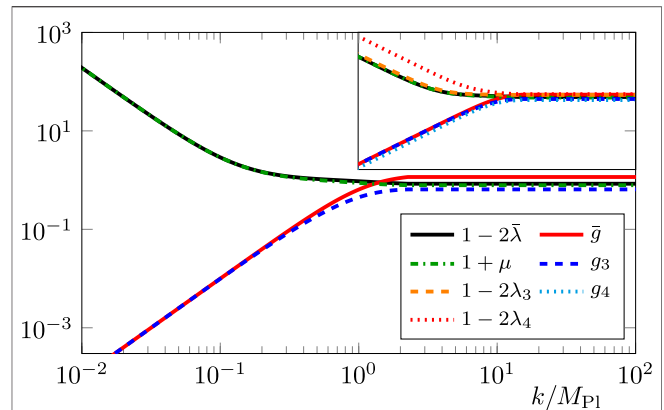


FIGURE 7 | Scale dependence of different fluctuation couplings along a trajectory from the UV fixed point 93 to the IR. In the IR, the couplings flow according to their canonical running. For small k , $\bar{g}$ and g_3 as well as $\bar{\lambda}$ and μ are related via the simplified NI (95). The inset shows the complete set of couplings. The results are taken from 184.

In particular, the results confirm that the latter captures the most important features in pure gravity. For example, the fixed-point value of the cosmological constant in the background field approximation is typically positive, which is comparable with the negative fixed-point value of μ in 93 ($\mu = -2\lambda_2$). Also, mostly three relevant directions are found in the background field approximation (see the reviews 7–16 and the very recent paper 189.)

A further extension, within the exponential split, has been investigated in 205. There, the dimensionless fluctuation potential V was approximated with $V = V_1(h) + \text{Tr}(h_{\text{TL}}^2)V_2(h)$, where h is the trace part and h_{TL} is the traceless part of the fluctuation graviton. The other graviton modes have been dropped. The results for the potentials V_1 and V_2 are displayed in Figure 6.

8.2 Ultraviolet–Infrared Trajectories

UV–IR trajectories in the fluctuation approach and hence the phase structure of quantum gravity have been discussed in 73, 184, 198, 199. In Figure 7, we display a trajectory from the UV fixed point (93) to the IR where all couplings run classically. In the displayed example, the graviton mass parameter runs to infinity, $\mu \rightarrow \infty$. In classical gravity and $\mu > 0$, the NIs entail that the cosmological constant is indeed given by $\Lambda = \bar{\Lambda} = -2\mu k^2$ in the limit $k \rightarrow 0$ and can take any negative value. This follows from

$$\frac{\delta\Gamma_k[\bar{g}, h]}{\delta\bar{g}} = \frac{\delta\Gamma_k[\bar{g}, h]}{\delta h}, \quad \text{for } \lim_{k \rightarrow 0} \mu \rightarrow \infty. \quad (95)$$

Moreover, the background Newton coupling and (all) the fluctuation Newton coupling agree. This can be seen for the dimensionless versions $\bar{\lambda}$, λ_2 and $\bar{g}$, g_3 in Figure 7. Solving the NIs for the higher couplings corresponds to a fine-tuning problem in terms of choosing an appropriate trajectory. However, a fully diffeomorphism-invariant solution including the higher order

avatars of the couplings has not been fine-tuned yet (see the inlay in **Figure 7**).

UV-IR trajectories with $\mu \rightarrow -1$ in the IR have also been investigated in 73, 184, 198, 199. Those trajectories are technically challenging since $\mu = -1$ corresponds to a pole in the propagator. We emphasize that the NIs and STIs are in this case nontrivial even for classical gravity: the classical effective action is the convex hull of the classical action, the latter not being convex for $\mu < 0$. This entails that $\lambda_2 = -2\mu$ cannot be identified with the cosmological constant $\lambda = \bar{\lambda}$ even though the sign of the latter must be also positive. Note also that any positive cosmological constant Λ can be obtained. The truncation triggered restriction to $\Lambda = 0$ at $k = 0$ in the background field approximation is lifted. From the physics point of view, these trajectories are appealing since they correspond to a positive cosmological constant in the IR.

8.3 Momentum Dependence and Unitarity

The momentum dependence of correlation functions have been discussed in 93, 184, 197–199, 202–204. This momentum dependence encodes the dynamics of the theory and is crucial for the question of unitarity. One of the advantages of the fluctuation approach in the flat vertex expansion is its easy access to the full momentum dependence of fluctuation correlation functions $\Gamma_k^{(0,n)}$ for all cutoff scales k . These momentum dependences carry the full dynamics of the underlying theory: all other quantities, ranging from the background correlation functions to diffeomorphism-invariant observables $\mathcal{O}[g]$, are built from the correlation functions. The latter observables are defined as expectation values $\mathcal{O}[g] = \mathcal{O}[\bar{g}, h = 0]$ of diffeomorphism-invariant operators $\hat{\mathcal{O}}[\bar{g}]$ with $\mathcal{O}[\bar{g}, \phi] = \langle \hat{\mathcal{O}} \rangle$. The $\mathcal{O}[\bar{g}, \phi]$ satisfy the flow equation for the expectation values of composite operators derived in 21,

$$\partial_t \mathcal{O}_k[g] = -\frac{1}{2} \text{Tr} \left[G_k \partial_t R_k G_k \mathcal{O}_k^{(0,2)} \right] [g], \quad (96)$$

at vanishing fluctuation field $h = 0$. Evidently, the flow **96** solely depends on the fluctuation field propagators and $\mathcal{O}[\bar{g}, \phi]$. (For applications and further investigations of **96**, see 21, 128, 133–135, 137, 138.)

Equation 96 entails in particular that *any* observable inherits its dynamics from that of the full field and momentum dependence of the fluctuation two-point function, or rather from the momentum dependence of the fluctuation correlation functions $\Gamma_k^{(0,n)}$ at a given field expansion point. It is in this sense that the momentum-dependent and RG-invariant vertex dressings $\mathcal{A}^{(\phi_1 \dots \phi_n)}(\mathbf{p})$ encode the dynamics of the theory. In particular, the symmetric point dressings $G_n(\bar{p})$ carry the meaning of momentum-dependent running couplings similar to those in standard quantum gauge theories and most notably in QCD. (For a detailed discussion in the latter case, see in particular 168, 170.) We emphasize that while in both cases these couplings are neither observables themselves nor even gauge- or diffeomorphism-invariant, they directly encode the dynamics of the theory, and in particular the dominance and/or decoupling of degrees of freedom. If done carefully, they can be also compared to scattering processes related to the respective

vertices. (For the SM, see the comparison of the QCD running (vertex) coupling to scattering experiments at accelerators [234].)

Moreover, the resolution of the momentum dependences of n -point functions gives at least indirect access to the question of unitarity of asymptotically safe gravity: From the Euclidean data, one can reconstruct Minkowski correlation functions and in particular the graviton spectral functions, both that of the fluctuation graviton and that of the background graviton (more details will be given in 235). Here, we simply comment on the physics content of the graviton spectral functions (see also 236). In this context, we will also use the analogy to the gluon in a non-abelian gauge theory as discussed in 237. (For a recent discussion of the challenges for unitarity in asymptotically safe gravity, see 16.)

To begin with, both the fluctuation graviton and the background graviton two-point functions are not diffeomorphism-invariant. Accordingly, they are not directly related to asymptotic states; even at low energies, gravity is weakly coupled and the theory exhibits a classical momentum and scale dependence (see **Figure 7**). The latter property suggests that if a Källén–Lehmann spectral representation of the graviton propagators exists, the graviton spectral functions may exhibit a particle-like spectrum for low spectral values. In turn, for large spectral values, we enter the UV fixed point regime, and the physics content of the spectral functions is unclear.

Note however that the same line of arguments would suggest that the gluon spectral function exhibits a particle-type spectral dependence in its perturbative regime for large spectral values. Instead, it can be shown that if a Källén–Lehmann spectral representation exists, the gluon spectral function is negative for large spectral values, and its spectral sum vanishes (Oehme–Zimmermann superconvergence relation). Moreover, it is also negative for small spectral values (see 237). These properties hold for both the fluctuation and the background gluon. Since these properties follow directly from the momentum dependence of the Euclidean correlation functions, we expect similar results for asymptotically safe gravity [235].

In summary, the spectral properties of diffeomorphism- or gauge-variant correlation functions only indirectly mirror the unitarity of the theory. This situation prohibits any direct conclusion of a lack of unitarity from the occurrence of negative parts of spectral functions including negative poles (ghost states). We also emphasize that the latter statement should not be taken as its converse. Of course, the occurrence of negative parts of spectral function requires a thorough investigation of the physics implications and may well be related to the lack of unitarity of the underlying theory. The example of the non-abelian theory simply indicates that this is not necessarily the case. Such an investigation requires the analysis of the spectral properties of diffeomorphism-invariant states. (For a recent discussion of such a setup, see 196.)

The discussion in this section so far emphasizes the importance of the computation of the momentum dependence of correlation functions both for the dynamics of observables and the intricate problem of unitarity. One of the advantages of the fluctuation approach is the direct access to momentum-dependent correlation functions with standard quantum field theory methods:

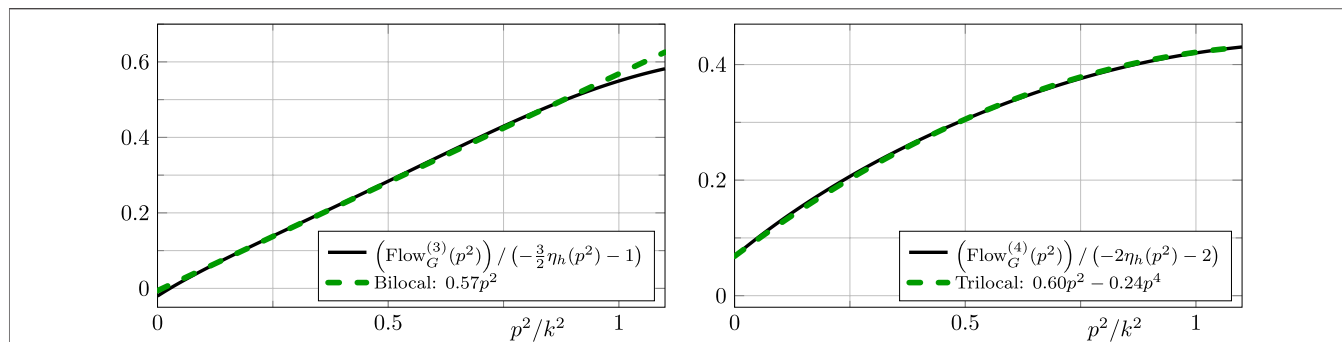


FIGURE 8 | Momentum dependence of the transverse traceless graviton three- and four-point couplings obtained by normalizing the vertex flow with $(-\frac{3}{2}\eta_h(p^2) - n + 2)$. The graviton three-point coupling (left panel) is well described with a linear p^2 function in the momentum range $0 \leq p^2 \leq k^2$. This momentum dependence stems from the R tensor structure. The absence of a p^4 behavior implies that the $R^2_{\mu\nu}$ tensor structure is suppressed. On the other hand, the graviton four-point coupling (right panel) shows a clear p^4 behavior, which is associated with R^2 tensor structure. The figures are taken from 184.

In 198, 199, the full momentum dependence of the graviton and ghost propagator was included via the anomalous dimensions. The computation of the momentum dependence was extended to the graviton three- [197] and four-point function [184] as well as to the scalar–graviton [93], the fermion–graviton [204], the gluon–graviton vertex [202], and the ghost–graviton vertex [203]. While only the momenta $0 \leq p^2 \leq k^2$ contribute to the flow, in all these works, the vertices have been computed at the symmetric point for the full momentum range $0 \leq p^2 < \infty$. This approximation ignores, in particular, the angular dependence of the vertex dressings. While the angular dependence is important for the discussion of the whole phase space of scattering experiments, it is averaged in the flow diagrams due to the angular loop integrations. The reliability of this approximation has been studied at length in QCD. (See 167–171 for detail.) There, it was shown that the above approximation is very accurate in the absence of resonant interaction channels, and so far, no indications have been found for such resonant channels. In conclusion, this analysis provides a nontrivial reliability argument for the approximation described above. Still, for a full reliability check, one has to study extended truncations.

In 184, 200, the momentum dependence was used to disentangle contributions from the couplings of the R^2 and $R^2_{\mu\nu}$ tensor structures. This was done in 200 with derivatives at vanishing momentum, while in 184, the momentum range $0 \leq p^2 \leq k^2$ was considered. Importantly, the transverse traceless graviton three-point function has overlap with $R^2_{\mu\nu}$ tensor structures and not with R^2 tensor structures, while the graviton four-point function has overlap with $R^2_{\mu\nu}$ and R^2 tensor structures. The momentum dependence of the couplings is obtained by normalizing the vertex flows with $(-(n/2)\eta_h(p^2) - n + 2)$. This is displayed in Figure 8. The three-point coupling is well described with a p^2 behavior. Thus, the $R^2_{\mu\nu}$ tensor structure is nontrivially suppressed. The four-point coupling shows a significant p^4 behavior. Due to the absence of a p^4 behavior in the three-point coupling, this suggests that they are related to R^2 tensor structures.

Recently, impressive progress has been made toward momentum-dependent computations in the background field

approximation [238–242]. There, the momentum dependence is captured via form factors W_k^i , for example, $\int_x R W_k^R(\Delta) R$ and $\int_x C_{\mu\nu\rho\sigma} W_k^C(\Delta) C^{\mu\nu\rho\sigma}$. This opens a path toward the comparison of the result of these two approaches. This will allow us to quantify the difference between a background field approximation and a fluctuation computation.

8.4 Curvature Dependence

The curvature dependence of correlation functions in the fluctuation approach has been discussed in 97, 107, 206. Most results in the fluctuation approach were computed on a flat background $\bar{g} = 1$. Results for generic backgrounds can be obtained from an expansion about the flat background. In 206, this was done with covariant heat kernel methods up to the first-order curvature couplings. Fixed-point values of all first-order curvature couplings were found and their gauge dependence investigated.

A different approach within the fluctuation approach was taken in 97, 107, where the fluctuation correlation functions were computed directly on a generic background with constant curvature. The computation reaches up to the graviton three-point function and also includes N_s scalar fields in 97. It was found that the curvature dependence of the fluctuation couplings counterbalances the explicit curvature dependence of the respective vertex, making the full vertex approximately curvature independent. This result supports results obtained on a flat background. Furthermore, it was explicitly shown that the background EoM differs from the quantum EoM at the UV fixed point (17). In particular, the background EoM does not have a solution at the UV fixed point, while the quantum EoM has two solutions, a minimum at negative curvature and a maximum at positive curvature, for all N_s that are accessible. This is displayed in Figure 2.

8.5 Gravity–Matter Systems

A theory of quantum gravity necessarily needs to include matter degrees of freedom to describe our universe. A central question is for which matter content, the UV fixed point exists and if certain types of matter field have a stabilizing or destabilizing effect. Most

studies have focused on analyzing SM matter fields within the minimally coupled approximation. In this approximation, the matter fields are considered without self-interaction and only couple to gravity via their kinetic term. There are works in the background field approximation [58, 183, 186, 243–249], in the hybrid approach [53, 219], and in a full fluctuation computation [93, 97, 201–204]. (For works beyond the minimally coupled approximation, see 47, 50, 83, 162, 186, 188, 210, 220–224, 228, 229, 232, 250–260, which also includes scalar–tensor theories and gravitational corrections to the running of matter couplings.)

A major keystone in the stability analysis of gravity–matter systems in the minimally coupled approximation was found in 201, 202. There, it was shown that minimally coupled gravity–matter systems in the Einstein–Hilbert truncation always show a Reuter fixed point as the system can be mapped to a pure gravity system at the level of the path integral. We emphasize that while the explicit computations in 201, 202 are done in the fluctuation approach, the conceptual investigation is general. (For a detailed discussion we refer to 202.) Here, we simply sketch the important steps: In minimally coupled gravity–matter systems, the matter part $S_{\text{mat}}[\bar{g}, \phi]$ of the full action $S = S_{\text{grav}} + S_{\text{mat}}$ is quadratic (or bilinear) in the matter fields. To find the Reuter fixed point, it is sufficient to discuss the UV limit of graviton correlation functions. Consequently, we consider vanishing matter sources, $J_{\text{mat}} \equiv 0$. After performing the Gaussian integration over the matter fluctuation fields $\hat{\phi}_{\text{mat}}$, the path integral of a minimally coupled matter–gravity system takes the schematic form,

$$Z[J] = \int \mathcal{D}\hat{\phi}_{\text{grav}} e^{-S_{\text{grav,eff}}[\bar{g}, \hat{\phi}_{\text{grav}}]} + \int d^4x \sqrt{\bar{g}} J_{\text{grav}}^a \hat{\phi}_{\text{grav},a}. \quad (97)$$

with

$$S_{\text{grav,eff}}[\bar{g}, \hat{\phi}_{\text{grav}}] = S_{\text{grav}}[\bar{g}, \hat{\phi}_{\text{grav}}] + \frac{1}{2} \text{Tr} \log S_{\text{mat}}^{(2)}[\bar{g}]. \quad (98)$$

Here, the full fluctuation field is split into $\phi = (\phi_{\text{grav}}, \phi_{\text{mat}})$ with $\phi_{\text{grav}} = (h_{\mu\nu}, c_\mu, \bar{c}_\mu)$, and the hatted field indicate the integration fields. In slight abuse of notation, we wrote $S_{\text{mat}}^{(2)}[\bar{g}]$ as the second derivative of the matter action with respect to the matter fields. Its argument is $\hat{g} = \bar{g} + \hat{h}$, the full metric that is integrated over. Hence, $S_{\text{mat}}^{(2)}$ is a covariant operator and the $\text{Tr} \log$ contribution is diffeomorphism-invariant.

The form of the generating functional in 97 is also obtained for UV-complete non minimally coupled matter theories such as Yang–Mills theories. Then, $S_{\text{grav,eff}}[\bar{g}, \hat{h}]$ is not of the form 98 but carries the full nonperturbative metric-dependent part of the effective action of Yang–Mills theories. The UV completeness within this procedure is required as otherwise the matter path integral cannot be performed. Trivially, minimally coupled systems are UV-complete. A useful analog for the study of the UV stability of minimally coupled gravity–matter systems is many-flavor QCD. There, the role of the graviton is taken by the gluon, and the quark action is bilinear.

The representation 97 emphasizes an intriguing and useful property of the fRG approach to quantum gravity (and beyond): The phase structure and in particular the fixed-point structure of a generic gravity–matter system can be accessed within pure

gravity. In particular, all fixed points are accessible within this setup, if a general fixed point effective action $\Gamma_k^*[\bar{g}, \phi_{\text{grav}}]$ is considered.

This intriguing property also carries an important intricacy of a generic fixed-point analysis: Seemingly, the parameterization 97 entails that generic gravity–matter systems are UV-stable if the matter part is UV-complete (with the assumption that the Reuter fixed point exists for pure gravity). This conclusion would apply directly to all minimally coupled gravity–matter systems. That this argument falls short can be seen at the example of many-flavor QCD. There, an (f)RG analysis reveals that the QCD β -function changes its sign for a large enough number of flavors. In the vicinity of this regime, interesting phenomena such as conformal scaling, instabilities, and the Caswell–Banks–Zaks fixed point occur (For fRG literature, see, e.g., 261–264 and references therein.) These findings are backed up by lattice results. The RG analysis in many-flavor QCD solely relies on the marginal operator $\text{tr} F_{\mu\nu}^2$. The quantum corrections from the integrating out of the quark fluctuations are proportional to

$$N_f \text{tr} F_{\mu\nu}^2 \log \frac{F_{\mu\nu}^2}{k^4}, \quad (99)$$

where N_f is the number of flavors. The analogous operators in gravity are the curvature-squared operators R^2 , $R_{\mu\nu}^2$, and $R_{\mu\nu\rho\sigma}^2$. The respective operators including matter quantum fluctuations are

$$N_{\text{mat}} \sqrt{\bar{g}} R^2 \log \frac{R}{k^2}, \quad (100)$$

and similar ones for $R_{\mu\nu}^2$ and $R_{\mu\nu\rho\sigma}^2$ and also covariant derivatives. Here, N_{mat} is the weighted sum over all species and flavors of matter fields.

The logarithmic RG running of the marginal operator $\text{tr} F_{\mu\nu}^2$ in QCD or R^2 in gravity necessarily triggers a field dependence of its coefficient as displayed in 99 and 100, respectively. In conclusion, the distinctive property of marginal operators is the inherent field dependence of the quantum corrections. In turn, the coefficients of (local) relevant and irrelevant operators are only scale-dependent. While the latter by definition are not important for a fixed-point analysis, the coefficients of the former ones, if present, can be readily absorbed in the respective pure gauge theory (or gravity) couplings. In the present example of many-flavor QCD, relevant operators are indeed absent. In gravity, this applies to the terms in the Einstein–Hilbert action, that is, the curvature term and the cosmological constant term.

In summary, from the perspective of the Yang–Mills system with the generating functional similar to that in 98, the marginal operator 99 introduces a new UV marginal (and hence physical) parameter N_f that cannot be absorbed in the Yang–Mills coupling. In gravity, this applies to the coefficients of the marginal operators R^2 , $R_{\mu\nu}^2$, and $R_{\mu\nu\rho\sigma}^2$. Thus, also here, the flavor number N_{mat} of a given matter field is a physical parameter. However, its relevance for the fixed-point analysis originates *solely* from the N_{mat} -dependent coefficients of the marginal operators R^2 , $R_{\mu\nu}^2$, and $R_{\mu\nu\rho\sigma}^2$. In contrast, N_{mat} of the relevant operators in the

Einstein–Hilbert action is not relevant for the fixed-point analysis. In particular, it cannot trigger instabilities.

The above properties imply that a fixed-point analysis of a given system within a truncation of the (f)RG flows that does not include the flows of the marginal operators should exhibit the respective fixed-point structure of the pure gravity system in the same truncation. In particular, this casts some doubt on any instability findings in the full truncation, if this instability survives in the absence of the marginal operators.

As an example of this statement, we consider now a minimally coupled gravity–matter system in the Einstein–Hilbert truncation. Without truncation, these systems have the path integral representation **97** with **98**. The Einstein–Hilbert truncation reduces $S_{\text{grav,eff}}$ in **98** to

$$S_{\text{grav,eff}}[\bar{g}, \hat{\phi}_{\text{grav}}] \rightarrow S_{\text{EH}}[g] + S_{\text{gf}}[\bar{g}, \hat{h}] + S_{\text{gh}}[\bar{g}, \hat{\phi}_{\text{grav}}] + \frac{1}{2} \text{Tr} \log S_{\text{mat}}^{(2)}[\hat{g}]|_{R,\Lambda}, \quad (101)$$

where the subscript $|_{R,\Lambda}$ stands for the reduction of the full one-loop determinant to its Einstein–Hilbert part with a curvature term and a cosmological constant term. The respective coefficients can be absorbed in a redefinition of the Newton constant and cosmological constant in $S_{\text{EH}}[g]$. (For more details, see **202**. Hence, **101** is equivalent to the Einstein–Hilbert truncation of the pure gravity system. The latter shows the Reuter fixed point and so should the minimally coupled system in this truncation.

The above result for minimally coupled systems has the direct consequence that the Einstein–Hilbert truncation to matter–gravity systems should also exhibit the Reuter fixed point for UV-complete matter systems, as the pure gravity system does. We add that this does not exclude the emergence of further fixed points in some N_{mat} regime.

This concludes our discussion of the fixed-point structure and stability properties of gravity–matter systems, its truncation dependence, and reliability requirements for truncations. The discussion enables us to formulate relevant properties that have to be considered for a conclusive stability analysis of matter–gravity systems:

- (1) The fixed-point analysis necessarily has to involve all (possibly) relevant operators of the theory under investigation, that is, **99** in many-flavor QCD and **100** in gravity–matter systems.
- (2) A fixed-point analysis within a given truncation is only fully reliable if it also reproduces the fixed-points of the pure gravity system in the same truncation excluding the marginal operators.

We now discuss the results in gravity–matter systems given the properties i) and ii): In **[201]**, the first full fluctuation computation for minimally-coupled systems was put forward. On the pure gravity side, the flows of the fluctuation graviton two- and three-point function were included. Importantly, a stabilizing mechanism for the fermionic contribution was found for general regulators: the graviton mass parameter is approaching its pole

$\mu \rightarrow -1$ and thus enhances the graviton contribution, in short: *gravity rules*. This is required from the discussion above. Technically, this simply means that the fermion contribution in this setup changes the parameters of the two- and three-point function within the stability regime of the phase diagram of pure gravity in the Einstein–Hilbert truncation. This stabilizing mechanism was also found in an extension of the truncation **[204]**, making the fermion–gravity system a showcase of the mechanism described above. In particular, with the existence of the Reuter fixed point for the minimally coupled system in the absence of marginal operators in the pure gravity subsystem, the flow equations of the fermion–gravity system satisfy the requirement ii). Consequently, a conclusive stability analysis of general fermion–gravity systems can be performed but requires the inclusion of the marginal curvature-squared operators.

In the same truncation applied to minimally coupled scalar–gravity systems, it was found within the fluctuation approach in **93**, **201** that the graviton anomalous dimension η_h grows with the number of scalars N_s and finally exceeds the value two beyond a critical flavor number $N_{s,\text{stab}}$: for $N_s > N_{s,\text{stab}} \approx 20$. For $\eta_h > 2$, the overall cutoff scaling of the graviton regulator goes with negative powers of the cutoff scales and effectively the—physical—cutoff decreases. For these large anomalous dimensions, we leave the reliability regime of the approximation. In short, the reliability bound on the truncation makes it impossible to see the stability of the system in this minimally coupled approximation. From the viewpoint of the pure gravity system, this simply means that the scalar contribution in this setup eventually moves the parameters of the two- and three-point function outside the stability regime of the phase diagram of pure gravity in the Einstein–Hilbert truncation. Consequently, the setup cannot be used for stability investigations in scalar–gravity systems. In **97**, it was suggested that an expansion about an on-shell background can lift this tension. In summary, at present, there is no conclusive stability analysis for scalar–gravity systems.

Applying the same truncation to minimally coupled gauge–gravity systems, it has been shown in **202** that depending on the regulator, the minimally coupled systems either behave similarly to the fermionic or the scalar system. This suggests that the truncation has to be improved. In summary, a stability analysis of gauge–gravity systems can be performed, but the results have to be taken with a grain of salt. A fully conclusive stability analysis for gauge–gravity systems requires an improvement of the truncations used so far in the literature.

In **Figure 9**, we display the state-of-the-art dependence of the fixed-point values on the number of scalar field N_s **[93]**, fermion field N_f **[204]**, and gauge fields N_v **[202]**. The truncations include the flow of the momentum-dependent graviton two- and three-point functions as well as the respective graviton–matter vertex. In the scalar case, the Newton couplings are diverging at $N_s \approx 52$. This is an artifact of the truncation, as described in the previous paragraphs and **202**. The fermion direction is stable for all N_f : the graviton mass parameter approaches its pole $\mu \rightarrow -1$ and the enhanced graviton contribution counterbalances the matter contribution. In the gauge case, the fixed point is disappearing

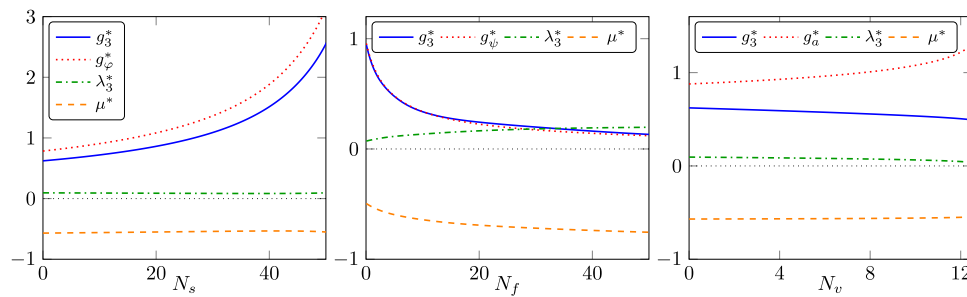


FIGURE 9 | Fixed-point values of the fluctuation couplings as a function of the number of scalar (left), fermion (middle), and gauge fields (right). All truncations include the graviton two- and three-point function as well as the respective graviton–matter vertex. In the scalar case, the Newton couplings, g_3 and g_ϕ , are diverging at $N_s \approx 52$. The fermionic case is stable for all N_f . In the gauge field case, the fixed point is disappearing in the complex plane at $N_v \approx 13$. It was explained in 202 that the vanishing of the fixed point is an artifact of the truncation and how it can be lifted in the gauge-field case. In 97, it was suggested that an expansion about a background that is a solution to the quantum EoM might remove the divergence in the scalar case. The result are taken from 93 (scalar), 204, (fermion), and 202 (gauge).

in the complex plane for $N_v \approx 13$. In 202, it was demonstrated that all numbers of gauge fields can be accessed with a different regulator, as discussed in the last paragraph.

Finally, we speculate on the stability properties of general gravity–matter systems based on the results obtained so far. To that end, we assume that there is a setup such that general minimally coupled gravity–matter systems in the Einstein–Hilbert truncation show UV stability with a Reuter fixed point similar to the one seen in the fermion–gravity system. This property allows for a consistent truncation as it satisfies ii). Now, we include tensor structures from curvature-squared terms, R^2 , $R_{\mu\nu}^2$, and $R_{\mu\nu\rho\sigma}^2$. It is convenient to parameterize this complete set of tensor structures in terms of the Ricci squared, the Weyl tensor squared, and the topological Gauß–Bonnet term,

$$\int d^4x \sqrt{g} \left(\frac{1}{g_{R^2}} R^2 + \frac{1}{g_{C^2}} C_{\mu\nu\rho\sigma}^2 + \frac{1}{g_E} E \right), \quad (102)$$

with the dimensionless couplings g_{R^2} , g_{C^2} , and g_E . The Gauß–Bonnet density E is defined in 81, and the Weyl tensor squared is in four dimensions given by

$$C_{\mu\nu\rho\sigma}^2 = R_{\mu\nu\rho\sigma}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2. \quad (103)$$

We concentrate on the Reuter fixed point with the assumption that it is dominated by the Einstein–Hilbert couplings in contradistinction to the perturbative R^2 fixed point. In 184, it has been observed that $R_{\mu\nu}^2$ contributions and hence C^2 contributions generated by the Einstein–Hilbert tensor structures at the Reuter fixed point are small. They are subleading in comparison to the R^2 tensor structure. This has been the topic of Section 8.3 (see Figure 8 and the respective discussion). With the assumption of the dominance of the Einstein–Hilbert couplings, it implies $1/g_{C^2}^* \approx 0$ and indicates the irrelevance of this operator at the fixed point.

Moreover, from the quartic term $c_4 p^4$ with $c_4 \approx -0.24$ in the running of the momentum-dependent coupling of the four-point function displayed in Figure 8, we deduce that its contribution

$c_{EH} g_{R^2}^2$ to the β -function $\beta_{g_{R^2}} = \partial_t g_{R^2}$ of the R^2 tensor structure is positive. (For a detailed discussion, see 184.) In turn, it is well-known that the R^2 coupling itself leads to a negative contribution $-c_{R^2}(g_{R^2})$, which is one-loop universal. We emphasize that both coefficients depend on the full fluctuation propagator. In combination, this leads us to a β -function

$$\partial_t g_{R^2} = \beta_{g_{R^2}} \approx c_{EH} g_{R^2}^2 - c_{R^2}(g_{R^2}). \quad (104)$$

Switching off the Einstein–Hilbert contribution leads us to the standard Gaußian fixed point for R^2 gravity. In turn, at the Reuter fixed point, we assume a small fixed-point value for $1/g_{R^2}$ that may also trigger a small, but nonvanishing fixed-point value for $1/g_{C^2}$. Combining these estimates for g_{R^2} and g_{C^2} , we arrive at

$$\frac{1}{g_{R^2}^*}, \frac{1}{g_{C^2}^*} \approx 0, \quad \text{and} \quad \frac{g_{C^2}^*}{g_{R^2}^*} \approx 0, \quad (105)$$

in pure gravity. We add that the relevance analysis in 184 suggests that the g_{R^2} coupling, while being small, is UV-relevant at the Reuter fixed point. This finding is corroborated by respective ones in the background approximation. (For higher derivative gravity work in the background field approximation, see, e.g., 49, 51, 54, 56, 58, 108–110, 172–174, 176, 183, 186, 189, 219, 265–269.)

We now proceed to the R^2 and C^2 contributions from matter fluctuations. Being short of a full fluctuation computation of these terms, we utilize the Nielsen identities in the presence of the cutoff (see 54 and 44 in Sections 5.2 and 6). The identity 44 comprises the difference between background-metric and fluctuation field derivatives, while the Nielsen identities 54 also take into account the difference introduced by the gauge fixing sector. For the present speculative analysis, it suffices to discuss 44. For example, we find for the R^2 contribution,

$$\left[\frac{\delta \Gamma_k}{\delta \bar{g}} - \frac{\delta \Gamma_k}{\delta h} \right]_{R^2} \approx \text{Tr} \left. \frac{\delta \sqrt{\bar{g}} R_k}{\sqrt{\bar{g}} \delta \bar{g}_{\mu\nu}} G_k \right|_{R^2} = \Delta g_{R^2}(\bar{g}) \int d^4x \sqrt{\bar{g}} R^2, \quad (106)$$

The right-hand side has a form similar to the flow equation itself and is UV- and IR-finite. Accordingly, $\Delta g_{R^2}(\bar{g})$ is a

dimensionless constant that depends on all couplings taken into account in the computation, summarized as vector $\vec{g}$. This includes the R^2 and C^2 couplings g_{R^2} and g_{C^2} themselves (or rather avatars thereof), as well as avatars of the dimensionless Newton coupling and the dimensionless cosmological constant (see **Section 8.1**). The scale derivative of **106** vanishes on a fixed point,

$$\partial_t \left[\frac{\delta \Gamma_k}{\delta \vec{g}} - \frac{\delta \Gamma_k}{\delta h} \right]_{R^2} \approx - \frac{\partial \Delta g_{R^2}(\vec{g}^*)}{\partial g_i} \beta_i(\vec{g}^*) = 0, \quad (107)$$

where we have used that the dimensionless coefficient $\Delta g_{R^2}(\vec{g}^*)$ cannot have an explicit k -dependence. Hence, *at a fixed point*, this result allows us to identify the matter contribution of the flow for R^2 tensor structures of fluctuation field vertices with that of the background field R^2 term. The same reasoning also applies to the C^2 term. In summary, the above arguments imply that the matter contributions to the curvature-squared couplings should be independent of the background-metric dependence of the regulator, as well as of the shape of the regulator. Moreover, since the ghost contribution to the curvature-squared couplings also does not depend on other scales than the cutoff scale, it should also be regulator-independent. The validity of these general statements can be checked explicitly with the results of 58, 183. There, different types of regulators have been investigated in $f(r)$ gravity: all couplings, except the R^2 -coupling, depend on the Laplacian used in the regulators. The results also confirm a regulator dependence of the graviton contributions, triggered by the Nielsen identities. As discussed above, this suggests that the pure gravity contributions to the flow should rather be computed within the fluctuation approach.

The above considerations allow us to discuss the generic structure of gravity matter flows within the fluctuation approach,

$$\begin{aligned} \beta_{g_{R^2}} &= \beta_{g_{R^2}}|_{\text{grav}} - c_{R^2} N_{R^2} g_{R^2}^2, \\ \beta_{g_{C^2}} &= \beta_{g_{C^2}}|_{\text{grav}} - c_{C^2} N_{C^2} g_{C^2}^2, \end{aligned} \quad (108)$$

where c_{R^2/C^2} are positive coefficients and N_{R^2/C^2} are weighted sums (positive weights) of the numbers of scalars, vectors, and fermions. All matter contributions have the same sign, which is the same as that of the gravity-ghost, which is computationally similar. (For explicit computations in the background field approximation, see, e.g., 58, 183, 186, 219.)

The quantitative evaluation of **108** depends on the full fluctuation flows in pure gravity including flow contributions from curvature-squared invariants. Here, we concentrate on the structure of the β -function of the R^2 coupling, $\beta_{g_{R^2}}$. The matter contributions are subtracted from the positive Einstein-Hilbert gravity contribution (see c_{EH} in **104**). For a critical number of matter fields, the complete contribution vanishes, and we are left with a system that resembles the pure gravity curvature-squared system. This mechanism is very similar, leading to the Caswell-Banks-Zaks fixed point in QCD discussed before. Note that in contradistinction to the minimally coupled system, the matter contribution cannot be absorbed in the pure gravity contributions, as they are related to

$R^2 \log(1 + R/k^2)$ terms. This is visible in the limit of large curvatures (see, e.g., 183). This qualitative analysis has to be sustained with a quantitative computation based on pure gravity flows including higher curvature terms. Such a computation requires improved truncations with the properties i) and ii).

We close this chapter with a brief overview of investigations of gravity-matter systems within the background field or hybrid approximations. In 219, gravity-matter systems in the minimally coupled approximation were investigated in a hybrid approach: while most contributions to the flow have been computed in the background field approximation, the matter parts of the anomalous dimensions have been computed in a fluctuation approach setup. Within this approximation, destabilizing effects for scalars and fermions and stabilizing effects for gauge fields were found. The destabilizing result for fermions in 219 is an artifact of the background field approximation, as discussed in destabilizing result for **Section 8.5**: the background-metric dependence of the regulator influences the (de)stabilizing property of minimally coupled fermions. However, this does not imply that the background field approximation breaks down for all gravity couplings. The results of 201, 202 showed that in particular, the most UV-relevant operators have to be taken from a fluctuation computation, that is, most importantly the graviton mass parameter μ . In turn, the background and the fluctuation Newton coupling behave rather similar under the influence of minimally coupled matter fields. The sign of leading-order contribution agrees: the scalar and fermionic contribution to the beta function of the Newton coupling at $\mathcal{O}(g^2)$ is positive, while the gauge contribution is negative.

In summary, the investigations of gravity-matter systems within the fluctuation approach open a systematic path toward reliable stability investigations of fully coupled matter systems as well as that of phenomenological consequences for high energy physics. Still, fully reliable results require a systematic and qualitative improvement of the current truncations. This is the subject of current work in the community.

8.6 Effective Universality

In the vertex expansion **68**, we have introduced the couplings g_n for each graviton n -point function as the running couplings of the Ricci scalar tensor structure $(\sqrt{g} R)^{(n)}$ (see **Section 8.3**). In a diffeomorphism-invariant approach, these couplings would agree. In turn, in the present gauge-fixed approach, these are different avatars of the Newton coupling. While not being identical, $g_i \neq g_j$, they are related by nontrivial mSTIs (**65**).

This is similar in non-abelian gauge theories, where different avatars of the running strong coupling $\alpha_s = g^2/(4\pi)$ can be derived from different correlation functions, both from pure glue vertices as well as glue-matter vertices. (For a detailed discussion, see 167, 168, 170 and the recent review 27.) The β -functions of all the avatars of the strong coupling are two-loop universal in mass-independent renormalization schemes, or may also define an RG scheme with the requirement that β -functions agree to all orders. However, the standard fRG renormalization scheme is mass-dependent, so even two-loop universality is not guaranteed. More importantly, identical β -functions do not necessarily lead to an identical momentum dependence.

Indeed, in non-abelian gauge theories, the momentum dependence of different avatars of the running strong coupling differs already at the universal two-loop order, which can be also shown from the STIs. Additionally, in the strongly correlated IR regime of a non-abelian gauge theory, the fRG β -functions and the momentum dependence of the running couplings differ significantly. Some of them, that is, the three-gluon coupling, even switch sign, while others, that is, the ghost-gluon and four-gluon coupling, stay positive [168].

In gravity, the situation is even more intricate. To begin with, the Newton coupling is dimensionful, and hence, the β -functions of the avatars of the Newton coupling are not universal, leaving aside an identical momentum dependence. Additionally, as already mentioned in the context of non-abelian gauge theories, the standard fRG renormalization schemes are typically mass-dependent, which adds to the differences, as do truncations.

Effective universality is the concept that in particular at the fixed point, where gravity is in a scaling regime, and the quantum theory is dominated by the diffeomorphism invariance of the underlying theory. If this scenario applies, the β -functions and the momentum dependence of different avatars of the Newton coupling should agree or are rather be close to each other on the asymptotically safe UV fixed point. This concept would apply to all couplings, and in particular, the λ_n can be understood as avatars of the cosmological constant. Additionally to the Newton couplings from the Ricci scalar tensor structure, we have further avatars of the Newton coupling stemming from the gravity-matter correlation functions.

Given the presence of truncations in explicit computations, the impact of nontrivial mSTIs and the nonperturbative nature of the UV fixed point, it is left to define a measure for effective universality. In 93, 203, it was quantified how these avatars differ at the UV fixed point using the measure

$$\varepsilon_{ij}(g, \mu, \lambda_3) = \left| \frac{\Delta\beta_{g_i} - \Delta\beta_{g_j}}{\Delta\beta_{g_i} + \Delta\beta_{g_j}} \right|_{g_i=g_j=g}, \quad (109)$$

where $\Delta\beta_{g_i}$ is the anomalous part of the β -function β_{g_i} obtained by subtracting the canonical running

$$\Delta\beta_{g_i} = \beta_{g_i} - 2g_i. \quad (110)$$

In 203, five avatars of the Newton coupling were included stemming from the three-point functions, $\Gamma_k^{(hhh)}$, $\Gamma_k^{(c\bar{c}h)}$, $\Gamma_k^{(\varphi\varphi h)}$, $\Gamma_k^{(\psi\psi h)}$, and $\Gamma_k^{(AAh)}$. Thus, the set of g_i is given by $i \in \{h, c, \varphi, \psi, A\}$, where $g_h = g_3$ in the previous notation. In 109, the β -functions are identical for $\varepsilon_{ij} = 0$, and we have full universality. A small value of ε_{ij} indicates almost identical β -functions and thus “effective universality.” In 203, these small values were estimated to be $\varepsilon_{ij} < 0.2$. This estimate is based on a systematic error estimate of the used truncations and the impact of the mSTIs. In turn, a larger value of ε_{ij} shows that universality is strongly broken and that the mSTIs are highly nontrivial.

The universality measures ε_{ij} are functions of all couplings, and we display them in **Figure 10** for $g_i = g_h^*$ as functions of μ and λ_3 . Remarkably, the UV fixed point lies in the green area, which signals $\varepsilon < 0.2$, and thus, effective universality holds. As discussed above, this statement is nontrivial since the mSTIs can introduce large differences between the avatars, in particular, if the fixed point is highly nonperturbative. In turn, this result gives a strong hint that the UV fixed point is in the semiperturbative region. Interestingly, a semiperturbative behavior was also found in large-order Ricci scalar expansions of the effective action in the background field approximation [108–110, 172, 173]. There it was found that the critical exponents of the high-order curvature invariants are close to their canonical values.

We emphasize that the observed effective universality is a highly nontrivial result. If it can be sustained in further analyses, it is presumably dynamical. This conjecture is supported by the following observation: for a marginal universal coupling, one may simply compute one avatar of the coupling and identify the other avatars with the computed one. In turn, in a theory like gravity, where the effective universality is potentially generated dynamically, this may only work in specific RG schemes. One may even define a natural RG scheme by $\varepsilon_{ij} \equiv 0$. This entails that in other RG schemes, only a subset of the couplings will have the natural β -functions. Note that the latter property is additionally triggered by the inherent truncations of explicit computations.

In any case, within a given RG scheme, some of the β -functions may satisfy $\varepsilon_{ij} \equiv 0$, while others may not. The identification of all avatars of the given coupling with a specific one will only work if the

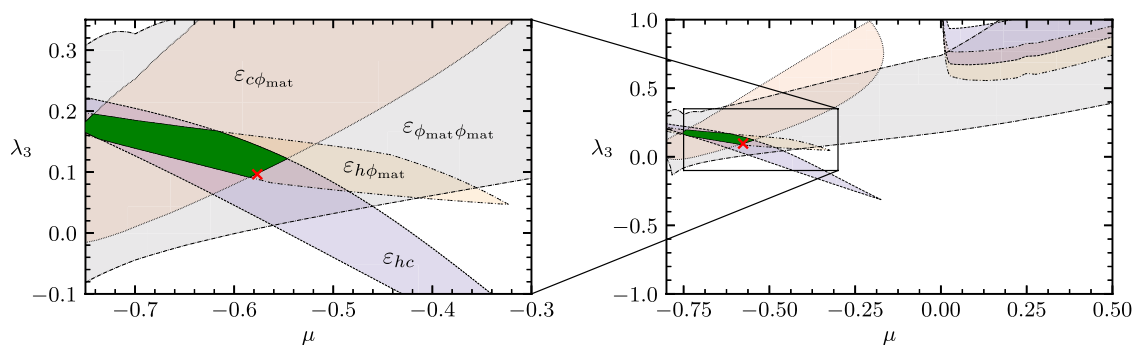


FIGURE 10 | Effective universality of the different avatars of the Newton coupling as a function of μ and λ_3 . The regions of effective universality are defined with $\varepsilon_{ij} < 0.2$ according to 109. The red cross indicates the UV fixed point, which lies in the region of effective universality. The figure is taken from 203.

latter coupling is chosen from the natural subset. Such an identification is an implicit way of enforcing the natural RG scheme. In turn, if all couplings are identified with an avatar which is not in the natural subset, the system may be corrupted. This can even lead to a loss of the fixed point.

In gravity–matter systems, we indeed observe, in given truncations, such a behavior: if all avatars of the Newton coupling are identified with the three-graviton coupling g_h , that is, $g_i = g_h$, the results are close to the full ones with multiple avatars of the Newton coupling. In turn, identifying all avatars of the Newton coupling with a gravity–matter avatar fails. In summary, this hints at a surprisingly complicated interaction structure in gravity–matter systems. Its origin is yet to be understood and may give us further valuable insights into the dynamics of these systems. In short, these investigations of effective universality indicate a *close perturbativeness* of the UV fixed-point regime of asymptotically safe gravity.

9 SUMMARY AND OUTLOOK

In this contribution, we have reviewed the state of the art of the *fluctuation approach* to quantum gravity. This approach is based upon the computation of the correlation functions of the dynamical graviton fluctuation field $h_{\mu\nu}$ within a systematic vertex expansion. This can be done within general parameterizations of the full metric, but most results have been achieved in the linear split, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. While the correlation functions of the fluctuation field are not observables by themselves and carry a gauge dependence, the computation of observables in quantum gravity requires the knowledge of the fluctuation correlation functions, and they indeed encode the dynamics of quantum gravity.

By now, the fluctuation approach has matured (see the overview of the results in **Section 8**). We see signs of *apparent*

convergence of the results in pure gravity. Moreover, by now, we can reliably evaluate the stability of general gravity–matter systems. In combination, the fluctuation approach now allows for reliable physics predictions for the UV regime of asymptotically safe gravity including its unitarity. The approach also allows for reliable physics predictions for the “IR” particle physics within the asymptotically safe standard model.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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APPENDIX A NOTATION

Our convention for functional derivatives are given by

$$\frac{\delta J_{\mu_1 \dots \mu_n}(x)}{\delta J_{\nu_1 \dots \nu_n}(y)} = \frac{1}{\sqrt{g}} \delta(x-y) \delta_{\mu_1}^{(\nu_1} \dots \delta_{\mu_n}^{\nu_n)}, \quad (\text{A1})$$

where the parenthesis in the superscript of the Kronecker- δ 's stands for the symmetrisation of the indices including a normalisation factor $1/n!$. For example we have

$$\frac{\delta h_{\mu_1 \mu_2}(x)}{\delta h_{\nu_1 \nu_2}(y)} = \frac{1}{\sqrt{g}} \delta(x-y) \frac{1}{2} (\delta_{\mu_1}^{\nu_1} \delta_{\mu_2}^{\nu_2} + \delta_{\mu_1}^{\nu_2} \delta_{\mu_2}^{\nu_1}). \quad (\text{A2})$$

This leads to the correlation functions of the fluctuation fields as given in Ref. 14.

The metric γ^{ab} in field space is diagonal for bosons φ , and is symplectic for fermions $\psi, \bar{\psi}$,

$$(\gamma_{\varphi}^{ab}) = 1, (\gamma_{\psi}^{ab}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A3})$$

with the Northwest-Southeast convention

$$\phi^a = \gamma^{ab} \phi_b, \phi_a = \phi^b \gamma_{ba}. \quad (\text{A4})$$

These definitions entail

$$\gamma_b^a = \delta_b^a, \text{ and } (\gamma_{\varphi})_b^a = \delta_b^a, (\gamma_{\psi})_b^a = -\delta_b^a, \quad (\text{A5})$$

more details can be found in Ref. 21.

APPENDIX B PONTRYAGIN INDEX IN U(1) GAUGE THEORIES

The Pontryagin index P of a four-dimensional $U(1)$ -gauge theory in flat space is a simply example for a topological index in quantum field theory. For general field configurations it is a non-vanishing integer on manifolds such as $\mathbb{T}^4$, the four-dimensional torus, e.g., underlying standard lattice simulation. We write in general

$$P[A, \theta] = \frac{1}{32\pi^2} \int_x \theta(x) F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad P[A, 1] \in \mathbb{Z}, \quad (\text{B6})$$

with the Pontryagin index $P[A] = P[A, \theta = 1]$. The (dual) field strength, $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, are given by

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \tilde{F}^{\mu\nu} = \frac{\epsilon^{\mu\nu\rho\sigma}}{2} F_{\rho\sigma}. \quad (\text{B7})$$

In momentum space $P[A, \theta]$ reads

$$P[A, \theta] = \frac{\epsilon^{\mu\nu\rho\sigma}}{16\pi^2} \int_{p,q} \theta(-(p+q)) p_{\mu} A_{\nu}(p) q_{\sigma} A_{\rho}(q), \quad (\text{B8})$$

The flow of θ has been studied in Ref. 193 for the topological charge in Yang-Mills theories. Two derivatives with respect to the gauge field in momentum space lead us from **Eq. B8** to

$$\frac{\delta P[A, \theta]}{\delta A_{\alpha}(p) \delta A_{\beta}(q)} = \frac{\epsilon^{\alpha\beta\rho\sigma}}{8\pi^2} p_{\rho} q_{\sigma} \theta(-(p+q)). \quad (\text{B9})$$

For a topological term with constant $\theta = \theta_{\text{top}}$ we have $\theta(l) = \theta_{\text{top}} (2\pi)^4 \delta(l)$. Inserting this choice into **Eq. B9**, the term vanishes with $\epsilon^{\alpha\beta\rho\sigma} p_{\rho} p_{\sigma} = 0$.



Renormalization of Group Field Theories for Quantum Gravity: New Computations and Some Suggestions

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We discuss motivation and goals of renormalization analyses of group field theory models of simplicial 4d quantum gravity, and review briefly the status of this research area. We present some new computations of perturbative Group field theories amplitudes, concerning in particular their scaling behavior, and the numerical techniques employed to obtain them. Finally, we suggest a number of research directions for further progress.

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1 INTRODUCTION

Group field theories (GFT) [1–3] are quantum field theories which aim at describing the fundamental quantum structures that constitute spacetime. They are quantum field theories of spacetime, rather than on spacetime. They are defined on group manifolds (hence the name), with an associated phase space given by the cotangent bundle of the same group.

$$\varphi : G^{\times d} \rightarrow \mathbb{C}.$$

Equivalently, in conjugate variables, the basic field maps d copies of the Lie algebra of the same group to the complex numbers, and should be understood in general as a non-commutative function (i.e., an element of a non-commutative algebra of functions), since the Lie algebra is in general a non-commutative manifold and this reflects on the algebra of functions defined on it. Depending on the specific model one is considering, various restriction can be imposed on the field, its domain, its target, and of course the choice of group manifold and ‘dimension’ d are also model-dependent. What is general, in all current GFT models, is that the basic quanta of the theory, corresponding to the basic field excitations, can be represented as abstract cells or polyhedra with the d algebraic data forming the domain of the field associated to their (boundary) faces. When d is chosen as the dimension of the spacetime to be reproduced in some approximation, the corresponding GFT quanta can be understood as $(d-1)$ -simplices with algebraic labels on their d $(d-2)$ -dimensional faces. This is the case we restrict to in the following. The other feature which constitutes another defining aspect of the formalism is the peculiar combinatorial structure of field interactions. The dynamics of the theory, which dictates how the fundamental GFT quanta interact forming extended spacetime structure, is specified by an action, first,

$$S(\varphi, \varphi^*) = \frac{1}{2} \int [dg] \varphi^*(\tilde{g}_i) \mathcal{K}(\tilde{g}_i, g_i) \varphi(g_i) + \frac{\lambda}{n!} \int [dg_{ia}] \varphi(g_{i1}) \dots \varphi(g_{in}) \mathcal{V}(g_{ia}) + c.c.$$

and then by its partition function (assumed here as being of statistical form):

$$Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{-S_\lambda(\varphi, \varphi^*)}.$$

Beside a quadratic local term (defined by an integral kernel convoluted with the two fields), the GFT action is determined by interactions (just one in the above example) that possess a characteristic ‘combinatorial non-locality’ in that the interaction kernels pair non-locally the field arguments (d variables being contributed by each field entering the given interaction term). Interaction kernels of order n can be associated with possible ways of gluing together n $(d-1)$ -simplices to form (the boundary of) a d -dimensional cell. The specific combinatorial patterns (i.e., the specific cells being associated to each fundamental interaction) and precise form of the interaction and kinetic kernels are part of the definition of each particular GFT model. However, from this generic aspect of the formalism follows one key fact: GFT Feynman diagrams Γ generated by the perturbative expansion of the GFT partition function,

$$Z = \int \mathcal{D}\varphi \mathcal{D}\varphi^* e^{-S_\lambda(\varphi, \varphi^*)} = \sum_{\Gamma \in \text{sym}(\Gamma)} \frac{\lambda^{N_\Gamma}}{\text{sym}(\Gamma)} A_\Gamma$$

obtained gluing interaction vertices (d -cells) along their $(d-1)$ -faces, are dual to d -dimensional cellular complexes, of arbitrary topology (since a priori there is no restriction on the allowed gluing). The Feynman amplitudes assign a probability amplitude for each such cellular complex, seen as an elementary interaction process of the fundamental GFT quanta.

The above can be taken as a sketchy definition of the TGFT formalism, but it is of course the specification of particular models which gives tentative physical meaning to it and this meaning will therefore change in different contexts. In particular, when the physical interpretation of a given model is grounded in its perturbative expansion, it will affect what we expect about the properties of the Feynman amplitudes A_Γ . In this contribution, we focus on GFT models for quantum gravity and in particular on the class of models closely related to simplicial gravity path integrals, spin foam models and loop quantum gravity. For this class of models we discuss the key features of the Feynman amplitudes in the next subsection. In particular, beyond their interpretation, we will discuss at length whether we should expect them to be finite or divergent when dwelling into the issue of renormalization of the corresponding quantum gravity models.

Before doing so, let us spend some words to clarify the choice of a statistical form for the GFT partition function. The foundations of the GFT formalism have received some attention only recently, and much remains to be understood. Both statistical and complex weights for GFT fields summed over in the definition of the partition function can be found in the literature, but in fact most of the literature until recently has been focusing only on the perturbative expansion of the models, where the non-perturbative definition is less directly relevant, thus avoiding the need to investigate it. In particular, for the class of models to be analyzed in the next subsection, the interpretation in terms of quantum sums over discrete geometries (and topologies) is guaranteed by the form of the Feynman amplitudes (which are either complex or real with oscillating behavior, in line with their interpretation as defining a quantum discrete path integral for gravity on a lattice, as we are going to

discuss). As for the non-perturbative definition of the GFT dynamics, instead, we have less guidelines, especially for the general definition of the formalism, before considering specific models. In the context of tensor models and tensorial group field theories treated as a generalization of usual local QFT framework, all non-perturbative analyses have simply assumed from the start a statistical definition, taken to be primary and in no special need of further justification; in particular, no attempt to understand it as the counterpart of an operatorial definition of the same model has been made. In non-quantum gravity applications, this is simply a choice of a ‘classical’ statistical field theory context, usually adopted for purely mathematical reasons (better chance of making sense of the path integral).

For quantum gravity applications, one may be interested also in a better conceptual foundation of the formalism. In this context, a complex weight involving the GFT action as a pure phase is of course a possibility, and maybe more in line with the intuitive idea of a quantum gravity path integral (even though the quantum gravity interpretation of the GFT field itself is not straightforward, while as we remarked already, this interpretation be consistently associated to the discrete structures appearing in the GFT perturbative expansion). However, we should remind ourselves that these quantum gravity models should not be expected to encode any global unitary evolution, as it is in fact true for any fundamental quantum gravity dynamics, and this removes one strong motivation to insist on this formulation of the path integral. Nor we have a complete derivation of the GFT path integral from a canonical quantum gravity dynamics or, alternatively, from some formal field theoretic gravitational path integral, that could dictate one choice over another. To date, the only tentative derivation of a GFT partition function ‘from first principles’, was given in ref. 4 where it was seen to arise from a *quantum* statistical definition of equilibrium for a system of quantized simplices (indeed, the basic quanta of such GFT models), under a requirement of maximization of entropy and a choice of macroscopic conditions to be imposed on average (this choice concurs to the specification of the resulting GFT action). If among these constraints one includes some appropriate counterpart of the Hamiltonian constraint of canonical gravity (adapted to the discrete setting), then the GFT partition function can be seen as a sort of grandcanonical partition function relaxing the imposition of such constraint (in the sense that configurations satisfying the constraint are assigned greater weight, but fluctuations off the constraint surface are allowed) as compared with a ‘microcanonical’ ensemble in which only solutions of the constraint are allowed. From a canonical quantum gravity perspective the latter, more restricted case would correspond to a definition of the physical inner product between quantum gravity states (with appropriate insertions of observables inside the GFT partition function), while in general the GFT formalism deals then with a broader class of quantum amplitudes. This scenario was also anticipated (more formally) in ref. 5, discussing the relation between GFT and canonical loop quantum gravity. This perspective also resonates (with many details still to be clarified, though) with the presentation of GFT from a quantum gravity perspective in ref. 6, where the ‘tree level’ (thus dominant,

from the perturbative point of view) GFT amplitudes were suggested to define the physical inner product of canonical quantum gravity, while the remaining GFT configurations were associated with topology changing processes (off-shell, from the canonical quantum gravity perspective).

1.1 Group Field Theories, Spin Foams and Other Quantum Gravity Formalisms

We recognize in this brief outline the straightforward generalization of how 2d surfaces are generated in the perturbative expansion of random matrix models. Indeed, GFTs can be seen as group-theoretic enrichment of random tensor models [7–9], to which they reduce if the Lie group domain is replaced by any finite set of N elements. The Feynman amplitudes become purely combinatorial, but the type of diagrams remains the same. Seen as tensors, GFT fields admit a natural action of unitary (and orthogonal) groups on their arguments. If one requires GFT interactions to be invariant under such unitary transformations, they can be fully classified, and we speak of *tensorial* GFT models. Most of the literature on GFT renormalization [10, 11] concerns these tensorial GFT models. More generally, focusing on tensorial aspects of GFTs allows to gain a greater control over the combinatorial structures of their states, diagrams and amplitudes and many of the results obtained in the simpler context of tensor models apply also to GFTs: the use of colors to encode the topology of Feynman diagrams, the large- N expansion, double scaling limits, universality results etc. The first two, in particular, are crucial for GFT renormalization.

In this contribution, we focus on GFT models which are ‘quantum geometric’: their fundamental quanta are quantized tetrahedra with a quantum geometry encoded in group-theoretic data. More precisely, the classical phase space of a single Lorentzian tetrahedron in 4d is chosen to be the cotangent bundle of 4 copies $SL(2, \mathbb{C})$, reduced by additional ‘geometrician’ constraints, and in turn this can be mapped, under the same constraints, to the cotangent bundle of 4 copies of $SU(2)$. At this classical level, this map amounts simply to a change of variables between two alternative parametrizations of the same classical geometry of an individual tetrahedron. We will give more details on the simplicial geometry in the next subsection (see also ref. 12). Appropriate gluing of five geometric tetrahedra on the boundary of a combinatorial 4-simplex can then be shown to provide a geometric characterization of the 4-simplex too, and of the whole simplicial complex obtained gluing geometric 4-simplices together. The same construction in the Riemannian case uses $Spin(4)$ instead of $SL(2, \mathbb{C})$. From the choice of classical phase space follows a choice of Hilbert space for individual quanta of the GFT model, given in one representation by $L^2(G^4)$ reduced by the quantum counterpart of the geometrician constraints, where G is one of the chosen groups mentioned above. More precisely, the natural Hilbert space for a single tetrahedron in this class of models would be $L^2((SL(2, \mathbb{C})^4)$ restricted by the geometrician conditions dictated by the underlying classical simplicial geometric understanding of the GFT quanta (and amplitudes);

this Hilbert space can be mapped, however, to the Hilbert space $L(SU(2)^4)$, with the geometrician constraints encoded, in a model-dependent manner, in the definition of the map. In this latter case, the covariance properties of states and amplitudes under the action of the Lorentz group as well as some of their geometric features become ‘hidden’ in the form of the kernels defining the GFT action or in the corresponding spin foam amplitudes, while boundary data (and GFT fields) only depend on $SU(2)$ data. Spin foam and GFT models defined using one or the other choice of fundamental Hilbert space are, in general, not equivalent, but the precise relation depends on the properties of the map being used (for example, its being isometric or not), but they would be equally justified from a simplicial geometric point of view. A definition of such map for the EPRL model has been given in ref. 13, together with an analysis of its properties, and a generalized definition of such map and analysis of its properties, valid for the whole class of models we deal with here, can be found in ref. 14. The complete GFT Hilbert space is the corresponding Fock space built on this single-quantum Hilbert space. We will give a few more details in the next subsection.

For this class of geometric models, the GFT formalism benefits from direct links to other modern quantum gravity approaches, which can, viceversa, benefit from GFT tools and results.

First, of all, when $SU(2)$ is used, the Hilbert space of a single GFT quantum is the same as that of a loop quantum gravity (LQG) [15] 4-valent spin network vertex. Generic GFT states, organized in a Fock space, will be populated by many such vertices and they will include, in particular, states corresponding to spin networks associated to closed graphs and gauge-invariant cylindrical functions for the same graphs. In fact, the LQG Hilbert space associated to any graph can be shown to be faithfully embedded in the GFT Fock space. The theories however differ in the way these graph-based Hilbert spaces are related, more precisely, in the scalar products between states associated to different graphs. Still, the correspondence, which can be extended to observables and quantum dynamics, allows to see GFTs as a 2nd quantized counterpart of LQG [5].

Next, for this class of models, the GFT Feynman amplitudes take the form of (non-commutative) simplicial gravity path integrals [16, 17], when written in (non-commutative) Lie algebra variables, which encode the discrete metric. The group variables, on the other hand, are understood as encoding the discrete gravity connection. They correspond indeed to discretizations of a classical formulation of gravity as a topological BF theory with added geometrician constraints, on the simplicial complex dual to the GFT Feynman diagrams. The specific way in which the BF action is discretized depends on the quantization map applied to Lie algebra variables, and different models correspond to different strategies for the imposition of the constraints and path integral measures.

In fact, when the same Feynman amplitudes are recast as functions of group representations, using Peter-Weyl or Plancherel decomposition, they take the form of spin foam models [18]. Spin foam models have been introduced as a covariant language for computing spin network dynamics, so they can be understood as a covariant counterpart of canonical

loop quantum gravity. A second perspective is to see spin foam amplitudes as a purely algebraic version of lattice gravity path integrals, or state sum models. In GFT, they arise as Feynman amplitudes. The correspondence is generic: for any given set of spin foam amplitudes associated to simplicial complexes (and admitting a local decomposition with respect to the complex), one can find a GFT action such that the perturbative expansion of the quantum partition function will produce the given amplitudes as Feynman amplitudes (and viceversa, any GFT action corresponds to a set of spin foam amplitudes). A complete definition of a spin foam model requires a prescription for the amplitudes to be associated to all possible cellular complexes (in some specified class) and an organization principle for them, i.e. one way of comparing, composing or selecting them, to obtain a single number for any observable one wants to compute. The GFT embedding provide one such clear organizing principle, by summing them in a QFT perturbative expansion. In addition, it provides a whole set of QFT tools that can be applied to study their mathematical foundations as well as for extracting physics. GFT renormalization can be seen, indeed, from this spin foam perspective.

1.2 Simplicial Group Field Theory Models for 4D Quantum Gravity

The starting point for the construction of simplicial GFT (and spin foam) models of 4d quantum gravity is the quantum geometry of a single tetrahedron in 4d [19].

The quantum geometry of this basic building block, and the extended structured built from it, can be described in various parametrizations [20, 21], and a number of generalizations can also be defined [22, 23] and imported in the GFT framework. Classically, one can use two equivalent characterization of a tetrahedral geometry, leading immediately to an algebraic translation. First, one can start with assigning four vectors $b_i^I \in \mathbb{R}^{3,1}$ to the four faces of the tetrahedron, forced to lie all in the same spacelike hypersurface with timelike normal V (thus satisfying $b_i \cdot V = 0$), and thinking of them as normal to the same faces, with their modulus identified with their area, $b_i^I = A_i n_i^I$ (with $|n| = 1$). The vectors are also forced to close to form the closed boundary of the tetrahedron, i.e. $\sum_i b_i = 0$. The vectors b_i , due to the constraints they satisfy, are actually elements of the vector space $\mathbb{R}^3$ which can be identified with the Lie algebra $\mathfrak{su}(2)$, after it has been endowed with the corresponding Lie bracket. The resulting space $\mathfrak{su}(2)^{\times 4}$ is then the space of geometries for a single tetrahedron. It can also be seen as the cotangent space of the phase space $(T^*SU(2))^{\times 4}$ which is then the phase space of a classical tetrahedron, purely expressed in terms of group-theoretic, algebraic data. The conjugate variables in $SU(2)^{\times 4}$ have the interpretation of parallel transports of a discrete connection along elementary paths from (the (bari)center of) the tetrahedron to the ((bari)center of its) boundary faces. The dual graph made of these paths becomes the graph associated to a single spin network vertex (with four outgoing ‘open links’). In group representation, the corresponding Hilbert space is thus $L^2(SU(2)^4)$ (with Haar measure).

An equivalent encoding of the classical geometry of a single tetrahedron uses directly the variables of discretized topological BF theory. All geometric quantities of a single tetrahedron can be computed starting from four bivectors $B_i^{IJ} \in \wedge^2 \mathbb{R}^{3,1} \simeq \mathfrak{sl}(2, \mathbb{C})$ which close $\sum_i B_i^{IJ} = 0$ and satisfy the simplicity constraints $V_I (*B_i)^{IJ} = 0$ ($*$ is the hedge dual), with respect to the same timelike normal vector V . The phase space of a single tetrahedron can be taken to be the cotangent bundle $T^*SL(2, \mathbb{C})^4$ and the Hilbert space to be $L^2(SL(2, \mathbb{C})^4)$. See also ref. 12 for more details. This second construction can be indeed seen as the discrete (and then, quantum) counterpart of the formulation of continuum General Relativity as a constrained BF theory (a topological field theory) in 4 spacetime dimensions. This amounts to adding suitable constraints, called ‘simplicity constraints’, to the BF action, resulting in the B field of the topological theory being equivalent to a tetrad field, in such a way that the insertion of the general solution of these constraints in the BF action gives the Palatini formulation of classical continuum gravity in terms of tetrad and connection fields, in turn equivalent, at the classical level, to the metric formulation (modulo subtleties concerning degenerate geometries). The ‘geometrician’ constraints we discussed above correspond to the combination of the (discrete counterpart of the) simplicity constraints and the gauge invariance constraints. For more details on the continuum formulation, see the cited references. The two geometric descriptions can be mapped into each other, as we have mentioned already. The simplicity constraints can be seen also, in fact, as determining such map [13, 14]. Since the Hilbert spaces indicated above admit a basis labeled by group representations, this correspondence can be seen also at that level, i.e. as specifying how the relevant representations of $SL(2, \mathbb{C})$ should be decomposed in $SU(2)$ representations, if they have to be understood as encoding the quantum geometry of a tetrahedron. Such representation labels are the variables in which spin foam amplitudes are expressed. Different spin foam (and GFT) models for 4d quantum gravity are specified (among other things) by the way they impose the simplicity constraints at the quantum level, and thus by the specific map between $SL(2, \mathbb{C})$ and $SU(2)$ entering their amplitudes, if used. In the Riemannian case, which will be our focus in this contribution, all the above applies, with $Spin(4)$ replacing $SL(2, \mathbb{C})$.

A spin foam amplitude, that is a GFT Feynman amplitude written in representation variables, will be assigned to any given simplicial complex, dual to a GFT Feynman diagram. The basic building block is an assignment of a quantum amplitude to each 4-simplex, i.e., a ‘vertex’ of the spin foam complex given by the GFT Feynman diagram, with this amplitude function of the algebraic data associated to the five tetrahedra on its boundary. These boundary data can be written as $SU(2)$ or $SL(2, \mathbb{C})$ data, using the mentioned map, and the vertex amplitude can be written as a function of both, featuring then the coefficients of the map, if used, and the geometrician constraints, in its expression. Thus the vertex amplitude will be a function of $SU(2)$ and $SL(2, \mathbb{C})$ representations associated to the triangles of the 4-simplex (faces of the dual complex), and intertwiners of both groups associated to the tetrahedra, following

the imposition of the closure conditions (equivalent to gauge invariance with respect to both groups). The data not used as boundary data are then summed over independently in each vertex amplitude. The spin foam amplitude associated to the whole simplicial complex can then be obtained by gluing together the amplitudes associated to its 4-simplices, with the gluing amounting to matching first and then tracing over the data associated to the tetrahedron shared by each pair of 4-simplices, possibly weighted by an additional gluing kernel. In the GFT context, the vertex amplitude and the gluing kernel are nothing else than the interaction kernel and the propagator (inverse of the kinetic kernel) defining the GFT action. The correspondence between GFT amplitudes and spin foam models, which could be motivated and defined independently, is thus very general [24].

One last comment about the discrete geometry of these models. The construction sketched above, at the classical level, leads to a full characterization of the discrete geometry of the 4d simplicial complex (to which the spin foam amplitudes are associated), equivalent to the more standard characterization in terms of edge lengths, as used in Regge calculus, even though it uses a different set of classical variables (it corresponds, indeed, to a formulation of classical simplicial geometry in terms of the discrete counterpart of the variables of BF theory, suitably constrained, or to so-called ‘area-angle’ Regge calculus [25]). The translation of the same characterization at the quantum level, and in particular the correct imposition of the geometric constraints on quantum states and amplitudes, is the crucial point for ensuring the correctness of the model from a discrete geometric point of view, and it is still subject to debate in the literature. In particular, one would expect to find back the Regge action for metric (edge length) variables, or an equivalent classical reformulation, in the semi-classical expansion of the spin foam amplitudes for a generic simplicial complex (or of the corresponding simplicial path integral). Many results are available (see the cited references) on this issue for a single 4-simplex and for extremely simple complexes, at least for the EPRL model, but the results are mixed, and the situation is especially unclear for larger complexes.

The general formula for the spin foam amplitudes, for all the models in this class, in the Riemannian setting, for given cellular complex $\mathfrak{m}$ dual to the Feynman diagram Γ is the following:

$$\mathcal{A}^\beta(\mathfrak{m}) = \sum_{J_f} \sum_{I_{ve} \in \mathcal{F}_{\mathfrak{m}}} \prod_{e \in \mathcal{F}} d_{J_f} \prod_{e \in \mathcal{F}} d_{I_{ve}} \prod_{v \in \mathcal{V}_{\mathfrak{m}}} \{15J_f\}_v \times \prod_{e \in \mathcal{E}_{\mathfrak{m}}} d_{I_{ve}} \sqrt{d_{I_{ve}}^{i_e, l/2}} (J_f, j_{ef}, k_e, \beta) \quad (1.1)$$

The coefficients f are matrix elements of the map between Spin(4) and SU(2) intertwiner spaces:

$$f_I^{i, l} (J_p, j_p, k, \beta) = \sum_{M_p m_p} (\mathcal{I})_{M_1 M_2 M_3 M_4}^{J_1 J_2 J_3 J_4} \times \left[\prod_{p=1}^4 C_{m_p}^{j_p, j_p} (k) w^l (J_p, j_p, \beta) \right] (\mathcal{I})_{m_1 m_2 m_3 m_4}^{j_1 j_2 j_3 j_4} \quad (1.2)$$

where we have a Spin(4) representation J_f labeling each face, a pair of Spin(4) four-valent intertwiners $I_{ve}, I_{v'e}$ for every edge

and an SO(3) spin j_{ef} for each edge in a given face, while C is the 3j-symbol, and the 15j-symbol is the one of the first type. We have indicated with w the function of group representations that characterizes the implementation of the simplicity constraints defining each model, depending on the representations of Spin(4) and SU(2) labeling each face of the complex. This depends also on the Immirzi parameter γ , through the combination $\beta = \frac{\gamma-1}{\gamma+1}$.

The amplitude's formula can be rewritten in terms of propagators as follows:

$$\mathcal{A}^\beta(\mathcal{G}) = \sum_{J_f} \sum_{I_{ve} \in \mathcal{F}_{\mathcal{G}}} \prod_{e \in \mathcal{E}_{\mathcal{G}}} d_{J_f} \prod_{e \in \mathcal{E}_{\mathcal{G}}} \sqrt{d_{I_{ve}} d_{I_{v'e}}} \mathcal{K}^{J_1 J_2 J_3 J_4} (I_{ve}, I_{v'e}, l, \beta) \prod_{v \in \mathcal{V}_{\mathcal{G}}} \{15J_f\}_v \quad (1.3)$$

where the propagator $\mathcal{K}$ is given by:

$$\mathcal{K}^{J_1 J_2 J_3 J_4} (I, l, \beta) = \sum_{j_1, \dots, j_4} \prod_{p=1}^4 d_{j_p} \sum_i \sqrt{d_{I'} d_i} d_i f_I^{i, l/2} (J_1, \dots, J_4, j_1, \dots, j_4, \beta) f_{I'}^{i, l/2} (J_1, \dots, J_4, j_1, \dots, j_4, \beta) = \sum_{j_1, \dots, j_4} \sum_i \sqrt{d_{I'} d_i} \prod_{p=1}^4 d_{j_p} w^l (J_p, j_p, \beta) \left\{ \begin{matrix} \bar{j}_1 & i^- & \bar{j}_2 \\ j_1^+ & i^+ & j_2^+ \\ j_1 & i & j_2 \end{matrix} \right\} \left\{ \begin{matrix} \bar{j}_3 & i^- & \bar{j}_4 \\ j_3^+ & i^+ & j_4^+ \\ j_3 & i & j_4 \end{matrix} \right\} \quad (1.4)$$

In order to derive the master Integral expression for a GFT Feynman $\mathcal{G}$ we have to write down the corresponding (regularized) full amplitude $\mathcal{A}(\mathcal{G}, \beta, \Lambda, \vec{T}_{\text{ext}})$ and then set to zero all the spin associated to the external and contractible internal faces. We are going to give one concrete example of this procedure in the next section, when studying the scaling of the corresponding amplitude.

The models we will deal with in the following are the EPRL model [18] and the Duflo BO model [17], whose defining maps are:

$$w_{\text{EPRL}} (j^-, j^+, j, \beta) = \delta_{j^- | \beta | j^+} \delta_{j (1+| \beta |)} j^+ \beta < 0 \quad (1.5)$$

$$w_{\text{EPRL}} (j^-, j^+, j, \beta) = \delta_{j^- | \beta | j^+} \delta_{j (1-| \beta |)} j^+ \beta \geq 0$$

$$w_{\text{Duflo}} (j^-, j^+, j, \beta) = \frac{(-1)^{j^- + j^+ + j}}{\pi \sqrt{(2j^- + 1)(2j^+ + 1)}} \quad (1.6)$$

$$\sum_{a=0}^{\lambda} (\text{Sign}(\beta))^a \left\{ \begin{matrix} a & j^- & j^- \\ j & j^+ & j^+ \end{matrix} \right\} \mathcal{T}_a^{j^- j^+} (| \beta |)$$

where the T function is given by:

$$\mathcal{T}_a^{j^- j^+} (| \beta |) = \frac{(-1)^a (2a+1)}{| \beta |} \int_0^{2\pi} d\psi \sin \frac{\psi}{2} \sin \frac{| \beta | \psi}{2} \chi_a^{j^-} (\psi) \chi_a^{j^+} (| \beta | \psi) \quad (1.7)$$

$$\lambda = 2 \text{Min} (j^-, j^+)$$

but it can also be given an expression purely in terms of representation labels. See ref. 17 for more details.

We note that the relative simplicity/complexity of these two models is highly dependent on the basis in which they are expressed, with the flux representation switching such relative complexity with respect to the spin representation given above.

We also point out that other spin foam models, obtained from alternative strategies of imposition of the same geometric constraints and thus also belonging to the same general class we are considering, can be cast in principle in the same general form, and studied by the same method we will illustrate in the following. Beside difficulties, for some of them, in achieving an explicit and manageable expression for their corresponding w coefficients in representation variables, that makes the analysis more cumbersome, it would indeed be very interesting to perform the same scaling analysis of amplitudes and compare with our results.

More details about the construction of spin foam amplitudes, as well as all the ingredients we mentioned as entering in such construction, in a language well adapted to their GFT embedding, can be found in ref. 17.

2 RENORMALIZATION OF GROUP FIELD THEORIES FOR 4D QUANTUM GRAVITY

Let us now discuss motivation and current status of renormalization of simplicial GFT (and spin foam) models for quantum gravity.

Beyond the connection to spin foam models and simplicial gravity path integrals, the general strategy for renormalization of GFT models [10, 11, 26] is to treat them as ordinary QFTs defined on a Lie group manifold, thus using the group structures (topology, Killing forms, etc) to define ‘scales’ and mode integration. A natural notion of scale, to be used to label the RG flow, is provided by group representations, which index the spectrum of differential operators on the group, e.g., the Laplace-Beltrami operator, in turn often used to define the propagator of GFT models. Cut-offs imposed as part of a renormalization group scheme are then imposed on representation labels; for example, in the case of $SU(2)$ cutting off the spectrum of the Laplacian operator means imposing the bound $\sum_{i=1}^d j_i (j_i + 1) \leq \Lambda^2$, for some real (large) number Λ . This fits well with the fact that divergences in spin foam amplitudes mostly come from the large representations regime. Still, a lot of non-trivial work (beside computational challenges) is needed to adapt for GFTs, whose Feynman diagrams are not graphs but cellular complexes, standard QFT notions, noticing also that any procedure for the contraction of divergent subgraphs of perturbative GFTs has the meaning, from the point of view of the simplicial gravity path integral or spin foam model corresponding to the Feynman amplitudes of the same, of a coarse graining scheme of the corresponding lattice theory.

For a proper renormalization group scheme, however, two more ingredients are needed: control over the theory space corresponding to a given GFT model, i.e. the space of allowed interactions; a detailed characterization of the combinatorics of (the cellular complexes dual to) GFT Feynman diagrams. On neither of these two points much is known for simplicial 4d

gravity models. As a result, most work in the context of GFT renormalization has been done focusing on tensorial GFT models, where the above limitations are not present.

Before discussing the goals of GFT renormalization, we spend a few words of caution concerning the physical interpretation of the renormalization group scheme and derived flows. With scales associated to group representation labels, the natural cutoffs entering as UV cutoffs are for large representations. The associated RG then flows from large to small representations (from UV to IR). In LQG and simplicial quantum geometry representation labels identify eigenvalues of geometric operators (e.g., triangle areas or tetrahedral volumes). Large representation labels correspond to large values of such geometric quantities. Thus we have an apparent inversion of roles here, with large distances/volumes playing the role of UV scales in GFT. Caution however should be exercised. In both LQG and simplicial geometry, we know an area of a surface, say, to result roughly speaking from the sum of the individual areas of all elementary surfaces forming the one under consideration, so that one has $A \approx \langle j \rangle \langle N \rangle$ with $\langle j \rangle$ the average area contribution and $\langle N \rangle$ the (average) number of contributions. Moreover, experience from classical Regge calculus and other simplicial gravity formalisms, leads us to expect continuum geometry to be reproduced when the number of elementary excitations contributing to a given continuum geometric quantity is very large, with each contribution smallish (but allowed to be orders of magnitude above Planck size). On the same basis, we expect continuum geometry, and with it any notion of large or small areas, volumes, distances etc, to be the result of coarse graining microscopic, fundamental degrees of freedom like the ones we deal with in the fundamental GFT (or spin foam) formalism. We would better refrain, then, from interpreting simplicial observables directly as geometric, in the sense we attribute to continuum spacetime geometry and physics. Finally, one more alert comes from recalling that, in GFT, the simplicial geometric observables and excitations are the ones associated to the Fock representation of the theory, and probably this is does not correspond to a fully geometric phase in which continuum gravitational (thus, spatiotemporal) physics is to be found, being best adapted to perturbation theory around the fully degenerate (from the point of view of geometry) Fock vacuum.

Also, let us comment on the importance of a better understanding of the symmetry properties of these 4d gravity models, both for the characterization of the corresponding theory space and for their relation to continuum gravity, which is of course crucially characterized by diffeomorphism symmetry. The issue of symmetries in GFT models is very important but also very much open. At the general level, we do not know much about symmetries of 4d GFT models, beyond the Lorentz invariance of the kinetic and interaction kernels and of the Feynman amplitudes (implemented as in usual lattice gauge theories, since the amplitudes are in fact lattice gauge theories for a (constrained) Lorentz connection). Because of the simplicity constraints and also of the simplicial combinatorics that characterize them, moreover, even the tensorial symmetry typical of tensor models is not present (or at least not manifest). Moreover, even for the few symmetries we know of,

in other models, the analysis of their consequences, for example in terms of conservation laws, is complicated by the non-local nature of the GFT interaction (see the analysis [27, 28]). Concerning diffeomorphisms, strictly speaking (being smooth transformations) they are not defined in a discrete context like that of GFT Feynman amplitudes, i.e. spin foam models and lattice gravity path integrals, and thus the question becomes whether we can identify some analogue of diffeomorphic symmetry that, in a continuum limit, could be then identified with the one characterizing GR. There are several analyses of such question for 3d (topological) models at the level of spin foam amplitudes [29], lattice gravity (see for example [30–32]) and corresponding GFT formulation [33], but nothing similar in the 4d gravity case (where the 4d counterpart of the symmetry identified in the 3d case is actually broken, at the discrete level [34]). When attempting a reconstruction of an effective dynamics of geometry in a continuum approximation, as done in the context of GFT cosmology, one has to proceed in terms of observables of the fundamental theory that have a chance to correspond to diffeomorphic invariant observables in GR, since all the structures of continuum GR on which diffeomorphisms act, e.g., manifold points, directions and coordinate functions, but also fields defined on the same manifold, are simply not present in the theory.

2.1 Quantum Consistency and Perturbative Renormalization

GFT models are first defined in perturbative expansion and it is in this perturbative formulation that spin foam amplitudes, and simplicial gravity path integrals, appear. The perturbative GFT amplitudes generically diverge and regularizations have to be imposed. It is this truncation that corresponds to working at a given ‘scale’. Is this definition of the quantum dynamics of GFT models consistent? is the spin foam description consistent? Here, consistency means first of all valid for all ranges of dynamical variables, under (controlled) removal of regulators. If not, the GFT model as defined in perturbative expansion, and thus the corresponding spin foam model (and simplicial path integral) cannot be trusted. In the GFT language, this is recognized immediately to be the issue of perturbative renormalizability of a given model. We should only trust, from the spin foam or lattice gravity point of view, only GFT models that turn out to be (perturbatively) renormalizable.

We note in passing that there should be no requirement that the model is finite (in the sense of presenting no divergence even before any renormalization); first, we have no obvious reason to expect it, if the model contains an infinite number of degrees of freedom; second, renormalizable models are usually more interesting, as QFTs, than finite ones, since they have a non-trivial RG flow and new effective physics at each scale.

Let us clarify further what we mean, here, to avoid possible misunderstandings. As a general point about field theories, we are saying that finiteness of the Feynman amplitudes associated to a given subset of diagrams, or even to all diagrams involved in a given “scattering process” is not so important, *per se*, and in fact not necessarily desirable. What is important is that the scattering

amplitudes can be -made finite- by suitable renormalization procedure (at any order in perturbation theory), if originally divergent in terms of bare couplings, and after resumming all the diagrams involved in their computation. The final renormalized scattering (or transition) amplitudes are what is physically relevant. A theory that is instead simply finite in the sense of not requiring any renormalization, even if clearly easier to deal with, would be less interesting from a physical point of view because this finiteness would probably indicate that the quantum dynamics is not very rich and it does not change much across scales (i.e. when more of its quantum degrees of freedom are accounted for). The consequence would also be a less interesting phase diagram. This should explain our comments about finiteness of GFT Feynman amplitudes. As for the finiteness of “quantum gravity scattering amplitudes”, what we should expect or desire depends on how we interpret the terms. If we take a given GFT model to be a tentative definition of full quantum gravity, then for sure we should hope that its “transition or scattering amplitudes” be finite, in the end, i.e., after renormalization. If the relation between GFT and quantum GR is as in the first case discussed above, i.e., the two are “equivalent”, then this also implies that the transition or scattering amplitudes of quantized GR will be finite, when properly defined and after renormalization (even if the renormalization procedure as well as the observables expressing the amplitudes may look very different in the two formulations). If the relation is as in the second case, and thus “quantum GR” is just an effective theory, then we do not have to expect that its transition or scattering amplitudes are finite tout court, but only within the domain of validity of the approximations or truncations leading to it within the fundamental theory.

Thus, the requirement of perturbative renormalizability is an important constraint, which helps removing from consideration inconsistent constructions. Here, the GFT embedding proves potentially very important also for spin foam models (and loop quantum gravity). All known GFT and spin foam models present several ambiguities, some intrinsic to any quantization procedure, others specific to simplicial GFT (and spin foam) models of quantum gravity. Requiring perturbative renormalizability means constraining such ambiguities. To name one, we have little constraints of the face amplitudes of spin foam models, even though they can drastically affect the scaling behavior of the GFT and spin foam amplitudes, to the point of allowing to achieve perturbative finiteness easily by simply fixing them to this end (which also shows why finiteness *per se* cannot be a goal, without a proper physical understanding), while perturbative renormalizability is a much trickier requirement. For example, see how simple modifications of the Barrett-Crane model (which is also the limit of both EPRL model and Duflo model for infinite Immirzi parameter) affect the resulting amplitudes [35, 36].

Let us list some of them. A first one is combinatorial: why restricting to simplicial complexes? These are the ones for which we have a better understanding of the discrete geometry underlying our models, and in particular of the simplicity constraints that characterize them. But what other cellular

complexes should be included in the theory for consistency, e.g., because corresponding to the counterterms required for taming the perturbative divergences? Others concern the underlying quantization and imposition of simplicity constraints. Being functions of the flux variables (which are non-commutative), they depend on which quantization map is chosen to quantize such variables. Different choices result in different discrete gravity actions and different simplicial path integral measures, thus different spin foam amplitudes. Also the very definition of the simplicity conditions as operator equations acting on quantum states depends on the chosen quantization map, from which follow thus different constraints on representation variables in the spin foam amplitudes. Further, the strategy by which simplicity constraints are imposed produces in general different models or versions of the same type of models (this is apparent in the Riemannian case, while in the Lorentzian one we only have experience of different versions of the EPRL model). These and other ambiguities are discussed, e.g., in ref. 17. Using $SL(2, \mathbb{C})$ or $SU(2)$ data to label quantum states, which is another choice to make, also leads to potentially different models and amplitudes. Nor one should think that these ambiguities are an artifact of the GFT or spin foam formulation. They can be convincingly argued to be the counterpart of ambiguities in the definition of the canonical Hamiltonian constraint operator and, in a way, failing to fix (at least most of) them via renormalizability conditions would be the counterpart, at the background independent level, of the problem of non-renormalizability of perturbative quantum gravity on a given spacetime geometry [37]. Perturbative GFT renormalizability is thus a crucial issue, also when one looks at it from the perspective of spin foam models, simplicial path integrals or canonical loop quantum gravity.

So, where do we stand, on this important issue? For simplicial GFT models of 4d quantum gravity the answer is, unfortunately, that we are only at the very beginning. The main reasons have been already mentioned. First, we do not know enough of their symmetries to characterize the relevant theory space. Second, the amplitudes for these models are very involved and technically challenging to compute, mostly due to the fact that the imposition of simplicity constraints makes them defined not simply on Lie group manifolds but on particular sub-manifolds of these (usually not even corresponding to homogeneous spaces). Third, dominant configurations (i.e., those giving the most divergent contribution to the amplitudes) are not just flat connections or similarly simple, but correspond to richer configurations from the point of view of simplicial geometry; possibly, they correspond to (or possibly include) the whole set of Regge geometries found as saddle-point configurations in the asymptotic analysis of spin foam amplitudes and corresponding simplicial path integrals. Therefore even power counting results are hard to obtain, and the brute force analysis of divergences is not advanced enough to indicate the needed counterterms, forming the theory space. If the theory space is hard to characterize also in the simpler 3d simplicial case (corresponding to topological BF theory), at least the amplitudes are manageable enough to obtain complete power counting theorems [38], identify some counterterms [39] and nice finiteness results [40].

So computational challenges are one big obstacle. It is on this aspect that we focus in the next section, presenting some new results in the Riemannian context. These new results should be added to other ones we have on the calculation of radiative corrections and basic divergences of both Riemannian and Lorentzian simplicial spin foam models, and on explicit evaluations of their building blocks (mainly the vertex amplitudes). For a partial list, see [41–49] and references therein. Future progress will build on these hard-won calculations. In turn these results build on the hard-won understanding, based on both analytical and numerical studies, regarding the asymptotic properties of $SU(2)$ recoupling coefficients and, more recently, of the $SL(2, \mathbb{C})$ recoupling invariants as well, following an extensive ongoing effort in the spin foam community to investigate the behavior of the spin foam transition amplitudes for various models in the constrained BF theory class. A tentative (and incomplete) list of interesting references is refs. 46–refs. 61.

For a comparison, one has to look at the amount of knowledge we have accumulated on tensorial (thus colored) GFT models [10, 11, 26]. Here we know several (classes of) models which are rigorously proven to be perturbative renormalizable, comprising both abelian and non-abelian models, on homogeneous spaces, with or without gauge invariance (closure condition), in different dimensions. Divergences are associated to bubbles, i.e. cells of the complex dual to the cellular complex associated to a GFT Feynman diagram, and typically the most divergent diagrams that form the relevant theory space of renormalizable theories are melonic ones, also singled out in tensor models. However, we also know example of TGFT models which are renormalizable outside the melonic truncation [62], and these examples may be relevant also for the case of simplicial GFT models, since the structure of their divergences presents some aspects of the simplicial case.

2.2 Continuum Limit and Non-Perturbative Renormalization

GFT models of quantum gravity are bona fide QFTs, thus they possess infinite degrees of freedom, as we expect quantum gravity to do (at least thinking of it naively as a quantum theory of the gravitational field). Control over a very large number of degrees of freedom can only be achieved step by step, within some truncation scheme. With the inclusion of more and more degrees of freedom, we can expect a richer and richer set of new phenomena to be unraveled, simply because the physics of many (quantum, interacting) degrees of freedom is very different from that of few of them. In particular, we expect new phases to be revealed. Controlling the full quantum dynamics is controlling the continuum limit of GFT models, and this implies mapping out as best as we can the phase diagram of the same models. In practical terms, it means being able to evaluate the full GFT partition function, for given values of coupling constants. This is the problem of computing the full non-perturbative renormalization group flow of any given GFT model.

Given the mentioned structural connections, understanding the non-perturbative renormalization of a quantum gravity GFT model implies controlling the continuum limit of the corresponding lattice gravity path integral and spin foam

model, and the full quantum dynamics of the corresponding canonical loop quantum gravity formulation. The characterization of the continuum quantum gravity phase diagram and the identification of one phase where an effective general relativistic dynamics of spacetime can be extracted is in fact the key outstanding open issue in the field [63–66].

This should already make clear why the precise relation between the (non-perturbative) renormalization of GFT models for 4d quantum gravity and the (non-perturbative) renormalization of continuum quantum GR treated as an ordinary field theory (as in the asymptotic safety approach) can only be envisaged in a very tentative manner. Let us give only some comment on our own tentative perspective on this. Take a given GFT model that can be fully defined at the non-perturbative level, thus associated with a continuum phase diagram where RG flow trajectories are well-defined from the deep UV (in the GFT sense) to the full IR (still in the GFT sense), and thus accounting for all the (infinite) degrees of freedom of the model; to achieve this situation is the goal of non-perturbative GFT renormalization, as explained. A matching with GR requires that one can also compute, in the same model, observables which characterize fully a 4d geometry and that can be shown to satisfy the GR equations in a classical approximation. Now, we can envisage two possibilities. If the rewriting is, in the appropriate sense, exact, i.e., if one can in principle go from the GFT formulation of the theory to the geometric “quantum GR” one, in the same continuum limit, then the GFT model could be seen, in fact, as a definition of “quantum GR”, without any change in dynamical degrees of freedom. In this case, one could expect that there exist a translation of the RG picture of the given GFT model into the one obtained by a non-perturbative RG treatment of GR, for example as provided (ideally) by the asymptotic safety scenario, and an isomorphism between their corresponding phase diagrams and RG trajectories. If the rewriting requires, instead, some truncation of the dynamical degrees of freedom of the GFT model, is valid only for a subset of the GFT observables, or some other drastic approximation to be valid, i.e., if “quantum GR” turns out to be only an effective, emergent description of some sector of the full quantum GFT, then the situation is different. In this case, we should not expect that the GFT phase diagram matches the GR one, and we can only expect that it will reproduce a portion of it, for scales and regime of couplings where the needed approximations and truncations hold. This regime will probably be the one corresponding to “low energies” from the standard GR and effective QFT perspective. Of course, all the above is very much tentative and it is hard to envisage the precise relation at the current stage of development of GFT as well as of “quantum GR”, even though a number of features of GFT models (e.g., the fact that they include a sum over topologies and not just geometries, at least at the discrete level) would suggest that the second scenario is more likely. Where do we stand, at the non-perturbative renormalization level? Beside work on the non-perturbative RG flow of tensor models [67, 68], a lot of activity has focused on the analysis of GFT models proper [11, 26]. Two main strategies have been followed. One is based on constructive methods, mostly focusing on the resummation of the perturbative series, e.g., showing Borel summability. The other is based on functional renormalization group analysis, either (mostly) based on the

Wetterich-Morris equation for the effective action, or the Polchinski equation for n -point functions. For the same reasons that limited work on perturbative GFT renormalization, little is known about the general RG flow of simplicial GFT models of 4d quantum gravity. Simplicial GFT models in 3d have been shown to be Borel summable [69, 70] and phase transitions for the GFT formulation of simplicial BF theory in any dimension has been shown to exist [71]. But no similar analysis has been carried over to the 4d gravity case, where, as mentioned, we even lack perturbative indications.

The tensorial GFT case, on the other hand, has been widely explored, mostly via functional renormalization techniques, with many results on a variety of models, again both abelian and non-abelian, with and without gauge symmetries, based on compact as well as non-compact Lie groups, in different dimensions. Concerning UV behavior, asymptotic freedom is found in many examples and asymptotic safety is found in others [72], in various truncations, and the perturbative results have been reproduced from a non-perturbative standpoint. More results on the relevance of truncation beyond the melonic sector have been found [73], and the use of Ward identities for studying the RG flow have been explored [26]. Concerning IR behavior (i.e., the actual continuum limit), work is more limited (and more difficult) at the analytic level, but hints have been found, in various truncations and for various models, of a non-trivial phase diagram. In particular, hints of the existence of Wilson-Fisher fixed points (often found alongside asymptotic freedom in the UV) and of broken (or condensate) phases have been obtained [62, 74], indirectly supporting parallel work on the extraction of continuum gravitational physics from such condensate phases [75–77].

Even if it is unclear, at this stage, which of these results holds also in the simplicial 4d models, with their additional quantum geometric intricacies, all this work on tensorial GFT models has certainly led to a better understanding of GFT renormalization group schemes (and flows). This will certainly turn out to be useful also for the analysis of full-blown quantum gravity models.

3 BUBBLE DIVERGENCES AND RADIATIVE CORRECTIONS IN GROUP FIELD THEORIES: SOME NEW RESULTS

In this section we report on some recent results concerning the leading order radiative corrections to N -point functions ($N \leq 6$) for the *Duflo model* and the EPRL model, whose amplitudes we have recalled above. We refer to the cited literature for more details on motivations, construction and features of these GFT (and spin foam) models. Also, we limit our presentation to a summary of results and procedures; a more detailed presentation will be left for a forthcoming publication.

3.1 A Warm-Up Example: The 2-Point Function of the Ooguri Group Field Theory Model

As a warm-up, we recall the general procedure to compute the degree of divergences of GFT amplitudes both in the holonomic

and spin formulation of GFT models, using the simpler case of the Ooguri GFT model for 4d topological BF theory with local $SU(2)$ invariance (that is, a simplicial GFT model on four copies of $SU(2)$ and with kinetic and interaction kernels made out of delta functions only).

Analytical evaluation in group variables - For models defined on the full group manifold or a corresponding homogeneous space (as it is the case in most tensorial GFT renormalization analyses), the evaluation is often conveniently done in the group representation. It can be carried out analytically, and it proceeds as follows. We compute first the bulk or amputated amplitude $\mathcal{A}_{\text{bulk}}$ by removing all the contributions from the external of the GFT diagram $\mathcal{G}$. This amounts to extracting only the dominant leading divergence of the amplitude (subleading divergences require a more refined procedure). Then we gauge fix the holonomic on all the edges of a maximal rooted tree of the graph $\mathcal{G}$. This reduces the evaluation to involve only a set of gauge-invariant variables. Next we drop the contribution from contractible internal faces¹ (if any). The expression so obtained is the irreducible *Master Integral* $\mathcal{I}(\mathcal{G})$ associated to the amplitude, which can then be regularized introducing appropriate cut-offs on the remaining integrals (e.g., by replacing the Dirac-delta functions with heat kernels). It is important to notice that different amplitudes contributing to different correlation functions might reduce to the evaluation of the same master integral. Last we evaluate the remaining integrals. This can be done analytically, exactly or approximately for example via saddle point methods (for example using the abelian asymptotic formula for the heat kernels), to find the Master integral scaling exponent $\omega(\mathcal{G})$, i.e. its superficial degree of divergence.

Numerical evaluation in the spin basis - When the analytic evaluation in group variables is not possible, it is often more convenient to pass to the equivalent expression in terms of group representations (like the ones given above for 4d gravity models), and then proceed numerically, along similar steps as in group variables. First we compute the bulk or amputated amplitude by setting to zero all the spins labeling the external faces and using the appropriate identities for degenerate recoupling coefficients. Next we compute the Master Integral $\mathcal{I}(\mathcal{G})$, expressed in group representations, by setting to zero the spins labeling internal contractible faces. Its expression can be easily regularized by putting a uniform cutoff Λ on all unbounded summations. Last we numerically evaluate the regularized master integral as a function of the cutoff. This can be done either using the full exact formula or using its approximate asymptotic formula (for large spins) obtained by uniform rescaling of all the spins. The amplitude degree of divergence ω can then be estimated by fitting the data. More precisely it is given by the angular coefficient of the linear best fit in a Log-Log data plot.

Let us illustrate the general procedure with an example. We consider the leading order radiative (melon) correction to the two-point function of this GFT model. The associated Feynman

diagram is shown in **Figure 1**. The diagram has four external and six internal faces, none of which is contractible.

In the holonomic formulation, the amplitude can be written as follows:

$$\mathcal{A}_{\text{Oo}}(\mathcal{G}_{\text{melon}}) = \int \left[\prod_{v \in \mathcal{G}} \prod_{e \ni v} dh_{ve} \right] \prod_{f \in \mathcal{F}_{\text{cl}}} \delta \left[\prod_{e \in f} h_{ve} h_{v,e}^{-1} \right] \prod_{f \in \mathcal{F}_{\text{ext}}} \delta \left[\prod_{e \in f} g_f h_{ve} h_{v,e}^{-1} g_f^{-1} \right] \quad (3.1)$$

In this case, with or without gauge fixing, we can perform all the integrations exactly. Neglecting the contributions from the external faces, without a regularization we would find the divergent result:

$$\mathcal{A}_{\text{bulk}}(\mathcal{G}_{\text{melon}}) = \mathcal{I}_{\text{Oo}}(\mathcal{G}_{\text{melon}}) = \delta(\mathbb{I}) \delta(\mathbb{I}) \delta(\mathbb{I}) \quad (3.2)$$

The master integral can be regularized either via a sharp cut-off or by heat kernels:

$$\begin{aligned} \delta(g) &\rightarrow \delta_{\Lambda}(g) = \sum_{j=0}^{\Lambda} (2j+1) \chi^j(g) \\ \delta(g) &\rightarrow K_{\alpha}(g) = \sum_{j=0}^{\infty} e^{-\alpha j} (2j+1) \chi^j(g) \end{aligned} \quad (3.3)$$

In both cases the amplitude's degree of divergence reads:

$$\mathcal{I}_{\text{Oo}}(\mathcal{G}_{\text{melon}}, \Lambda) \propto \Lambda^9 \quad \omega(\mathcal{G}_{\text{melon}}) = 9 \quad (3.4)$$

The same result can be recovered by evaluating the amplitude in the spin basis. We have:

$$\mathcal{A}_{\text{Oo}}(\mathcal{G}_{\text{melon}}) = \sum_{j_f | f \in \mathcal{F}_{\text{cl}}} \sum_{i_{ve} f \in \mathcal{F}_{\text{cl}}} \prod_{e \in \mathcal{E}_{\text{ext}}} d_{j_f} \prod_{v \in \mathcal{V}} (\mathcal{I})_{m_{ef}}^{j_{ve}} \prod_{v \in \mathcal{V}} \{15j_f\}_v \quad (3.5)$$

Since there are no contractible faces, by setting to zero all the spins labeling the external faces we immediately obtain the expression of the regularized master integral $\mathcal{I}_{\text{Oo}}$. Upon using the appropriate identity for the degenerate $15J$ -symbol we find:

$$\begin{aligned} \mathcal{A}_{\text{bulk}}(\mathcal{G}_{\text{melon}}, \Lambda) &= \mathcal{I}_{\text{Oo}}(\mathcal{G}_{\text{melon}}, \Lambda) \\ &= \sum_{j_1 j_3 j_4 j_7 j_8 j_9} d_{j_1} d_{j_3} d_{j_4} d_{j_7} d_{j_8} d_{j_9} \left\{ \begin{matrix} j_7 & j_3 & j_1 \\ j_9 & j_4 & j_8 \end{matrix} \right\}^2 \\ \mathcal{I}_{\text{asy}}(\mathcal{G}_{\text{melon}}, \Lambda) &= \Lambda^5 \sum_j \Lambda^6 d_j^6 \left\{ \begin{matrix} j & j & j \\ j & j & j \end{matrix} \right\}^2 \approx \Lambda^5 \sum_j \Lambda^6 j^6 j^{-3} \approx \Lambda^9 \end{aligned} \quad (3.6)$$

Equation 3.6 is the result of applying the identity A.9 to the Ooguri model's two point amplitude $\mathcal{A}_{\text{Oo}}(\mathcal{G}_{221}, \Lambda)$ (see the combinatorics of the diagram in **Figure 1**) after setting to zero the spins labeling the external faces of the graph $\mathcal{G}_{221}$.

Thus the degree of divergence can be obtained by evaluating the master integral's exact formula, or approximately from the above asymptotic formula, by combining the volume factor (replacing the redundant summations) and the face weights with the large- j behavior of the Wigner $6J$ -symbol, obtaining:

$$\omega_{\text{full}}(\mathcal{G}_{\text{melon}}) = 8.92 \quad \omega_{\text{asy}}(\mathcal{G}_{\text{melon}}) = 9 \quad (3.8)$$

¹An internal face of $\mathcal{G}$ is contractible if it has at least one internal edge which is not shared with any other internal face.

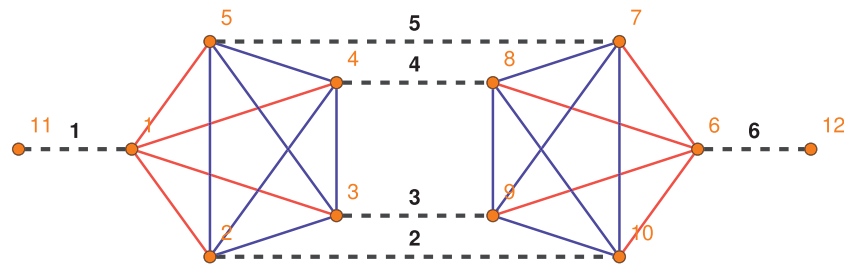
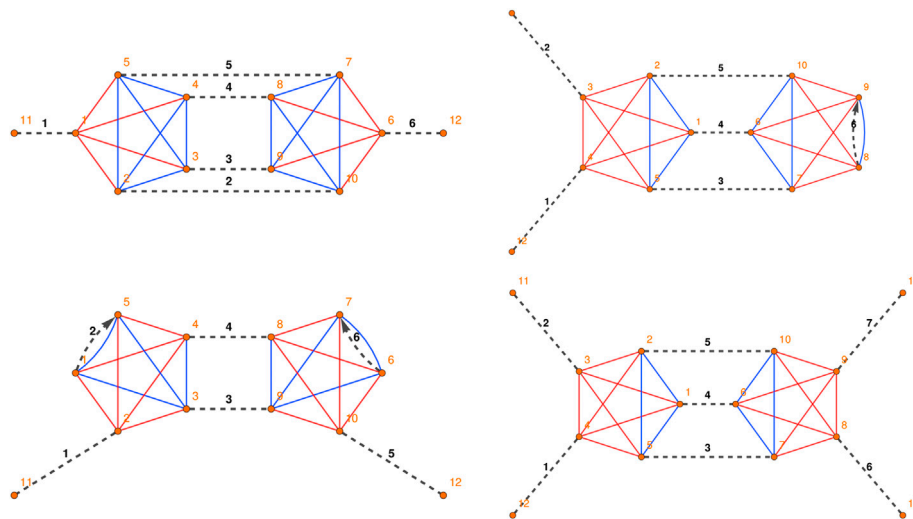


FIGURE 1 | The picture shows the LO (melonic) radiative correction to the simplicial Ooguri model two-point function $\mathcal{W}_2$. The dashed black lines represent propagators. The solid lines encode the internal structure of each simplicial GFT vertex. A face is an alternate sequence of dashed and solid lines. Blue lines are associated to internal faces while red lines belong to the external ones. The diagram has four external and six internal faces.

TABLE 1 | The 1PI diagrams contributing to the LO expansion of the 2-point and 4-point functions $\mathcal{W}_2$ and $\mathcal{W}_4$. Each GFT Feynman diagram is labeled according to the number of external edges, the number of vertices and the position in the list. For example, the first graph on the left is called $\mathcal{G}_{221}$. The dashed lines represent the propagators. The solid lines denote the internal structure of each simplicial GFT vertex. Blue lines are associated to the internal faces, while red lines belong to the external ones. A face is a one-color alternating sequence of solid and dashed lines. The labels for internal and external faces have not been shown.

Leading order corrections to the 2-point and 4-point functions.



in agreement with the analytical result obtained in the group formulation.

3.2 Radiative Corrections in Simplicial Group Field Theories Models for Quantum Gravity

We now report on some recent results concerning the leading order radiative corrections to N -point functions ($N \leq 6$) for the Dufo model and the EPRL model.

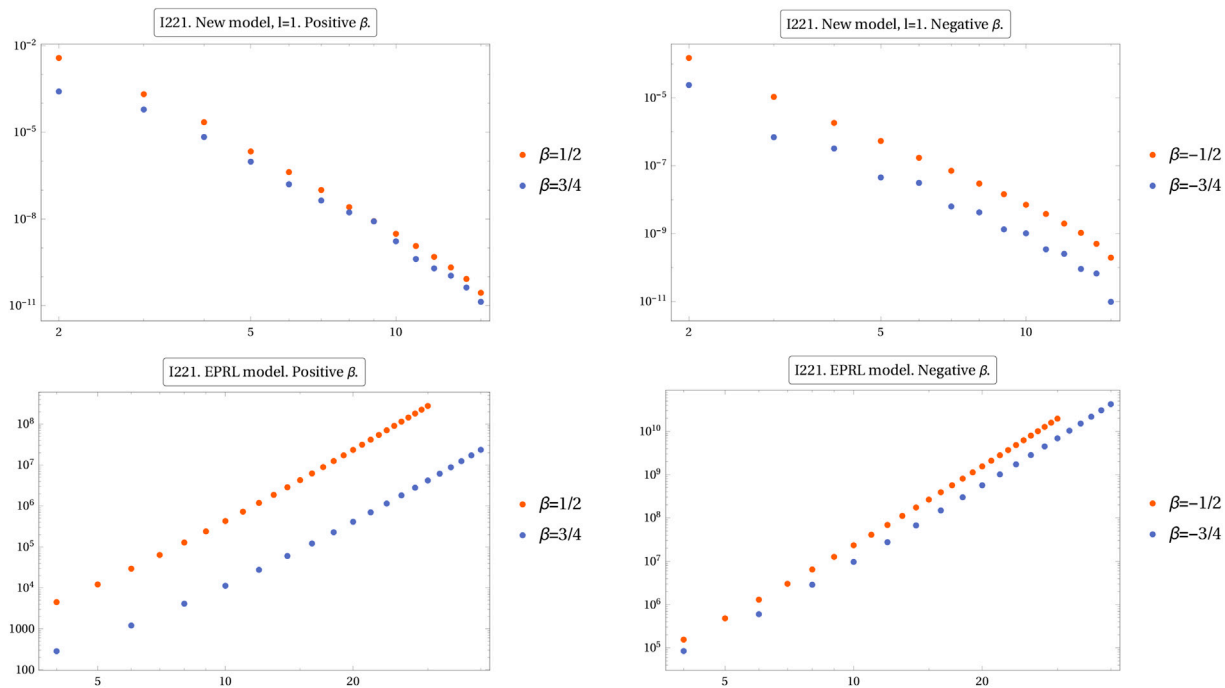
We identify the relevant 1PI Feynman diagrams, compute the corresponding master integral formulae and use them to evaluate the master integrals' scaling (i.e., the diagrams' superficial degree of divergence) as a function of the cutoff. We also comment on the diagrams' combinatorial properties and on the structure of the corresponding counterterms. Finally we show how these

results can be applied 'beyond perturbation theory' to characterize to all orders the scaling of the necklace graphs (an important subclass of diagrams appearing in the radiative correction, also identified as the relevant graphs for renormalizability in the tensorial model of ref. 62). We will derive first general formulae that applied to all models in the chosen class (again, simplicial models constructed from constraining those for topological BF theory) and then specialize to the models of interest by choosing the relevant form for the coefficients w , encoding the geometric conditions characterizing them.

3.2.1 Leading Order Corrections to the N -Point Functions

The relevant 1PI GFT diagrams, appearing in the perturbative expansion of the 2-point and 4-point functions at the leading

TABLE 2 | Top: numerical evaluation of the expression $\mathcal{I}_{221}(\Lambda_{\max}) - \mathcal{I}_{221}(\Lambda)$ as a function of the cutoff $\Lambda \in [2, 16]$ in a Log-Log scale for different values of β . Here $\mathcal{I}_{221}$ is given by the master integral (3.15). **Bottom:** numerical evaluation of the EPRL model's master integral (3.16) as a function of the cutoff Λ and β . The cutoff ranges are $\Lambda \in [4, 30]$ for $\beta = \pm \frac{1}{2}$ and $\Lambda \in [4, 40]$ (only even values) for $\beta = \pm \frac{3}{4}$.



order in the GFT coupling constant λ , are shown in **Table 1** with the notation explained in the caption.

Let us make a few remarks before moving to the analysis of specific diagrams.

Selected diagrams - The diagrams showed in **Table 1** are the only potentially divergent 1PI GFT diagrams at the leading order. All the other leading order corrections to the 3-, 5- and 6-point functions, labeled $\mathcal{G}_{311}$, $\mathcal{G}_{511}$ and $\mathcal{G}_{621}$, are manifestly convergent. The diagram $\mathcal{G}_{511}$ has no internal faces, thus no potentially divergent summation over representations. The diagrams $\mathcal{G}_{311}$ and $\mathcal{G}_{621}$ have only one internal contractible face each. Therefore their corresponding Feynman amplitudes are again finite.

2-point diagrams - The diagram $\mathcal{G}_{221}$ is melonic and therefore also tracial. It has six internal faces and four external ones and it could be expected to be the most divergent LO contribution to the 2-point function. Its dominant (or leading) divergence², if any, can be subtracted by mass renormalization (as done in ordinary QFT). The associated irreducible master integral will be denoted as $\mathcal{I}_{221}$. The diagram $\mathcal{G}_{222}$ has four external faces and four internal

faces; one of them, the tadpole face, is contractible. The diagram $\mathcal{G}_{223}$ has four internal faces and four external ones. Although the diagram $\mathcal{G}_{222}$ and $\mathcal{G}_{223}$ are not isomorphic, their Feynman amplitudes can be reduced to the evaluation of the 4-point function's master integral $\mathcal{I}_{421}$.

The 4-point diagram - The diagram $\mathcal{G}_{421}$ is the first melonic correction to the 4-point function. It has three internal faces (forming a bubble) and eight external ones. Its master integral $\mathcal{I}_{421}$, associated to the bubble subgraph, controls the UV scaling of the LO non-melonic 2-point diagrams and of all the necklace diagrams contributing to the N -point functions with $N = 4, 5, 6$.

To summarize: in order to determine the scaling behavior and divergent structure of the leading corrections to the 2- and 4-point functions we only need to study the independent master integrals $\mathcal{I}_{221}$ and $\mathcal{I}_{421}$, whose expression we will give and evaluate in the following.

3.2.2 The 2-Point Function

To derive the expression of the master integral $\mathcal{I}_{221}$ for the leading (melonic) correction to the 2-point function, as explained, we first write down the full regularized amplitude $\mathcal{A}(\mathcal{G}_{221}, \beta, \Lambda, \vec{\mathcal{I}}_{\text{ext}})$ and then we set to zero all the spins associated to the external faces. Exploiting the identities (A.1, A.10, A.12) and a number of algebraic simplifications, we obtain:

²The first melonic correction to the self energy might also have a subleading divergence as in the case of the Ooguri model and (Lorentzian) EPRL model. Such divergence is responsible for the wave function renormalization. This in turns requires a modification of the covariance with a second-order derivative term in order to account for the new counterterm.

$$\begin{aligned} \mathcal{I}(\mathcal{G}_{221}, l, \beta, \Lambda) = & \sum_{J_f | f \in \mathcal{F}_d} \Lambda d_{J_1} d_{J_3} d_{J_4} d_{J_7} d_{J_8} d_{J_9} \\ & \times \mathcal{K}^{J_3 J_1 J_7}(l, \beta) \mathcal{K}^{J_4 J_7 J_8}(l, \beta) \mathcal{K}^{J_8 J_9 J_3}(l, \beta) \mathcal{K}^{J_9 J_1 J_4}(l, \beta) \left\{ \begin{matrix} J_7 & J_3 & J_1 \\ J_9 & J_4 & J_8 \end{matrix} \right\}^2 \end{aligned} \quad (3.9)$$

where the integer l denotes the number of simplicity constraint insertions and the propagator³ $\mathcal{K}$ is given by Equation A.13. As anticipated, in the above formula, all the model-dependent features are encoded by the single-link fusion coefficients w appearing in the expression $\mathcal{K}$, which we will refer to as the ‘propagator’ in the following. Thus the above result is valid for any simplicial GFT model for constrained BF theory.

Before applying this formula to specific models, let us give a few more details on how it derived, the procedure being in fact the same for the other diagrams.

The melonic 2-point diagram $\mathcal{G}_{221}$ has four external faces and six non-contractible internal faces. Furthermore each internal edge of the graph $\mathcal{G}_{221}$ is shared by four faces (three internal and one external). The associated amplitude is given by:

$$\begin{aligned} \mathcal{A}(\mathcal{G}_{221}, l, \beta, \Lambda) \propto & \sum_{J_f | f \in \mathcal{F}_d} \Lambda \sum_{I_{ve}} \prod_{All f} d_{J_f} \prod_{All (ve)} \sqrt{d_{I_{ve}}} \mathcal{K}^{J_6 J_1 J_3 J_7}(I_{12}, I_{22}, l, \beta) \\ & \mathcal{K}^{J_7 J_2 J_4 J_8}(I_{13}, I_{23}, l, \beta) \\ & \times \mathcal{K}^{J_8 J_5 J_9 J_4}(I_{14}, I_{24}, l, \beta) \mathcal{K}^{J_9 J_4 J_1 J_{10}}(I_{15}, I_{25}, l, \beta) \\ & \left\{ \begin{matrix} I_{11} & J_2 & I_{13} & J_4 & I_{15} \\ J_6 & J_7 & J_8 & J_9 & J_{10} \end{matrix} \right\} \left\{ \begin{matrix} I_{26} & J_2 & I_{23} & J_4 & I_{25} \\ J_6 & J_7 & J_8 & J_9 & J_{10} \end{matrix} \right\} \\ & \left\{ \begin{matrix} J_1 & I_{12} & J_3 & I_{14} & J_5 \end{matrix} \right\} \left\{ \begin{matrix} J_1 & I_{22} & J_3 & I_{24} & J_5 \end{matrix} \right\} \end{aligned} \quad (3.10)$$

Hence after setting to zero the spins $J_i = (j_i^-, j_i^+)$ labeling the external faces (in this case J_2, J_5, J_6, J_{10} according to the labeling conventions adopted in the paper), the four internal propagators will depend only on three spins. Thus in order to derive the Master Integral expression $\mathcal{I}_{221}$ we first need to compute the formula for the degenerate propagator with one vanishing triad which in turns requires the identity for a *type-A* NineJ symbols. Upon using the identity A.1 we obtain the degenerate propagator’s formula A.12 which together with the A.9 leads us to the Master integral expression 3.9. The same line of reasoning applies to the derivation of the Master Integral formula for any GFT graph with one and only one external leg for each simplicial vertex (like the melonic 2-point function, upon setting the external spins to zero the internal propagators of these graphs will only depend on three spins).

For completeness, the full formula for the propagator is given by:

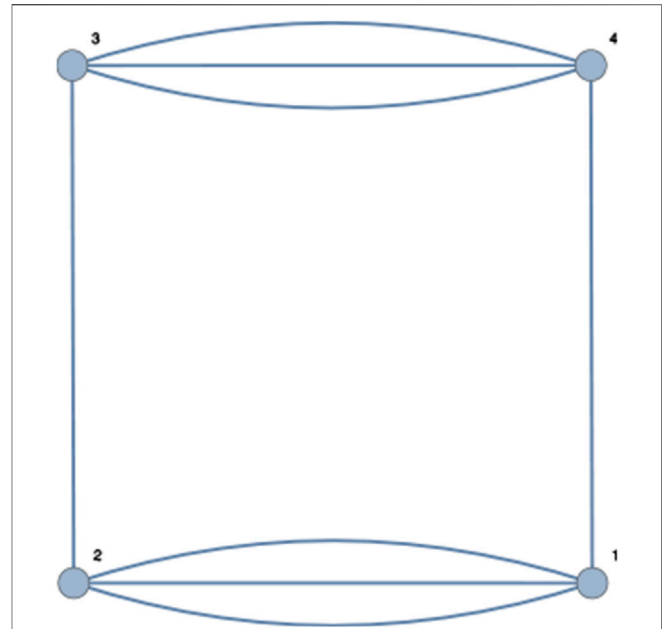


FIGURE 2 | The quartic tensorial bubble interaction B_{41} .

$$\begin{aligned} \mathcal{K}^{J_1 J_2 J_3 J_4}(I, l, \beta) = & \sum_{j_1, \dots, j_4} \prod_{p=1}^4 d_{j_p} \sum_i \sqrt{d_{I_p}} d_{j_p} \\ & f_i^{i/2}(J_1, \dots, J_4, j_1, \dots, j_4, \beta) f_i^{i/2}(J_1, \dots, J_4, j_1, \dots, j_4, \beta) \\ = & \sum_{j_1, \dots, j_4} \sum_i \sqrt{d_{I_p}} d_{j_p} \prod_{p=1}^4 d_{j_p} w^l(J_p, j_p, \beta) \left\{ \begin{matrix} j_1^- & i^- & j_2^- \\ j_1^+ & i^+ & j_2^+ \\ j_1 & i & j_2 \end{matrix} \right\} \\ & \left\{ \begin{matrix} j_3^- & i^- & j_4^- \\ j_3^+ & i^+ & j_4^+ \\ j_3 & i & j_4 \end{matrix} \right\} \left\{ \begin{matrix} j_1^- & i^- & j_2^- \\ j_1^+ & i^+ & j_2^+ \\ j_1 & i & j_2 \end{matrix} \right\} \left\{ \begin{matrix} j_3^- & i^- & j_4^- \\ j_3^+ & i^+ & j_4^+ \\ j_3 & i & j_4 \end{matrix} \right\} \end{aligned} \quad (3.11)$$

The formula for the degenerate propagator (with one vanishing triad) can be written as follows:

$$\begin{aligned} \mathcal{K}^{J_1 J_2 J_3 0}(I, I, l, \beta) \equiv & \mathcal{K}^{J_1 J_2 J_3}(I, I, l, \beta) \\ = & \delta_{I J_3} \delta_{I J_3} \sum_{j_1 j_2 j_3} \prod_{p=1}^3 d_{j_p} w^l(J_p, j_p, \beta) \left\{ \begin{matrix} j_1^- & j_2^- & j_3^- \\ j_1^+ & j_2^+ & j_3^+ \\ j_1 & j_2 & j_3 \end{matrix} \right\}^2 \end{aligned} \quad (3.12)$$

Upon setting $j_i^\pm = j^\pm$ the reduced degenerate propagator reads:

$$\begin{aligned} \mathcal{K}^{j^- j^+}(i^\pm, i, l, \beta) = & \delta_{i^\pm j^\pm} \delta_{i^\pm j^\pm} \sum_{j_1 j_2 j_3} \prod_{p=1}^3 d_{j_p} w^l(j^-, j^+, j_p, \beta) \\ & \left\{ \begin{matrix} j^- & j^- & j^- \\ j^+ & j^+ & j^+ \\ j_1 & j_2 & j_3 \end{matrix} \right\}^2 \end{aligned} \quad (3.13)$$

The above formulas are completely general. For the euclidean EPRL model they can be further simplified yielding the equations

³According to the diagram’s connectivity, each internal edge of the graph $\mathcal{G}_{221}$ is shared by four faces (three internal and one external). Hence, after setting to zero the external spins, the internal propagators depend only on three variables.

TABLE 3 | Summary of the estimated scaling exponent $\bar{\omega}$ for leading order melonic graph $\mathcal{G}_{221}$. The EPRL values we found are in excellent agreement with other analytical results already available in the literature [42].

$\mathcal{I}_{221}$	Numerical estimates of the degree of divergence $\bar{\omega} \pm \sigma$			
	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$
Duflo model, $l = 1$	-11.50 ± 0.85	-11.40 ± 0.51	-8.83 ± 1.06	-10.40 ± 1.94
Duflo model, $l = 2$	-10.77 ± 0.93	-10.99 ± 0.82	-10.23 ± 0.97	-10.17 ± 0.96
Duflo model, $l = 1$. Asym	-2.40	-1.88	-2.28	-2.20
Duflo model, $l = 2$. Asym	-13.24	-13.4	-13.56	-13.72
EPRL model	6.10 ± 0.04	5.95 ± 0.06	6.26 ± 0.026	6.29 ± 0.035
EPRL model. Asym	6	6	6	6

A.14 – A.16 provided in Appendix, as we are going to use in the following.

Let us now focus on the *Duflo model*. In this case the expression (3.9) is still too complicated to be evaluated exactly, even numerically, as it stands. In order to simplify it, we use the asymptotic formula for the $9J$ -symbol (A.8). Upon introducing a new coefficient Ω_{al}

$$\Omega_{al}(\bar{j}^-, j^+, \beta) \equiv \sum_{j=|\bar{j}^--j^+|}^{\bar{j}^++j^+} (2j+1)^a w_{Duflo}^l(\bar{j}^-, j^+, j, \beta) \quad (3.14)$$

we can rewrite the master integral as follows:

$$\begin{aligned} \mathcal{I}_{Duflo}(\mathcal{G}_{221}, l, \beta, \Lambda) &= \sum_{\text{All } \bar{j}^-} \sum_{\text{All } j^+} \frac{d_{j_1^+}}{d_{j_1^-} d_{j_9^-} d_{j_3^+} d_{j_4^+} d_{j_8^+}} \left\{ \frac{\bar{j}_7^-}{\bar{j}_9^-} \frac{\bar{j}_3^-}{\bar{j}_4^-} \frac{\bar{j}_1^-}{\bar{j}_8^-} \right\}^2 \\ &\quad \left\{ \frac{j_7^+}{j_9^+} \frac{j_3^+}{j_4^+} \frac{j_1^+}{j_8^+} \right\}^2 \Omega_{0l}^2(\bar{j}_1^-, j_1^+, \beta) \Omega_{0l}^2(\bar{j}_7^-, j_7^+, \beta) \Omega_{0l}(\bar{j}_3^-, j_3^+, \beta) \\ &\quad \Omega_{1l}(\bar{j}_3^-, j_3^+, \beta) \Omega_{0l}(\bar{j}_4^-, j_4^+, \beta) \Omega_{1l}(\bar{j}_4^-, j_4^+, \beta) \Omega_{0l}(\bar{j}_8^-, j_8^+, \beta) \\ &\quad \Omega_{1l}(\bar{j}_8^-, j_8^+, \beta) \Omega_{0l}(\bar{j}_9^-, j_9^+, \beta) \Omega_{1l}(\bar{j}_9^-, j_9^+, \beta) \end{aligned} \quad (3.15)$$

The coefficient Ω_{al} can be easily tabulated using the analytic formula for the coefficient w_{Duflo} (see Equation 1.6). The master integral (3.15) can now be numerically evaluated⁴ as a function of the cutoff for different values of the parameters β and l .

In the case of the EPRL model, instead, the general formula (3.9) simplifies rather drastically⁵, yielding the following result

$$\begin{aligned} \mathcal{I}_{EPRL}(\mathcal{G}_{221}, l, \beta, \Lambda) &= \sum_{\text{All } j^+}^{\Lambda} d_{j_1^+} d_{j_3^+} d_{j_4^+} d_{j_7^+} d_{j_8^+} d_{j_9^+} d_{|\beta|j_1^+} d_{|\beta|j_3^+} \\ &\quad d_{|\beta|j_4^+} d_{|\beta|j_7^+} d_{|\beta|j_8^+} d_{|\beta|j_9^+} \times \mathcal{K}_{3j_1^+j_7^+}^{\Lambda}(\beta) \mathcal{K}_{4j_3^+j_8^+}^{\Lambda}(\beta) \mathcal{K}_{8j_9^+j_3^+}^{\Lambda}(\beta) \mathcal{K}_{9j_1^+j_4^+}^{\Lambda}(\beta) \\ &\quad \left\{ \frac{j_7^+}{j_9^+} \frac{j_3^+}{j_4^+} \frac{j_1^+}{j_8^+} \right\}^2 \left\{ \frac{|\beta|j_7^+}{|\beta|j_9^+} \frac{|\beta|j_3^+}{|\beta|j_4^+} \frac{|\beta|j_1^+}{|\beta|j_8^+} \right\}^2 \end{aligned} \quad (3.16)$$

⁴The main limitation on range of cutoff values we can test depends on the computational resources available. Here we choose $\Lambda_{\max} = 16$, $l = 1, 2$ and multiple values of β . When possible, we check for stability under extension of the range.

⁵The $9J$ -symbol in the propagator (A.15) depends only on three spins, and one can use an equilateral formula, rather than the non-equilateral asymptotic formula (A.8) as for the Duflo model. Notice also that in the EPRL case there is no need to specify l , since the simplicity constraints act as a projector.

where the (degenerate) propagator $\mathcal{K}_{EPRL}$ is given by Equation A.15.

A sample of the results of the numerical evaluation of the relevant master integrals⁶ is in **Table 2**.

The degree of divergence $\omega(\mathcal{G}_{221})$ is given by the angular coefficient of the linear best fit of the data plotted in logarithmic scale. The mean value⁷ of the scaling exponent $\bar{\omega}$ and its standard deviation for the studied cases are summarized in **Table 3**.

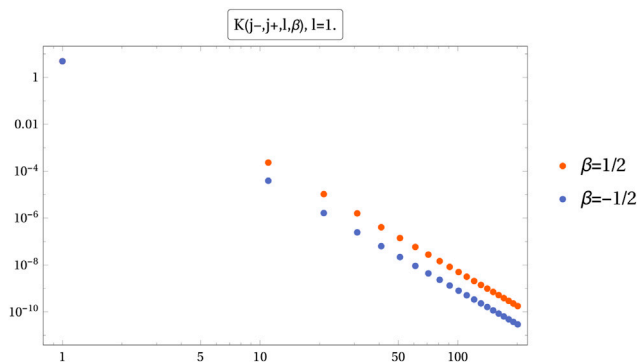
In the case of the Duflo model, some further subtleties arise in the evaluation, due to the more involved nature of the simplicity or geometrician coefficients. These subtleties require additional care in the numerical evaluation of scaling exponents, which are worth emphasizing here, since they are of more general validity in this class of spin foam amplitudes. The master integral formula (3.15) relies on the use of the asymptotic formula for the $9J$ -symbol (A.8). This might not be very accurate for relatively small spins, like the ones we can concretely explore in our numerical evaluations. Furthermore, since the (A.8) has a stronger suppression rate than other approximate formulas for the $9J$ -symbol used in scaling analyses, e.g., the equilateral one $\{9j\} \approx j^{-\frac{3}{5}}$ (in which also all the j s are identified, which we cannot do in the Duflo model), the full expression (3.15) is expected to provide *only* a lower bound for $\bar{\omega}$. To test this expectation and also to cross-check the known EPRL results, obtained using the equilateral scaling, we repeat our analysis using in both cases a different asymptotic formula.

In order to derive an asymptotic formula for the melonic master integral $\mathcal{I}_{221}$, we localize its expression Equation 3.9 around a background configuration $(\bar{j}^-, j^+)$ by setting $\bar{j}_i^- = \bar{j}^-$ and $j_i^+ = j^+$. For the EPRL model this also implies an homogeneous identification of all the spins j_i due to the peculiar form of the EPRL's simplicity coefficients. This procedure has been applied and tested for a number of different simplicial models, including a different version of the EPRL model [41], and it seems to be reliable [44–47, 49].

⁶They will be discussed in detail, alongside the results for other values of the various parameters, in a follow-up publication [78].

⁷To derive an estimate for $\bar{\omega}$ we took the statistical average of the values obtained by fitting the data points in the cutoff ranges $\Lambda \in [\bar{\Lambda}, \Lambda_{\max}]$ with $\bar{\Lambda} = 7$. The lower threshold for $\bar{\Lambda}$ corresponds to the point from where the value of $\bar{\omega}$ appears to be stable within a 10% error margin (i.e., the digits to the left of the decimal point are steady).

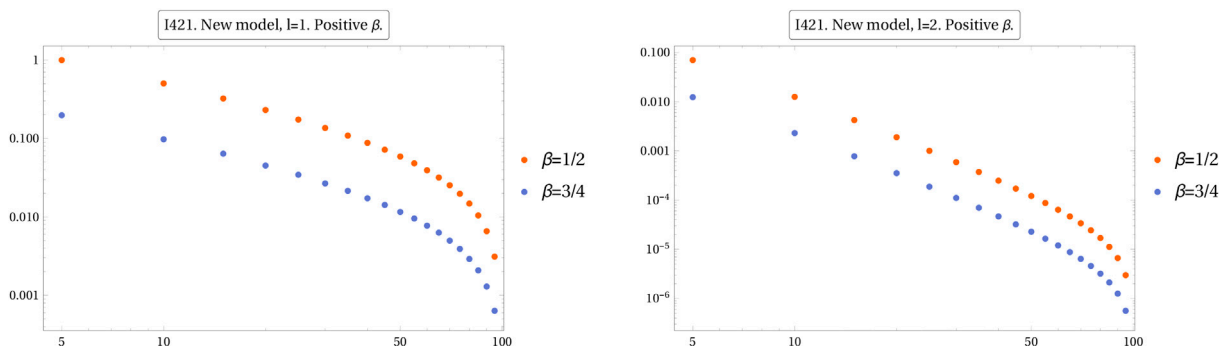
TABLE 4 | Top panel: numerical evaluation of the propagator $\mathcal{K}_{\text{Duflo}}$ with $l = 1$ and $j^- = |\beta|j^+$ in a logarithmic scale. **Bottom panel:** computed estimates of the propagator's scaling exponent α for various values of β and l .



Numerical values of α

l	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$
$l = 1$	4.85	4.72	4.82	4.80
$l = 2$	7.56	7.60	7.64	7.68

TABLE 5 | Numerical evaluation of the expression $\mathcal{I}_{421}(\Lambda_{\max}) - \mathcal{I}_{421}(\Lambda)$ as a function of the cutoff $\Lambda \in [5, 100]$ in a Log-Log scale for different values of l and β . Here $\mathcal{I}_{421}$ is given by the Master integral formula Equation 3.20.



After appropriate simplifications, the general formula Equation 3.9 becomes.

$$\mathcal{I}_{\text{asy}}(\mathcal{G}_{221}, l, \beta, \Lambda) \simeq \Lambda^{5\mu} \sum_{j^- = 0}^{\Lambda} \sum_{j^+ = 0}^{\Lambda} d_f^6 d_{j^+}^6 [\mathcal{K}^{j^- j^+}(l, \beta)]^4 \left\{ \begin{matrix} j^- & j^- & j^- \\ j^- & j^- & j^- \end{matrix} \right\}^2 \left\{ \begin{matrix} j^- & j^+ & j^+ \\ j^+ & j^+ & j^+ \end{matrix} \right\}^2 \quad (3.17)$$

where $\mu = 1, 2$, for the EPRL and Duflo models. The reduced propagator $\mathcal{K}^{j^- j^+}$ is given by Equation (A.14). To determine the amplitude's degree of divergence we combine the scaling of the various factors. The scaling of the equilateral $6j$ -symbol is given by the Regge formula $\{6j\} \approx j^{-\frac{3}{2}}$. According to our analysis, in the large- j regime the propagator (A.14) can be very well approximated by the following expressions:

TABLE 6 | Numerical values of the divergence's degree for the 4-point amplitude $\mathcal{I}_{\text{Duflo}}$.

$\mathcal{I}_{421}$	Numerical estimates of the degree of divergence $\bar{\omega} \pm \sigma$	
β	$\pm \frac{1}{2}$	$\pm \frac{3}{4}$
Duflo model, $l = 1$	-2.36 ± 0.24	-2.35 ± 0.24
Duflo model, $l = 2$	-3.67 ± 0.20	-3.67 ± 0.20

$$\mathcal{K}_{\text{Duflo}}^{j^- j^+}(l, \beta) \simeq \frac{\delta_{j^-} |\beta| j^+}{(2j^+ + 1)^\alpha} \quad \mathcal{K}_{\text{EPRL}}^{j^- j^+}(\beta) \simeq \frac{\delta_{j^-} |\beta| j^+}{(2j^+ + 1)^{\frac{3}{2}}} \quad (3.18)$$

$$\alpha = \alpha(l, \beta)$$

The EPRL formula can be analytically derived, as shown in **Supplementary Appendix A** (see A.16, A.17). The corresponding

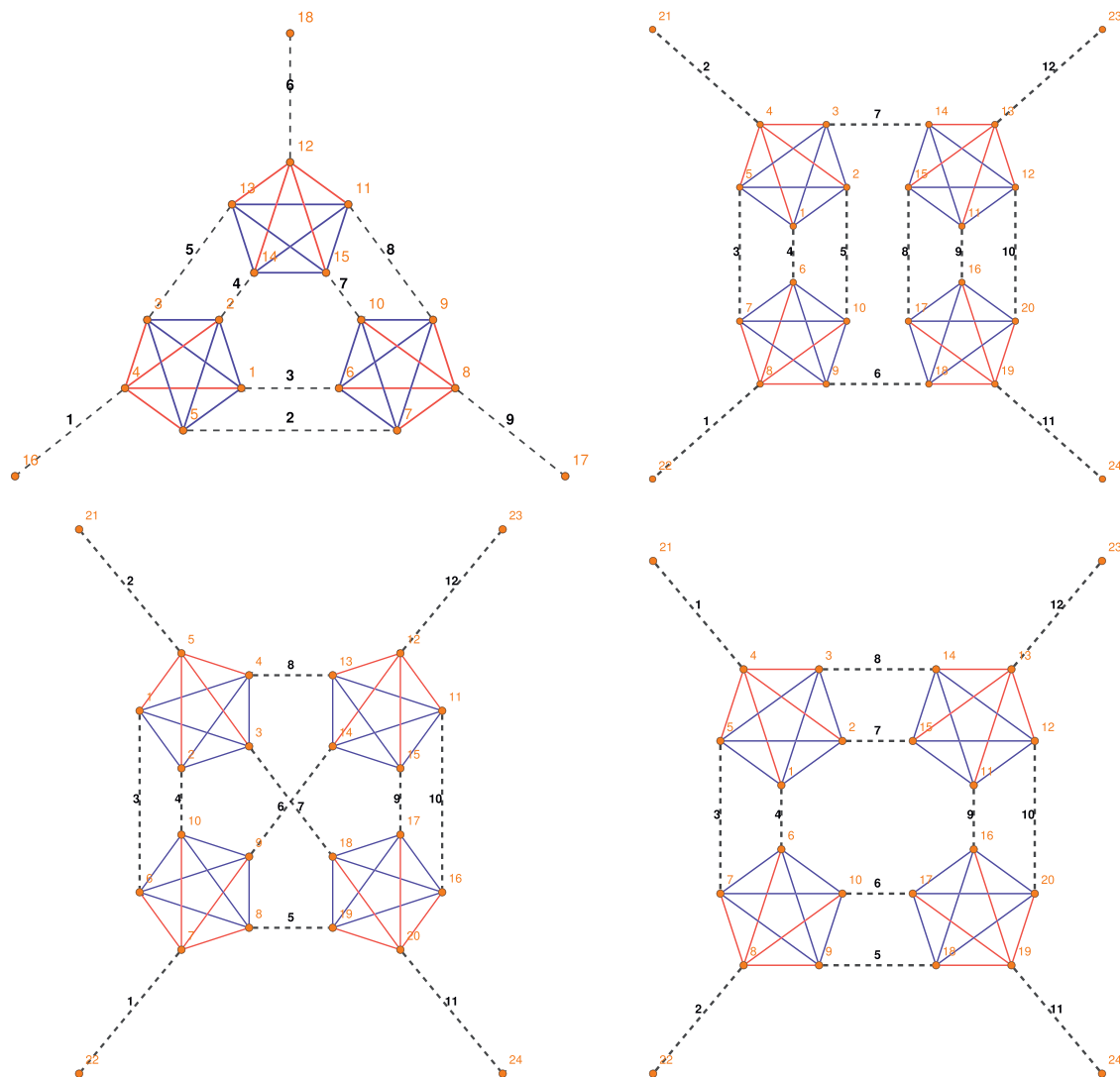


FIGURE 3 | Example of GFT diagrams of the type $\mathcal{G}_{N_v, N_e, n}$ contributing to the 3- and 4-point functions at the NLO.

formula for the Duflo model follows from a direct numerical evaluation of the propagator. It is worth noticing that also the Duflo propagator, containing the much more involved Duflo geometrician coefficients Equation 1.6, peak on the same configurations of $Spin(4)$ representations (j^+, j^-) . This is due to the asymptotic behavior of such coefficients, which contains also (but not only) the configurations corresponding to the EPRL configurations among the dominant ones [79].

The computed values of the scaling exponent α can be found in **Table 4**.

The values of ω , obtained by substituting the identities Equation 3.18 into the master integral formula Equation 3.1, are listed in **Table 3**.

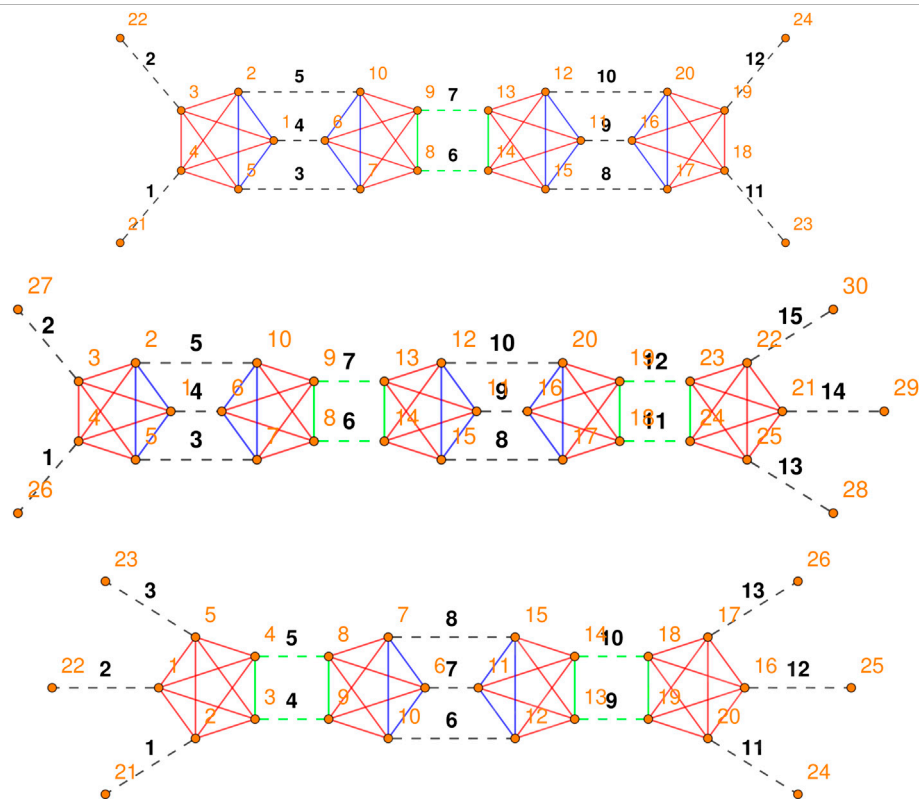
To summarize: the leading order (melonic) correction to the self-energy Equation 3.9 appears to be convergent for the Duflo model and divergent for the EPRL one. The degree of divergence we computed for the EPRL model is in excellent

TABLE 7 | Degree of divergence for the diagrams in **Figure 3** computed from the power counting formula Equation 3.27.

Degree of divergence. NLO graphs

$\mathcal{G}$	$\mathcal{G}_{331}$	$\mathcal{G}_{441}$	$\mathcal{G}_{442}$	$\mathcal{G}_{443}$
Duflo model, $l = 1, \beta = \frac{1}{2}$	-15.1	-15.8	-31.8	-19.8
EPRL model	0	3	-9	0

agreement with known analytical results in the literature [41, 42]. Concerning the *Duflo model* for the case $l = 1$ the data clearly indicates that the use of the non-equilateral formula (A.8) in Equation 3.15, as appropriate for this model, strongly suppresses the amplitude scaling leading to a more convergent result. This might also be true for the case $l = 2$ although we cannot state it with full confidence at the moment based on the small cutoff range we tested (recall that $\Lambda_{\max} = 16$ in the full

TABLE 8 | Examples of necklace diagrams contributing to the 4-, 5- and 6-point functions.**GFT necklace graphs**

formula and $\Lambda_{\max} = 100$ in the asymptotic formula). The limited range of Λ values we explored might also explain why the difference between the values of ω for $l = 1, 2$ computed from Equation 3.15 (first and second row of **Table 3**) is smaller than the difference between the corresponding values obtained from the asymptotic formula (third and fourth row of **Table 3**).

3.2.3 The 4-Point Function

We now focus on the leading (melonic) correction to the 4-point function. The corresponding GFT Feynman diagram $\mathcal{G}_{421}$ is depicted in **Table 1**. In order to derive the master's integral expression we follow the same strategy used in the previous section. After setting to zero the spins labeling the external faces and performing the appropriate simplification we find:

$$\mathcal{I}(\mathcal{G}_{421}, l, \beta, \Lambda) = \sum_{j_1^-, j_1^+} \sum_{j_{31}^+, j_{41}^+, j_{51}^-} \frac{d_{j_{31}} d_{j_{41}} d_{j_{51}}}{d_{j_1^-} d_{j_1^+}} w^l(j_1^-, j_1^+, j_{31}^+, \beta) w^l(j_1^-, j_1^+, j_{41}^+, \beta) w^l(j_1^-, j_1^+, j_{51}^-, \beta) \quad (3.19)$$

Once more, the above formula is completely general and valid for any simplicial GFT (spin foam) model for constrained BF theory. For the EPRL and Duflo models it specializes to:

$$\mathcal{I}_{\text{Duflo}}(\mathcal{G}_{421}, l, \beta, \Lambda) = \sum_{j_1^-, j_1^+} \frac{1}{d_{j_1^-} d_{j_1^+}} [\Omega_{1l}(j_1^-, j_1^+, \beta)]^3 \quad (3.20)$$

$$\mathcal{I}_{\text{EPRL}}(\mathcal{G}_{421}, \beta, \Lambda) = \sum_{j_1^-, j_1^+} \frac{(2(1-\beta)j_1^+ + 1)^3}{(2j_1^+ + 1)(2|\beta|j_1^+ + 1)} \quad (3.21)$$

where in the first expressions we used the same notations of Equation 3.15.

The degree of divergence of the Master Integral in the Equation 3.20 can be computed by fitting the data shown in **Table 5**. The resulting values of ω are reported in **Table 6**.

The scaling of the EPRL 4-point amplitude can be directly read off from the corresponding formula Equation 3.21.

$$\omega(\mathcal{G}_{421}) = 2 \quad \beta \neq 0, \pm 1 \quad (3.22)$$

To summarize: the leading order radiative correction to the 4-point function converges for the Duflo model while it diverges quadratically in the EPRL model. Neglecting possible ambiguities in the definition of both models (which, as we emphasized earlier, could affect the face amplitudes and thus the precise scaling behavior) it would then seem that the Duflo model does not require renormalization, at least at this order, while the EPRL model does. But of course higher orders are needed to establish the renormalizability of both models, thus it is hard to draw too

many conclusions from this result. More than the divergence degree in itself, it is important to notice that, since the $\mathcal{G}_{421}$ is melonic (and thus tracial), the corresponding counterterm that is required to absorb the divergence, when present, is proportional to a tensor invariant quartic interaction term (more precisely to the bubble B_{41} vertex in **Figure 2**).

Such counterterm is incompatible with a pure simplicial theory space (e.g., a strictly simplicial EPRL model would be non-renormalizable), and this signals the need to extend the theory space of geometric GFT models beyond the simplicial ansatz to include tensorial bubble interactions.

3.2.4. Next-To-Leading Order Corrections to the N -point Functions

We now show how to generalize the master integral formulas for the melonic 2-point function $\mathcal{G}_{221}$ to an important class of higher order GFT Feynman diagrams.

The master integral expressions Equations 3.9, 3.17 rely on the property that each internal link of the diagram is shared exactly by one external face and three internal ones. Hence, after setting to zero the spins labeling the external faces we are left with: i) a pair of $6J$ -symbols for each vertex (coming from a pair of degenerate $15J$ -symbols); ii) propagator of the form (A.13) for each internal edge. A pair of face weights $d_{j_i} d_{j_i^*}$ for each internal face.

The above combinatorial property is true for any tadpole-free GFT diagram with one and only one external link for each simplicial vertex (here denoted as $\mathcal{G}_{N_v N_e n}$). Some examples of these diagrams are shown in **Figure 3**. Therefore the expression Equation 3.17 can be generalized as follows:

$$\mathcal{I}(\mathcal{G}_{N_v N_e n}, l, \beta, \Lambda) = \sum_{J_f \in \mathcal{F}_{cl}} \prod_{f \in \mathcal{F}_{cl}} d_{J_f} \prod_{e \in \mathcal{E}_{int}} \mathcal{K}^{J_{f_l} J_{f_r} J_k}(l, \beta) \prod_{v \in \mathcal{V}} \{6J_{vf}\} \quad (3.23)$$

$$\mathcal{I}_{asy}(\mathcal{G}_{N_v N_e n}, l, \beta, \Lambda) = \Lambda^{(N_f-1)\mu} \sum_{j^-, j^+} d_{j^-}^{N_f} d_{j^+}^{N_f} [\mathcal{K}^{j^- j^+}(l, \beta)]^{N_e} \quad (3.24)$$

$$\left\{ \begin{matrix} j^- & j^- & j^- \\ j^- & j^- & j^- \end{matrix} \right\}^{N_v} \left\{ \begin{matrix} j^- & j^+ & j^+ \\ j^+ & j^+ & j^+ \end{matrix} \right\}^{N_v}$$

with $\mu = 1, 2$ respectively for the EPRL and the Duflo model.

Upon using the identities Equation 3.18, the asymptotic master integral Equation 3.24 takes the following form:

$$\mathcal{I}_{asy}(\mathcal{G}_{N_v N_e n}, l, \beta, \Lambda) \simeq \Lambda^{(N_f-1)\mu} \sum_{j^+} (j^+)^{2N_f - \alpha N_e - 3N_v} \quad (3.25)$$

$$= \Lambda^{(\mu+2)N_f - 3N_v - \alpha N_e - \mu+1} \quad (3.26)$$

For the Duflo model the value of α must be computed on case by case basis (see **Table 4**), while for the EPRL model we have $\alpha = \frac{3}{2}$.

The degree of divergence ω can also be written in terms of the number of vertices N_v

$$\omega(\mathcal{G}_{N_v N_e n}) = (\mu + 2)N_f - (2\alpha + 3)N_v - \mu + 1 \quad (3.27)$$

$$N_e = \frac{5N_v - N_{ext}}{2} = 2N_v$$

For the diagrams in **Figure 3** the above formula give us the following results, reported in **Table 7**: the Duflo model amplitudes for all four diagrams are finite; for the EPRL model⁸, the first and last diagrams might be logarithmically divergent and therefore require a deeper analysis [78]. The diagram $\mathcal{G}_{442}$ converges, while the melonic diagram $\mathcal{G}_{441}$ diverges as a cubic power of the cutoff. The corresponding counterterm is proportional to the quartic bubble vertex B_{41} , suggesting again the need for a suitable extension of the theory space to incorporate the relevant tensorial interactions.

Once more, this is in fact the crucial lesson we draw from this analysis of divergences, more important, we think, than the precise scaling of the amplitudes, for the reasons already explained.

3.2.5. Beyond Perturbation Theory: The Necklace Diagrams

In the previous two subsections we discussed the all the leading order and a subclass of next-to-leading order radiative corrections to the N -point functions with $N \leq 6$. Now we show how to use some of the results we found to estimate the scaling of the so-called necklace diagrams to all orders in perturbation theory.

A (connected) GFT Feynman diagram belongs to the necklace class if (and only if) it consists of an open chain of vertices where each vertex (except the first and last ones) is connected only to its two closest neighbors. Here we restrict to the set of k -necklace diagrams with $4 \leq k \leq 6$ where k denotes the number of external links, since all other higher order diagrams are either 1-particle reducible or manifestly convergent. Three examples of necklace diagrams are shown in **Table 8**.

The necklace diagrams share the following remarkable property: the set of internal faces of a k -necklace diagram can always be decomposed into the direct sum of two subsets

$$\mathcal{F}(\mathcal{G}_{k\text{-necklace}}) = \overline{\mathcal{F}} \oplus \tilde{\mathcal{F}} \quad \overline{\mathcal{F}} = \oplus_{i=1}^{N_b} \mathcal{F}_{B_{421}} \quad (3.28)$$

$$\tilde{\mathcal{F}} = \{f_j \mid f_j \text{ contractible } \forall j\}$$

where the (three) faces in each set $\mathcal{F}_{B_{421}}$ form the bubble subgraph B_{421} of the melonic 4-point diagram $\mathcal{G}_{421}$ while the faces in $\tilde{\mathcal{F}}$ are contractible. The integer N_b , namely the number of disjoint bubbles B_{421} depends on the connectivity of the necklace diagram itself (i.e. on its number of vertices and external links). Thus after setting to zero the spins labeling the external and the contractible internal faces, the amplitude of k -necklace diagram factorizes into N identical copies of the master integral $\mathcal{I}_{421}$ given in Equation 3.19. More in detail, we have:

$$\mathcal{A}(\mathcal{G}_{k\text{-necklace}}, l, \beta, \Lambda) = [\mathcal{I}(\mathcal{G}_{421}, l, \beta, \Lambda)]^{\frac{N_v - \text{mod}(k,2)}{\text{Floor}(k/2)}} \quad (3.29)$$

To summarize: the master integral $\mathcal{I}_{421}$ encodes the scaling of the necklace diagrams to all orders in perturbation theory. For the EPRL model a consistent (recursive) subtraction of all

⁸We point out again that we are studying both models under specific choices fixing the various ambiguities that enter the construction of the spin foam amplitudes. These ambiguities affect, in general, the scaling results.

divergences associated to k -necklace graph requires an extension of the theory space to include the appropriate tensor invariant interactions of order four and six. A more detailed analysis of the combinatorial structure of the required counterterms is left for future work [78].

4 THE ROAD AHEAD: SOME SUGGESTIONS

We close with a set of suggestions for research directions to be pursued, toward a complete understanding the renormalization group flow of simplicial GFT models for 4d quantum gravity.

4.1 Scaling and More Scaling, and the Discrete Geometry of Divergent Configurations

The first suggestion is stating the obvious: compute, compute, compute⁹. We need to (pardon us the word pun) scale up the effort in investigating the scaling of simplicial GFT amplitudes, at the same time trying to import insights from tensorial GFT amplitudes. We need to know much more about the divergent configurations and their dependence on the combinatorics of the underlying cellular complex. Lacking better tools, hard brute-force computations of spin foam amplitudes are the inevitable duty, and these in turn can only build on a better control of the relevant building block the vertex amplitude or, in GFT language, the vertex kernel (which plays an important role also in non-perturbative calculations). Brute force alone will not lead us far, however. On the one hand, we need to develop a more refined analytic understanding of these kernels and resulting amplitudes, and to identify simplified expressions that capture the relevant scaling properties, and their behavior under coarse-graining. On the other hand, where analytical methods do not reach, we need numerical ones to take over; numerical tools for the evaluation of GFT amplitudes are thus badly needed. Packages like have been developed for local QFT Feynmanology would be of course most welcome. On the analytic side, another important objective should be to characterize in detail the (simplicial) geometric meaning of the dominant, most divergent configurations. This is needed to understand the nature of the needed counterterms, but it may also provide insights on the physical features of such GFT models, even beyond their discrete formulation.

⁹To be clear: 1-loop and 2-loop calculations would be probably enough to extract a lot of interesting properties from the perturbative expansion of GFT models for 4d quantum gravity. They may even be sufficient, since we actually expect that most interesting physics should be looked for elsewhere, i.e., within different regimes and approximations of the fundamental quantum dynamics, and not in the perturbative (spin foam) expansion. Moreover, as we stressed, such perturbative calculations should have as main goal to identify the theory space within which the relevant GFT models should be placed, more than simply pushing the perturbative analysis for its own sake. We emphasize, however, that, in fact, a lot of computational effort is required to perform such 1-loop and 2-loop calculations because one needs to consider many and complicated GFT diagrams, and involved spin foam amplitudes, already at this order. These are the calculations we intend to encourage.

The goal here is not to so much to be able to compute GFT Feynman diagrams to arbitrary order (e.g., in vertex or loop expansions). Even in standard QFT, for the physical questions for which the perturbative expansion is the correct approximation scheme, we need to compute (very) many diagrams, but there is often no need nor possibility to go beyond some (usually low) order of approximation and beyond a certain (usually small) number of physical degrees of freedom in the chosen boundary states (of course, the two restrictions go together, since for highly populated boundary states, even the simplest diagrams are of high order). A clear physical picture behind this approximation scheme is as important as computational power. Moreover, coming to the specific quantum gravity case, we would argue that the perturbative GFT expansion, and the description of the dynamics in terms of elementary processes involving few of the fundamental quanta, i.e. the usual spin foam language, is not the most convenient approximation to capture the effective continuum physics of quantum gravity, e.g., concerning early cosmology or quantum black holes.

Nor the goal of such analysis of perturbative GFT divergences is establishing that one specific GFT model is finite. Not only renormalizability is a more subtle and possibly interesting feature than finiteness, but one can imagine playing with the ambiguities entering the construction of any given model to modify its scaling behavior and turning it into a finite one. This could be a way to fix or constrain such ambiguities, of course, but it also shows that finiteness per se probably should not be a goal, and that physical conditions fixing the same ambiguities are needed. The main goal of actual computations of GFT amplitudes should then be to provide solid indications on the general power counting of divergences, on the way to a renormalizability proof, and, even more, to indicate the relevant counterterms to be added to the model, and thus the relevant theory space of the starting GFT model. More generally, the goal of such perturbative calculations should be to provide information and tools to be employed to go beyond the perturbative setting and dwell into non-perturbative GFT renormalization. It is only the latter that can provide us with the insights and the results we need to truly explore the formal solidity and effective continuum physics of GFT models of quantum gravity.

4.2 Group Field Theories Theory Space, Colors and Relation Between Simplicial and Tensorial Models

We have emphasized several times already the importance of defining the relevant theory space of simplicial GFT models, in order to set up a proper renormalization scheme (perturbative and non-perturbative). Much more work should be devoted to this issue, in particular understanding more about the symmetries of such quantum gravity models. One question is whether the yet to be identified theory space of simplicial GFT models relates to the one of tensorial GFTs. We speculate that they do and, in fact, some hints that the two may largely coincide are known. First, taking seriously the tensorial nature of GFT fields implies coloring (thus distinguishing and ordering) their arguments and, as a consequence, their Feynman diagrams. As

we noted, this coloring allows a precise control over the topology of the cellular complexes dual to these Feynman diagrams [7] and, in turn, this greater control allowed for many results that are central in renormalization analyses of tensorial GFTs (e.g., large- N expansions) [10, 11]. A precise control over the topology of the Feynman diagrams, i.e., the cellular complexes on which spin foam amplitudes are based, is needed also in simplicial GFTs, if one aims at identifying the nature of divergences, leading to precise power counting results. It is also needed for identifying key symmetries, as we know already in the case of topological BF models [80], where the complete power counting also relied on the full topological information on the underlying cellular complex [38]. Thus we have a strong argument for relying on colors also in simplicial GFT models of 4d quantum gravity; the form of the corresponding spin foam amplitudes would remain unchanged, but they would now be defined on full 4d cellular complexes, rather than just their 2-skeleton. Assuming we work on colored simplicial GFT models, we then have two preliminary results that suggest a close relation with tensorial models. One is that using colors one can identify similar symmetries in the simplicial case than one finds in the tensorial one [28]. The other is that integrating out all fields except one in a colored simplicial GFT model (in any dimension, with trivial kinetic term) produces an equivalent tensorial GFT model for the remaining field (with the same coupling constant for all interactions) [81]. A third general fact pointing in the same direction is that divergences in simplicial GFT models for topological BF theory, which is the starting point of the construction of simplicial 4d gravity models, are associated to bubbles in the cellular complex, which are in fact the cells associated to allowed interactions in tensorial GFTs with the same base group manifold. These results, in our opinion, suggest that there could be a single theory space containing both (colored) simplicial models and tensorial ones, with interaction kernels in the tensorial directions yet to be identified.

4.3 Group Field Theories Models With Local Directions

The third suggestion for further research is to devote attention to the renormalization of GFT models which combine the combinatorially non-local pairing structure on geometric variables, in GFT interactions, with the presence of local directions. This includes both simplicial GFT models and tensorial ones, with the distinction referring to the pairing of geometric variables.

There are two main examples of such ‘mixed’ models. One is the tensorial models used to describe SYK-like many-body systems [82], whose renormalization has been in fact studied in several cases. Here the non-local, tensorial indices are usually reduced to finite sets (we have thus simple tensor models, rather than full GFTs) and the single local direction is a time variable. The standard SYK models are indeed quantum mechanical models in $0 + 1$ dimensions, with generalizations to higher dimensions (thus, with more local directions) having been proposed. The other class of mixed models is the extension of

(simplicial) GFT quantum gravity models to include scalar fields coupled to gravitational degrees of freedom [83]. These extended models have been studied in particular in the context of GFT condensate cosmology [75–77, 84, 85], with the additional scalar fields playing (also) the role of clock and rods that allow to define relational, diffeo-invariant observables in terms of which an effective cosmological dynamics can be extracted from the GFT hydrodynamics.

The potential physical interest of these models, and of their renormalization analysis, is thus obvious. They present several interesting issues. The presence of both local and non-local directions may modify sensibly the renormalization flow and the structure of divergences, thus leading to different dominant diagrams and effective dynamics in both UV and IR sectors. One can also envisage setting up an altogether different renormalization group scheme, adopting a notion of scale tied to the scalar (local) directions, rather than the group manifold (or involving both), potentially producing very different results. Such focus on the flow parametrized by variables with a (tentative) physical interpretation as relational time/space variables may also allow a more direct physical interpretation of the renormalization flow itself, e.g., in a cosmological context (even though similar cautionary remarks as for the usual renormalization scheme would apply here).

4.4 Relation With Lattice Spin Foam Renormalization

We have emphasized how renormalizing a GFT model is tantamount to renormalizing (and studying the continuum limit of) the corresponding discrete gravity path integral and spin foam amplitudes, from a different standpoint. But the GFT formalism is only one way to provide a complete definition of spin foam models, the other being to view them as a peculiar (because background independent) lattice theory and setting up some appropriate refinement procedure. Therefore, it would be very important to compare results obtained in the context of GFT renormalization, especially for simplicial quantum gravity models, with the results and techniques developed for renormalizing spin foam amplitudes from a lattice gauge theory perspective [65, 66, 86–88].

In this lattice-focused approach to spin foam renormalization, a cut-off is also imposed on representation variables, but the notion of ‘scale’ is rather given by the combinatorial complexity of the underlying lattice, and the renormalization group flow is driven by refinement/coarse-graining steps ordered by such complexity. Refinement/coarse-graining steps affect both bulk lattices and boundary graphs, and the flow of quantum amplitudes is constrained by the requirement of their consistency under restriction to coarser boundary states.

Despite their differences, the two renormalization schemes share several, since also GFT subtraction moves amount to lattice coarse-graining steps, and corresponding maps between associated amplitudes are also built-in in the (perturbative) QFT renormalization steps used in the GFT context. Still, a detailed work of translation between the two frameworks would be very useful. This work may require, on the GFT

side, a combination of functional renormalization group techniques, since we are interested in the continuum limit of spin foam models, and perturbative expansions, given that spin foam models arise in such expansion. This comparison would be beneficial for both approaches; in particular, it would emphasize the role of combinatorial complexity of boundary states in the GFT renormalization flow. This work should be carried out for all models that have been studied in both settings (also in the lattice renormalization approach work has been confined mostly to highly simplified models), aiming of course at unraveling the continuum phase diagram of 4d quantum gravity from two perspectives at once.

4.5 Group Field Theories Renormalization Via Tensor Networks

One powerful set of techniques coming from the theory of quantum many-body systems, that have been already applied in the context of lattice-based renormalization of spin foam models, uses the language of tensor networks [89, 90]. This language is useful both for numerical studies and for emphasizing the role of entanglement in the renormalization group flow [91, 92]; in particular, it allows to unravel topological quantum phases of many-body systems.

In the case of GFT models, the interest in importing techniques from tensor networks goes beyond these general facts, and stems also from the fact that GFT states themselves can be seen as generalized tensor networks [93], and by the related fact that entanglement is responsible for the basic connectivity between GFT quanta that gives rise to extended discrete structures labeled by quantum geometric data. The many facets of the GFT formalism, moreover, would allow for a manifold application of tensor network techniques. On the one hand the basic GFT field is a tensor and its quantum states are tensor networks, as mentioned; on the other hand, it remains a QFT, calling for continuum tensor network techniques

as employed, say, in standard scalar quantum field theory [94]. At the same time, its Feynman amplitudes are lattice gauge theories, to which a different set of tensor network techniques can be applied [95] (as developed in the context of spin foam lattice renormalization). And they remain quantum many-body systems, peculiar for their background independent nature, but still conventional enough to allow the deployment of tensor network methods taken from their natural context.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding authors.

AUTHOR CONTRIBUTIONS

All authors have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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SUPPLEMENTARY MATERIAL

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Renormalization Group Approach to the Continuum Limit of Matrix Models of Quantum Gravity With Preferred Foliation

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This contribution is not intended as a review but, by suggestion of the editors, as a glimpse ahead into the realm of dually weighted tensor models for quantum gravity. This class of models allows one to consider a wider class of quantum gravity models, in particular one can formulate state sum models of spacetime with an intrinsic notion of foliation. The simplest one of these models is the one proposed by Benedetti and Henson [1], which is a matrix model formulation of two-dimensional Causal Dynamical Triangulations (CDT). In this paper we apply the Functional Renormalization Group Equation (FRGE) to the Benedetti-Henson model with the purpose of investigating the possible continuum limits of this class of models. Possible continuum limits appear in this FRGE approach as fixed points of the renormalization group flow where the size of the matrix acts as the renormalization scale. Considering very small truncations, we find fixed points that are compatible with analytically known results for CDT in two dimensions. By studying the scheme dependence of our results we find that precision results require larger truncations than the ones considered in the present work. We conclude that our work suggests that the FRGE is a useful exploratory tool for dually weighted matrix models. We thus expect that the FRGE will be a useful exploratory tool for the investigation of dually weighted tensor models for CDT in higher dimensions.

Keywords: matrix model, functional renormalization, causal dynamical triangulation, tensor model, continuum limit

1 INTRODUCTION

The construction of a unified theory that contains the two most successful branches of modern physics, i.e. General Relativity (GR) and Quantum Field Theory (QFT) in a curved spacetime, as appropriate limits has been ongoing for more than 80 years and sparked many approaches to the so-called problem of quantum gravity. A complete list of these approaches goes beyond the scope of this introduction. The approaches range from the very conservative application of QFT [2, 3] methods to theories of gravity and the asymptotic safety conjecture [4–6] over refined applications of quantization rules, such as loop quantum gravity [7], spin foams [8], group field theories [9] and tensor models [10–13] to significantly less conservative approaches like emergent gravity [14], holographic duality [15] and to searches for so-called theories of everything such as string theory [16]. Despite the significant diversity, no approach has produced a completely satisfactory answer to the problem of quantum gravity as of now. However, when comparing different approaches, one is lead to the general observation that most of them possess some built-in features that one expects from

quantum gravity, but all known approaches come with intrinsic short-comings that have to be overcome before qualifying the particular approach as a candidate theory of quantum gravity. This observation suggests to combine formulations with different built-in strengths with the goal of obtaining a new approach that mixes the best of both.

The present contribution intends precisely this by combining the systematic renormalization group investigation of tensor models for quantum gravity with the success of CDT in producing phases in which the partition function is dominated by extended geometries [17]. This combination is most straightforwardly possible when CDT is formulated as a dually weighted tensor model. Before going into detail, let us take a step back and describe the big picture schematically:

Tensor models of quantum gravity are based on the principles of Euclidean lattice quantum gravity. Euclidean lattice quantum gravity is a partition function approach, where the partition function is obtained as a sum over Boltzmann factors for spacetimes that are constructed from discrete building blocks. The continuum limit of these partition functions is taken as the limit in which the size of the building blocks approaches zero, while the total volume of the spacetime is held fixed. This implies that the number of building blocks has to diverge when taking the continuum limit, thus indicating that one needs to consider the renormalization group flow of these partition functions, such that the possible continuum limits are identified as the classes of IR-relevant deformations of UV-attractors (in most cases fixed points) of the renormalization group flow. A particularly useful tool for the systematic investigation of non-perturbative renormalization group flow is the FRGE, which takes the form of a simple one-loop equation that describes an interpolation between a bare action and the quantum effective action [18]. To apply this powerful tool to Euclidean lattice gravity it is very useful to exploit the duality between the Feynman-graphs of (un)-colored¹ tensor models and discrete geometries. This duality allows one to identify the Feynman amplitude of the (un)-colored tensor model with the Boltzmann factor of the associated discrete gravity partition function and hence allows a translation from the tensor model action to the discrete gravity action, which takes the form of a Regge action [19]. Thus, the duality relates the large N -limit of the tensor model with the continuum limit of the discrete gravity partition function. Hence, the investigation of continuum limits of lattice quantum gravity is readily translated into the investigation of the possible large N -limits of tensor models, which can be investigated systematically using the FRGE.

This rigorous connection between the continuum limit of Euclidean lattice quantum gravity and the large N -renormalization group flow of tensor model actions is an invaluable intrinsic feature of the tensor model approach to quantum gravity; and the systematic investigation of these continuum limits with the FRGE is particularly convenient. Unfortunately, the extended geometries that are approximated in the continuum limits that have been investigated so far possess

dimension two or less [20]. In other words, so far no state sum model of discrete geometry is known to coarse grain to a model of extended spacetime geometry in more than two dimensions.

There are however numerical indications that d -dimensional CDT and its counter part Euclidean Dynamical Triangulations (EDT) do coarse grain to extended higher dimensional geometries (for $2 \leq d \leq 4$) [21]. This can be heuristically understood as the fact that the foliation in CDT and the volume term in EDT implement additional terms in the Boltzmann factor for discrete geometry which change the universality class of the model. Moreover, there exist tensor models that implement critical features of CDT and EDT partition functions in the literature. The novelty in these models is that they possess a nontrivial propagator, which implements a dual weighting of the Feynman graphs of these tensor models. Hence, one can use the FRGE to investigate the continuum limits of CDT and EDT by studying the renormalization group flow of tensor models with dual weights. This is the motivation for the work presented in the present contribution.

As a first step, we consider a dually weighted matrix model proposed by Benedetti and Henson whose partition function is dual to two-dimensional CDT [22]. By doing this we follow a strategy that was used when first applying the FRGE to tensor models [23], where matrix models for two-dimensional Euclidean quantum gravity were considered to introduce the setup, develop the technique and to compare with the analytic results known from constructive approaches to two-dimensional Euclidean quantum gravity, which serve as a benchmark. This allows us to test a setup (the systematic FRGE investigation to dually weighted tensor models) that is readily available in higher dimensions, in particular in $3 + 1$ dimensions [22], but at the same time is understood analytically, providing benchmark results for the FRGE calculation which we can use to gauge this setup.

This contribution is organized as follows: In the following section (*Preliminaries*) we provide the necessary background on dually weighted tensor models, the particular model proposed by Benedetti and Henson and the foundations of the application of the FRGE to tensor models. We provide the derivation of the beta functions in β -functions. We perform a fixed point analysis and study of scheme dependence in *Fixed Point Analysis and Scheme Dependence*. We summarize our results in *Conclusion* and briefly discuss their implications for future investigations on dually weighted tensor models for quantum gravity and provide a short recipe for the calculation in the appendix.

2 PRELIMINARIES

Random tensor models are by now an established approach to Euclidean quantum gravity [24, 25]. However, to fully appreciate the way in which dually weighted matrix models provide an approach to quantum gravity with a preferred time slicing one needs to take a step back and consider the foundations of random tensor models.

2.1 Tensor Models and Dual Weights

The random tensor model approach to quantum gravity is based on the basic observation that the Feynman graphs of so-called

¹The term (un)-colored is explained in the next section.

uncolored random tensor models possess a geometric interpretation in terms of tessellations of piece-wise linear pseudo-manifolds, as do some so-called colored tensor models. The uncolored models are defined for tensors $T_{i_1 i_2, \dots, i_k}$ and their complex conjugates $\bar{T}_{i_1 i_2 i_3, \dots, i_k}$ through the symmetry of the action $S[T, \bar{T}]$ under the $U^k(N)$ transformations

$$T_{i_1 i_2, \dots, i_k} \rightarrow (U_1)_{i_1}^{j_1} (U_2)_{i_2}^{j_2} \dots (U_k)_{i_k}^{j_k} T_{j_1 j_2, \dots, j_k}. \quad (1)$$

This symmetry implies that the action can be expanded in terms of generalized trace invariants in which the first index i_1 of each tensor T must be contracted with the first index of a complex conjugated tensor $\bar{T}$, and similarly the second index i_2 and all further indices i_l . These generalized traces can be represented as colored graphs where each tensor T is represented by a white vertex and each complex conjugate tensor $\bar{T}$ is represented by a black vertex and each contraction of vertices by an index i_l is represented by an edge of color l connecting the vertices associated with the two contracted tensors. Such colored graphs are then dual to piecewise linear pseudo-manifolds: Each vertex is associated with a $(k-1)$ -simplex and the adjacent edges are associated with a gluing of the colored $(k-2)$ -simplices in the boundary of the two $(k-1)$ simplices. Moreover, closed two-colored sub-graphs are associated with the gluing of $(k-3)$ -simplices in the boundary of the associated $(k-2)$ -simplices. Analogously, closed three- and more-colored subgraphs are associated with the gluing of simplices of even lower dimension. The generalized trace-invariants of a rank k tensor model can thus be interpreted as tessellations of piecewise linear $(k-1)$ -dimensional manifolds.

We can perform the analogous identification for the Feynman graphs generated by the rank k random tensor model through realizing that the Feynman graphs of a rank k tensor model possess a graphical representation in terms of $(k+1)$ -colored graphs, where a new color is associated with the propagator. This provides the desired geometric interpretation of the Feynman graphs of an uncolored rank k tensor model with tessellations Δ of piecewise linear pseudo-manifolds of dimension k . It follows that the partition function of these random tensor models possesses a geometric interpretation

$$Z = \sum_{\Delta} A(\Delta) = \sum_{\Delta} e^{-\ln(A(\Delta))}, \quad (2)$$

Where $A(\Delta)$ denotes the Feynman amplitude associated with the Feynman graph dual to Δ . This resembles the random lattice partition function for Euclidean quantum gravity

$$Z_{\text{grav.}} = \lim_{a \rightarrow 0} \sum_{\Delta} \exp(-S_E[\Delta, a]), \quad (3)$$

When the gravity action $S_E[\Delta, a]$ is identified with $-\ln(A(\Delta))$. The Feynman amplitude depends on the details of the random tensor model, but one can generally say that they depend on the number of N_k of k simplices and the number N_{k-2} of $k-2$ simplices in Δ as well as the tensor size N and the coupling constants λ_r . An example amplitude for $k=3$ with one coupling is

$$A(\Delta) = N^{N_1 - 3N_3/2} (\bar{\lambda})^{N_3/2}, \quad (4)$$

Where the couplings N, λ possess a simple relation with the couplings in the Regge-expression of General Relativity in three dimensions $S_R[\Delta] = \kappa_3 N_3 - \kappa_1 N_1$. Hence, $\kappa_1 = \ln(N)$ and $\kappa_3 = 3/2 \ln(N) - 1/2 \ln(\bar{\lambda})$ establishes a relation with the discrete General Relativity coupling constants.

The total volume is $\langle V \rangle = \langle N_d \rangle a^d V_o$, where V_o is the filling factor of the geometric building blocks. Hence, one can take the lattice continuum limit $a \rightarrow 0$ at fixed total volume $\langle V \rangle$ by tuning to a point where the expectation value of the total volume diverges. This requires that $\langle N_d \rangle$ diverges. It turns out that this in turn requires that $N \rightarrow \infty$. However, to obtain simultaneously a finite value of the total volume and of Newton's constant, one needs to tune λ and N simultaneously. Since Z diverges for $N \rightarrow \infty$ one can only obtain a finite result when λ approaches a critical point λ_* as $N \rightarrow \infty$ is approached. Hence, we can write down the required behavior of $\lambda(N) = \lambda_* + cN^{-\theta}$, where c is an arbitrary constant and θ the critical exponent. In other words, the conjecture is that the continuum limit of lattice quantum gravity can be investigated by studying the critical points in the large N behavior of random tensor models.

So far we have only considered a canonical quadratic term $T_{i_1 i_2, \dots, i_k} \bar{T}_{i_1 i_2, \dots, i_k}$, as is implied by $U^k(N)$ invariance. This kinetic term leads to an index-independent propagator $\propto \delta^{i_1 j_1} \dots \delta^{i_k j_k}$, so each closed loop of indices will contribute with a factor of N to the amplitude, but can not depend on the number of vertices that are crossed when going around this loop. However, we will see shortly that such a dependence of the amplitude can be motivated geometrically. To construct tensor models whose amplitude depends non-trivially on the number of vertices crossed by a closed index loop. Before motivating these so-called “dually weighted” tensor models, we will consider the general setup of the FRGE for tensor models [26].

2.2 Application of the FRGE to Tensor Models

One of the most convenient tools to investigate critical behavior is the functional renormalization group equation (FRGE). In the usual setting the FRGE

$$\partial \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left(\frac{\dot{R}_k}{\Gamma_k^{(2)}[\phi] + R_k} \right) \quad (5)$$

Describes how the effective average action

$$\Gamma_k[\phi] := \sup_J \{ J\phi - \ln(Z_k[J]) \} - \frac{1}{2} \phi R_k \phi \quad (6)$$

Changes when the IR suppression scale k is changed. This IR-suppression scale is introduced through a modification of the bare action by the scale dependent mass term $\Delta_k S[\phi] = 1/2 \phi R_k \phi$, which is designed to give a mass of $\mathcal{O}(k)$ to modes in the IR of the scale k while not significantly affecting modes in the UV of this scale. Heuristically, one can argue as follows: $\Delta_k S[\phi]$ dominates the path integral in the limit $k \rightarrow \infty$ and hence the saddle point approximation of the path integral becomes exact in this limit and shows that the effective average action coincides with the bare

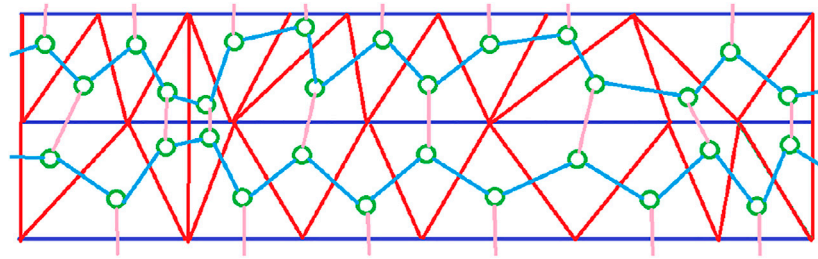


FIGURE 1 | Part of a dual triangulation to a Feynman graph. The solid colors red and blue indicate the time- and space-like boundaries of dual triangles, the light colors the dual propagators in the Feynman graph and the green circles the vertices in the Feynman graph. The important fact to note is that having precisely two pink propagators in each closed loop implies that the blue lines foliate the entire Feynman graph. Notice that we drew the propagators as single lines to not clutter the picture too much; the usual depiction of the matrix model propagator would be though a double line, i.e. line one for each contacted index.

action when $k \rightarrow \infty$. Hence, one finds critical points as UV fixed points of the FRGE and can study the critical behavior by studying the linearized flow near the fixed points.

The usual FRGE arguments outlined above relies heavily on the mass dimension and scaling with a scale k that possesses units of mass. Such a mass scale is missing in the random tensor setup, instead one wants to study the scaling of the couplings with the dimensionless tensor size N . This requires one to identify 1) the scaling of the IR-suppression term with N and 2) the scaling of the coupling constants with N . It turns out that the requirement that the RHS of the FRGE admits a $1/N$ -expansion imposes significant restrictions on the scaling with N , but it does not fix it completely. To obtain a completely determined scaling with N one needs to impose that the bare action possesses a geometric interpretation. Essentially, the requirements are that (1) the bare propagator and the modified propagator (after including the IR-suppression term $\Delta_N S[T]$) possess the same scaling for large index values and 2) that the interaction term possesses the scaling necessary for the geometric interpretation. These two initial conditions, together with the restrictions that result from the $1/N$ -expandability of the RHS of the FRGE fix the setting that is sufficient to investigate the large- N -critical behavior.

2.3 2D Causal Matrix Model

A matrix model that enforces a preferred time slicing in its Feynman-graphs was proposed by Benedetti and Henson in [1]. This model is constructed using two dynamical $N \times N$ matrices, A and B , representing the spacelike and timelike edges of a triangle, and a constant matrix C which implements the dual weighting. The partition function is

$$Z = \int dA dB e^{-N \text{Tr} \left(\frac{1}{2} A^2 + \frac{1}{2} (C^{-1} B)^2 - g A^2 B \right)}, \quad (7)$$

Where in the large N limit the matrix C must satisfy the condition

$$\text{Tr}(C^m) = N \delta_{2,m}, \quad (8)$$

With $m \in \mathbb{N}$. The partition function (7) with a weighting matrix C that implements (8) generates Feynman diagrams that possess the geometric interpretation of polytopes with an arbitrary number of space-like edges and only two time-like edges

(see **Figure 1**). This is clear by analyzing the free propagators ($g = 0$) of the model

$$\langle A_{ij} A_{kl} \rangle_0 = \frac{1}{N} \delta_{il} \delta_{kj}, \quad (9)$$

$$\langle B_{ij} B_{kl} \rangle_0 = \frac{1}{N} C_{il} C_{kj}, \quad (10)$$

$$\langle A_{ij} B_{kl} \rangle_0 = 0. \quad (11)$$

As we can see in **Figure 1**, this restriction implements a foliation of the discrete geometries that appear in the expansion of the partition function, thus introducing the necessary structure for the implementation of CDT in tensor models.

We proceed by integrating out matrix B since it is a gaussian integral, obtaining

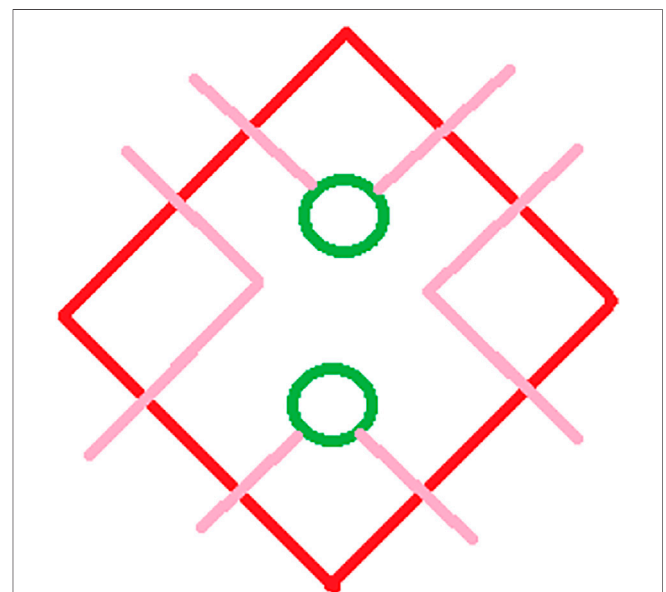


FIGURE 2 | Resulting vertex after the integration over matrix B . The double pink lines indicate the propagator and the green circles indicate the insertion of the matrix C in the interaction.

$$Z = \int dA e^{-N \text{Tr} \left(\frac{1}{2} A^2 - \frac{g^2}{2} (A^2 C)^2 \right)}. \quad (12)$$

This partition function together with condition (8) determines our starting point in this work.

One can understand the integration over matrix B as the gluing of triangles along their spacelike edges. This gives rise to a model of squares only with timelike edges (Figure 2). This produces an anisotropic quadrangulation with rigidity associated with condition (8).

We identify the matrix model action that implements CDT as

$$S = \frac{1}{2} \text{Tr} (AA^\top) - \frac{g^2}{2} \text{Tr} (AA^\top CAA^\top C), \quad (13)$$

Which takes the form of the Euclidean action [25], except for the presence of matrix C . The appearance of the dual weighting matrix C changes the symmetry of the matrix model. Let us consider an $N \times N$ orthogonal matrix, O , and the transformation

$$\begin{aligned} A &\rightarrow AO \\ A^\top &\rightarrow O^\top A^\top. \end{aligned} \quad (14)$$

Since the combination $AA^\top$ is invariant $A \rightarrow OA$ ant (14) is a symmetry of the Euclidean and the CDT action, however the conjugate symmetry under and $A^\top \rightarrow A^\top O^\top$ is *only* a symmetry of the Euclidean action and *not* of the CDT action. This shows an explicit difference with the real Euclidean matrix model. In the language of the renormalization group: the CDT action (13) lives in a different theory space, which is governed by a different symmetry.

Weighting Matrix

The matrix C implements the weighting of closed loops of propagators in the Feynman graph expansion, i.e. the dual weighting of the Feynman graphs. In principle one could define this matrix abstractly only through the property (8) and only use Eq. 8 whenever the matrix occurs in a Feynman diagram. However, in order to do practical calculations with the FRGE, it is very useful to have an explicit representation of C at ones disposal.

For a $N \times N$ diagonal matrix, X , with eigenvalues $\{x_i\}$ we can write its characteristic polynomial as

$$P_X(t) = \sum_{k=0}^N (-1)^k e_k t^{N-k}, \quad (15)$$

Where e_k is

$$\begin{aligned} e_0(x_1, \dots, x_N) &= 1, \\ e_1(x_1, \dots, x_N) &= \sum_{i=1}^N x_i, \\ e_2(x_1, \dots, x_N) &= \sum_{1 \leq i < j \leq N} x_i x_j, \\ &\vdots \\ e_N(x_1, \dots, x_N) &= x_1 x_2 \dots x_N, \end{aligned}$$

Then Newton identities allow us to write this coefficients in terms of the k th power of the trace of X , p_k , in the form

$$k e_k = \sum_{i=1}^k (-1)^{i-1} p_i e_{k-i}, \quad (16)$$

So, C can be found by solving

$$P_C(t) = t^N - \frac{1}{2} N t^{N-2} + \frac{1}{8} N^2 t^{N-4} + \dots = 0. \quad (17)$$

These solutions exist by the fundamental theorem of algebra and one can use ones preferred approximation scheme to obtain these. One scheme that suggests itself in particular when one wants to gain insights into the effects of dual weightings is to build a matrix C from smaller blocks of matrices C_o , so $C = \text{diag}(C_o, \dots, C_o)$. The matrix obtained in this way does not implement the entire tower of Eq. 8, but permits traces periodically. This approach is particularly interesting to study, since it allows us to study how many of the equations one needs to enforce to attain the phase transition between the Euclidean matrix model and the CDT matrix model. The first three matrices C_o are

$$\begin{aligned} \cdot k = 2 : \\ C_o = \text{diag}(-1, 1), \end{aligned} \quad (18)$$

$$\begin{aligned} \cdot k = 4 : \\ C_o = \text{diag}(-1.09 - 0.45i, -1.09 + 0.45i, \\ 1.09 - 0.45i, 1.09 + 0.45i), \end{aligned} \quad (19)$$

$$\begin{aligned} \cdot k = 6 : \\ C_o = \text{diag}(1.02 + 0.70i, 1.02 - 0.70i, 1.37, \\ -1.02 + 0.70i, -1.02 - 0.70i, -1.37). \end{aligned} \quad (20)$$

Putting these together as blocks to build an $N \times N$ matrix gives for example for $k = 2$

$$C_{jj} = (-1)^j, \quad (21)$$

And for $k = 4$

$$C_{jj} = \begin{cases} (-1)^k \gamma - i\xi, & j = 2k - 1, \\ (-1)^k \gamma + i\xi, & j = 2k, \end{cases} \quad (22)$$

That are $N \times N$ matrices formed by 2×2 and 4×4 blocks, and where $\gamma = 1.09$ and $\xi = 0.45$.

Functional Renormalization Group for Matrix Models

Let us briefly review the application of the FRGE to matrix and tensor models. One can follow the fundamental presentation of [18] and apply it to matrix models as done in [23]. The starting point is the definition of the effective average action $\Gamma_N[\varphi]$ in the presence of an IR-suppression term $\Delta S_N[\phi]$:

$$\Gamma_N[\varphi] = \inf_J \{ W_N[J] + J\varphi - \Delta S_N[\varphi] \}, \quad (23)$$

Where

$$\exp(-W_N[J]) = \frac{1}{N_N} \int_{\Lambda} \mathcal{D}\phi e^{-S[\phi] + J\phi - \Delta S_N[\phi]}, \quad (24)$$

And φ represents the expectation value of a quantum field ϕ , while the term

$$\Delta S_N[\phi] = \frac{1}{2} \phi_{ab} R_N^{abcd} \phi_{cd} \quad (25)$$

Represents an IR-suppression term in so far as it is designed to give a mass term of order N to “IR” degrees of freedom of the matrix. Since matrix and tensor models do not implement a fundamental scale, there is no canonical identification of which degrees of freedom are “IR”. Rather one needs to implement by hand a division of theory space according to an RG scale k . The simplest assignment is to identify the upper-left corner of the matrix with index values below the scale k as “IR”. Once the IR-suppression term is chosen, one can proceed as in [18]; one arrives at the FRGE

$$\partial_t \Gamma_N = \frac{1}{2} \text{Tr} \left(\frac{\partial_t R_N}{R_N + \Gamma_N^{(2)}} \right), \quad (26)$$

Where $t = \ln N$. The solutions to (26) are functionals of the $N \times N$ matrix ϕ and hence of infinitely many degrees of freedom in the large N -limit. Practically one resorts to finite truncations of the effective average action, i.e. one performs an expansion of the effective average action into monomials

$$\Gamma_k[\varphi] = \sum_i \bar{g}_i(k) \mathcal{O}_i[\varphi]. \quad (27)$$

Then one truncates this expansion at a manageable set of operators $\mathcal{O}_i$. In this way the computation is reduced to the study of the projected flow in the space of coupling constants $\bar{g}_i$. The quality of the FRGE results depends critically on the operators that are included in the truncation. In matrix models it turned out that surprisingly good approximations to the FRGE flow were obtained in [23] by considering the flow of single trace operators. The analogous truncation in the presence of the matrix C is

$$\Gamma_N = \frac{Z}{2} \text{Tr}(AA^\top) + \sum_{n=2}^P \frac{\bar{g}_{2n}}{2n} \text{Tr}((AA^\top C)^2 (AA^\top)^{n-2}), \quad (28)$$

Which only includes operators with $AA^\top$, which is invariant under (14), and two C matrices. In the present contribution we will truncate this to the ansatz that contains the bare action and the single trace operator that can directly contribute to the beta functions of the bare action at one loop. This truncation is:

$$\begin{aligned} \Gamma_N = & \frac{Z}{2} \text{Tr}(AA^\top) + \frac{\bar{g}_4}{4} \text{Tr}(AA^\top CAA^\top C) \\ & + \frac{\bar{g}_6}{6} \text{Tr}(AA^\top CAA^\top CAA^\top). \end{aligned} \quad (29)$$

We introduce the dimensionless couplings

$$\begin{aligned} \bar{g}_4 &= Z^2 N^{\alpha_4} g_4 \\ \bar{g}_6 &= Z^3 N^{\alpha_6} g_6 \end{aligned} \quad (30)$$

Where α_4 and α_6 are as of yet undetermined, since the matrix model does not include an intrinsic notion of scale. The scale is later fixed by imposing that the beta functions admit a $1/N$ expansion.

To make the calculation concrete, we choose the explicit form of the IR-suppression term R_N to take the form

$$R_N^{abcd} = Z \left(\frac{N}{a+b} - 1 \right) \theta \left(1 - \frac{a+b}{N} \right) \delta^{ac} \delta^{bd}, \quad (31)$$

Which has the advantage of being a diagonal and field independent tensor, so we can readily invert the kinetic term to obtain the propagator. It is practically useful to split the second variation of the effective average action into a field independent term G and a field dependent term F :

$$R_N + \Gamma_N^{(2)} = G_N + \bar{g}_4 F_N^{(4)}[A] + \bar{g}_6 F_N^{(6)}[A], \quad (32)$$

Which allows us to expand the RHS of the Wetterich equation as a geometric series, using only the propagator $P = G^{-1}$ and the F -term:

$$\begin{aligned} \frac{1}{2} \text{Tr} \left(\frac{\dot{R}_N}{R_N + \Gamma_N^{(2)}} \right) &= \frac{1}{2} \sum_{k=0} (-1)^k \text{Tr}(\dot{R}P(FP)^k) \\ &= \frac{1}{2} \text{Tr}(\dot{R}P) - \frac{1}{2} \text{Tr}(\dot{R}PFP) + \frac{1}{2} \text{Tr}(\dot{R}PFPFP) \\ &\quad - \frac{1}{2} \text{Tr}(\dot{R}PFPFPFP) + \dots \end{aligned} \quad (33)$$

The upshot of this P - F expansion is that each F term contributes more field operators. Hence a truncation that contains polynomial operators with only a finite number of fields terminates the geometric series at a finite number of terms. With the proposed truncation (29), the first term in (33) is of order zero in the Feynman diagrams expansion since there is no field contribution, the second term gives rise to 2-vertex and 4-vertex diagrams which contribute to η and β_4 , the third one to 4-vertex and 6-vertex diagrams which contribute to β_4 and β_6 and so on.

2.5 Benchmark Results

We use the FRGE to find fixed points of the RG flow and to investigate the universality class associated with this fixed point. This is done by calculating the critical exponents θ at the fixed point, i.e. by considering the linearized FRGE-flow at the fixed point, where the critical exponents appear as the eigenvalues of the Hessian of the beta functions. We chose our truncation in such a way that we can resolve the fixed point that known as the double scaling limit in the matrix model literature. This fixed point possesses a single positive critical exponent, which is usually expressed in terms of the string susceptibility γ_{str} :

$$\theta = \frac{2}{2 - \gamma_{str}}. \quad (34)$$

For Euclidean Matrix Models [27] $\gamma_{str} = -1/2$, while for CDT [28] $\gamma_{str} = +1/2$, which leads to the following critical exponents

$$\theta_{MM} = \frac{4}{5}, \quad \theta_{CDT} = \frac{4}{3}. \quad (35)$$

3 β -FUNCTIONS

In this section we summarize the steps that we took to obtain the beta functions of the matrix model for CDT.

3.1 Operator Products

The structure of the beta functions is determined by the operator products of the F -terms that we showed in (33). The terms $F_N^{(4)}$ and $F_N^{(6)}$ are the second variations of the operators whose contribution to the effective average action is measured by the coupling constants $\bar{g}_4$ and $\bar{g}_6$. The second variations take the form

$$F_N^{(4)} = C^{ac} (A^\top CA)^{db} + (CA)^{ad} (CA)^{cb} + (CAA^\top C)^{ac} \delta^{db} \quad (36)$$

And

$$\begin{aligned} F_N^{(6)} = & \delta^{ac} (A^\top CAA^\top CA)^{db} + A^{ad} (CAA^\top CA)^{cb} \\ & + (AA^\top C)^{ac} (A^\top CA)^{db} + (AA^\top CA)^{ad} (CA)^{cb} \\ & + (AA^\top CAA^\top C)^{ac} \delta^{db} + C^{ac} (A^\top CAA^\top A)^{db} \\ & + (CA)^{ad} (CAA^\top A)^{cb} + (CAA^\top C)^{ac} (A^\top A)^{db} \\ & + (CAA^\top CA)^{ad} A^{cb} + (CAA^\top CAA^\top)^{ac} \delta^{db} \\ & + C^{ac} (A^\top AA^\top CA)^{db} + (CA)^{ad} (AA^\top CA)^{cb} \\ & + (CAA^\top)^{ac} (A^\top CA)^{db} + (CAA^\top A)^{ad} (CA)^{cb} \\ & + (CAA^\top AA^\top C)^{ac} \delta^{db} \end{aligned} \quad (37)$$

By looking at (33), we notice that traces of products of (36) and 37 give rise to a big range of operators which are not present in the truncation ansatz (29), such as $(\text{Tr}(CA))^2$, $\text{Tr}(CAA^\top C^2)$, $\text{Tr}(A^\top CA)$, etc. However, a reasonable projection rule onto the truncation should not project these operators onto the beta functions of the truncation. We therefore analyze which operators can contribute to the beta functions in the truncation, i.e. to β_4 and β_6 . Considering for example the trace

$$\text{Tr}(F_N^{(4)}) = \text{Tr}(C) \text{Tr}(A^\top CA) + \text{Tr}(CAA^\top C) \text{Tr}(\delta), \quad (38)$$

We see that each term contains the matrix C , while we know from the structure of the P-F-expansion that these are the only terms generated by the restriction of the FRGE to the truncation that contain two matrices A . Hence, the restriction of the FRGE to the truncation does not generate terms $\sim \text{Tr}(AA^\top)$ and hence does not generate any term that contributes to the anomalous dimension η . To generate a contribution to the anomalous dimension, one needed to include a term with a single C matrix in the truncation. This term would then be generated at one loop by the first term in (38) and in turn contribute to η at one loop. The investigation of this kind of secondary effect however goes beyond the scope of this first investigation.

This analysis relies on the fact that our projection rule is able to discern the structure in which the matrices A are contracted with the constant weighting matrix C , so at first sight one might worry that such a projection does not exist. However, one can consider the appearance of the matrix C in the operators as a special case of operators with index-dependence, i.e. operators whose variations w.r.t. A can not be expressed in terms of A and δ_{ij} , which can be discerned by a suitable projection rule. Hence it is not only

possible, but even prudent to use a projection rule that discerns the different ways in which the matrix C is contracted.

To make this distinction, we mark in the following the terms that contribute to the beta functions in our truncation by putting a box around them. Subsequently, we will impose the use of a projection rule that only retains these operators and thus consider only the contributions of the boxed terms.

$$\begin{aligned} \text{Tr}(F_N^{(6)}) = & \text{Tr}(\delta) \text{Tr}(AA^\top CAA^\top C) \\ & + 3 \text{Tr}(AA^\top CAA^\top C) \text{Tr}(\delta) \\ & + 2 \text{Tr}(C) \text{Tr}(A^\top CAA^\top A), \end{aligned} \quad (39)$$

$$\begin{aligned} \text{Tr}(F_N^{(4)} F_N^{(4)}) = & \text{Tr}(CC) \text{Tr}(A^\top CAA^\top CA) \\ & + \text{Tr}(CAA^\top CCAA^\top C) \text{Tr}(\delta), \end{aligned} \quad (40)$$

$$\begin{aligned} \text{Tr}(F_N^{(4)} F_N^{(6)}) = & 2 \text{Tr}(CC) \text{Tr}(A^\top CAA^\top AA^\top CA) \\ & + 3 \text{Tr}(CAA^\top CAA^\top CAA^\top C) \text{Tr}(\delta) \\ & + \text{Tr}(C) \text{Tr}(A^\top CAA^\top CAA^\top CA), \end{aligned} \quad (41)$$

$$\begin{aligned} \text{Tr}(F_N^{(4)} F_N^{(4)} F_N^{(4)}) = & \text{Tr}(C^3) \text{Tr}(A^\top CAA^\top CAA^\top CA) \\ & + \text{Tr}(CAA^\top CCAA^\top CCAA^\top C) \text{Tr}(\delta). \end{aligned} \quad (42)$$

When considering $\text{Tr}((F_N^{(4)})^n)$ with $n > 2$ we see that all resulting operators contain at least three C matrices which are not present in the original proposed action (29), this means that in this truncation the β -functions do not possess contributions coming from these traces.

3.2 General Form of the β -functions

Now that we have identified the terms that can contribute to the β -functions, we can write down the general structure of the beta functions. To do so, we introduce the constants D_i , E_i and F_i , which depend on the details of the projection rule. Repeating the same argument as in the previous subsection for the single trace truncation 28 we obtain

$$\eta = 0, \quad (43)$$

For i odd

$$\beta_{2i} = (i\eta - \alpha_{2i})g_{2i} + D_i g_{2(i+1)} + E_i g_{2i} g_4, \quad (44)$$

For i even

$$\beta_{2i} = (i\eta - \alpha_{2i})g_{2i} + D_i g_{2(i+1)} + E_i g_{2i} g_4 + F_i g_{2(i+2)}. \quad (45)$$

We can see in particular that in this truncation tadpoles and 2-vertex diagrams contribute.

4 FIXED POINT ANALYSIS AND SCHEME DEPENDENCE

By using the obtained general form of the beta functions for the single trace truncation at our disposal we can discuss fixed points. We first consider the fixed point structure analytically, before inserting particular truncation rules, which provide numerical values for the critical exponents, which allows us to discuss the scheme dependence of our calculation.

TABLE 1 | Critical points with its corresponding pair of critical exponents.

Critical point	Critical exponents
(0,0)	α_4
$\left(\frac{\alpha_4}{E_2}, 0\right)$	α_6
$\left(\frac{\alpha_6}{E_3}, \alpha_6 \frac{\alpha_4 E_3 - \alpha_6 E_2}{D_2 E_3^2}\right)$	$-\alpha_4$
	$\frac{\alpha_6 E_2 - \alpha_4 E_3}{E_2}$
	$\frac{-2\alpha_6 E_2 + \alpha_4 E_3 - \sqrt{4\alpha_6 (E_2^2 - E_2) - 4\alpha_4 (E_2 E_3 - E_2^2) + \alpha_4 E_3^2}}{2E_3}$
	$\frac{-2\alpha_6 E_2 + \alpha_4 E_3 + \sqrt{4\alpha_6 (E_2^2 - E_2) - 4\alpha_4 (E_2 E_3 - E_2^2) + \alpha_4 E_3^2}}{2E_3}$

4.1 Analytic Fixed Point Analysis

By setting our truncation to (29), we obtain the following beta functions

$$\beta_4 = (-\alpha_4)g_4 + D_2 g_6 + E_2 g_4^2, \quad (46)$$

$$\beta_6 = (-\alpha_6)g_6 + E_3 g_4 g_6, \quad (47)$$

Where η has been set to zero in accordance with our previous analysis. The set of fixed points of this system of beta functions is

$$(g_4^*, g_6^*) = \left\{ (0, 0), \left(\frac{\alpha_4}{E_2}, 0\right), \left(\frac{\alpha_6}{E_3}, \alpha_6 \frac{\alpha_4 E_3 - \alpha_6 E_2}{D_2 E_3^2}\right) \right\}, \quad (48)$$

And the Hessian matrix (defined as $H_{ij} = -\partial\beta_i/\partial g_j$) is

$$\begin{pmatrix} \alpha_4 - 2E_2 g_4 & -D_2 \\ -E_3 g_6 & \alpha_6 E_3 g_4 \end{pmatrix}. \quad (49)$$

Hence, the critical exponents, i.e. the eigenvalues of the Hessian, evaluated in each of the fixed points take the form.

The canonical dimensions α_4 and α_6 are not fixed by themselves, but tadpole diagrams show that α_6 has to be one dimension of N greater than α_4 . We identify the Gaussian fixed point (0,0), which will not have a relevant direction. There are two non-Gaussian fixed points: the first fixed point $(\alpha_4/E_2, 0)$ contains one relevant and one irrelevant direction if $E_2/E_3 > \alpha_4/\alpha_6$, and the third fixed point in **Table 1** possesses a relevant and an irrelevant direction if $E_2/E_3 < \alpha_4/\alpha_6$. These two points are our candidates for a double scaling limit. Next we will examine them using particular schemes.

4.2 Scheme Dependence

To find concrete critical exponents, we supplement the projection rule with the evaluation of both sides of the FRGE at preferred test matrices A . Moreover, we consider the two rigidity matrices obtained by constructing a block diagonal matrix from (18) (19), namely (21) and (19) respectively. The specific test matrices that we use for the projection are

$$A_{ab}^\delta = \delta_{ab} \theta(N - a), \quad (50)$$

$$A_{ab}^{\delta-mod} = a \delta_{ab} \theta(N - a), \quad (51)$$

$$A_{ab}^{\delta-IR} = \delta_{1,a} \delta_{1,b}. \quad (52)$$

Using three different matrices A allows us to estimate a lower bound for the scheme dependence. We expect this because the three field configurations contain two field configurations with distinct UV behavior and one manifest IR field configuration.

One often assumes that the scheme dependence is an actual approximate measure for the quality of the fixed point analysis, however when comparing with analytic results, we will see that this underestimates the truncation error.

The corresponding obtained critical exponents are shown in **Table 2** and the fixed points are shown in **Table 3**. The euclidean values are computed as done in [23] using (50), (51) and (52) as test fields.

We see that the relevant critical exponents are all close to 1, while the irrelevant critical exponents spread a bit wider between -0.858 and -1.215 . Moreover, we observe that all critical exponents lay within the spread obtained by scheme dependence. This means that we can not distinguish the Euclidean models from the Causal models, built from the 2×2 and 4×4 matrices, based on the present derivation of the critical exponents.

In order to attempt to obtain more accurate numerical values for the critical exponents we use the fixed point approximation. This consists in first finding the zeros of the beta functions, then evaluating the anomalous dimension, η , in the fixed point g_4^* and substituting this numerical value in the beta functions to find the critical exponents. The critical exponents obtained by using the fixed point approximation are shown in **Table 4**.

Since the numerical values reported in **Table 2** were found to have a strong scheme dependence, it is important to compare the renormalization scheme dependence vs. the causal-euclidean results in the latter ones in **Table 4**. We compute the average of the difference between the critical exponents obtained in the different renormalization schemes and the “Causal vs. Euclidean” results with each of both methods.

In **Table 5** we observe that, while the first method (full) shows a stronger renormalization scheme dependence, with the fixed point approximation method the “Causal vs. Euclidean” relation is more significant than the renormalization scheme dependence. Regarding the accuracy of the values for the critical exponent obtained with both methods compared to the theoretical values (35), we observe that the Causal ones differ more from the theoretical value than the Euclidean critical exponents. Therefore we can conclude that in this case the fixed point approximation is more useful for differentiating the Causal from the Euclidean results, while the full method reproduces more accurate numerical results.

CONCLUSION

This contribution is motivated by the observation that the application of the FRGE to tensor models with dual weights could lead to an approach to quantum gravity that combines the advantages of the systematic search of continuum limits with the FRGE with the physically promising phase diagrams of CDT and EDT. The systematic development of these tools and the systematic investigation of these models is a very ambitious task. In this contribution we took a first step into this direction and considered the FRGE flow of a matrix model for CDT in $1 + 1$ dimensions proposed by Benedetti and Henson. This model implements a foliation through a dual weighting of

TABLE 2 | Numerical values obtained for critical exponents. Causal² corresponds to values computed using (21) and Causal⁴ corresponds to the ones computed using (LABEL:c4).

	δ			δ -mod.		δ -IR	
	Causal ²	Causal ⁴	Euclidean	Causal ²	Euclidean	Causal ²	Euclidean
θ	1.033	1.008	1.046	1.024	1.033	1.052	1.065
θ'	-0.928	-1.215	-1.080	-0.858	-0.959	-1.086	-1.050

TABLE 3 | Numerical values obtained for critical points. Causal² corresponds to values computed using (21) and Causal⁴ corresponds to the ones computed using (LABEL:c4).

	δ			δ -mod.		δ -IR	
	Causal ²	Causal ⁴	Euclidean	Causal ²	Euclidean	Causal ²	Euclidean
g_4^*	-0.435	-0.300	-0.288	-0.902	-0.588	-0.339	-0.202
g_6^*	-0.118	-0.094	-0.06	-0.387	-0.208	-0.040	-0.026

TABLE 4 | Numerical values obtained for critical exponents using the fixed point approximation. Causal² corresponds to values computed using (21) and Causal⁴ corresponds to the ones computed using (LABEL:c4).

	δ			δ -mod.		δ -IR	
	Causal ²	Causal ⁴	Euclidean	Causal ²	Euclidean	Causal ²	Euclidean
θ	0.722	0.902	0.630	0.649	0.544	0.658	0.602
θ'	-0.953	-1.222	-1.116	-0.880	-0.995	-1.105	-1.090

TABLE 5 | Renormalization scheme dependence and Causal-Euclidean difference with both methods.

	Renorm. Scheme	Causal vs. Euclidean
Full	0.019	0.012
Fixed p. a.	0.049	0.084

Feynman graphs, which introduces an action with an index-dependent propagator. This action, upon integration of an auxiliary field, reduces to an action with an index dependent interaction.

Recalling the critical exponents analysis in *Analytic Fixed Point Analysis* and the values in **Table 1**, we see that the model uses a rigidity matrix C which is chosen in such a way that one of the conditions for the beta function polynomials $E_2/E_3 > \alpha_4/\alpha_6$ or $E_2/E_3 < \alpha_4/\alpha_6$ is satisfied. In this case we obtain a relevant and an irrelevant direction simultaneously, implying that the only Feynman diagrams that contribute to the partition function are the ones where all C matrices are contracted as $Tr(C^2)$, which is precisely the condition that implements a foliation. In this contribution we considered a single trace truncation in which we included operators that contain two C -matrices with the pattern prescribed by the interaction in the Benedetti-Henson model and calculated the beta functions for this truncation. We found that wave function renormalization does not occur in this truncation, since the one-loop structure of the FRGE can only remove 1 C matrix. This technical observation has far reaching consequences for the structure of the beta

functions, which change significantly compared to the Euclidean model, which is obtained by setting C to the identity matrix. In other words: the structure of the beta functions is more complicated and in particular includes wave function renormalization if C is replaced with the identity matrix.

We then investigated this system of beta functions in a truncation in which we included only a four- and a six-point interaction. Despite the significant difference in the structure of the beta functions, we found that this truncation contains fixed points that possess the properties of the double scaling limit. We investigated these fixed points numerically using three distinct field configurations for projection.

This numerical investigation revealed a practical challenge: To obtain numerical values for the beta functions one can not resort to an abstract definition of the rigidity matrix C , since the calculation requires an explicit numerical expression of C . We took this as an opportunity to investigate the weakening of the condition $Tr(C^m) = \delta_{m \bmod k, 2}$ for $k = 2, 4, 6$. This has the implication that not all Feynman diagrams without a foliation structure are suppressed, but only a part of these. In particular the case $k = 2$ does not introduce any new restriction at the level of Feynman diagrams of the Euclidean model, however, since we used the structure of the beta function for general C , we still obtained equations that differ from the Euclidean matrix model. The numerical investigations however revealed that we can not discern the Euclidean and the CDT model on the basis of the critical exponents at the fixed point associated with the double scaling limit. These results are summarized in **Table 2**. The obtained relevant critical exponent (θ) in the Benedetti-

Henson model differs from the exact values in *Benchmark Results* by 0.31 from the CDT value and 0.27 from the Euclidean one. We also observe a 4% spread of these values depending on the field configuration used for projection, which is a significantly lower spread than the difference with the analytic values. This is however consistent with the results obtained in [23], where a similar difference from the analytic values was found. This was also to be expected, because the presented calculation is technically very similar to the calculation done in [23]. We therefore expect that the truncation error improves in a similar way as in [23] when the truncation is gradually increased. This means that we expect the truncation error to improve over the range of a few percent as one enlarges the truncation, but also that there will remain a significant deviation from the analytic results until the broken unitary Ward-Identity, stemming from the variation of the kinetic term, is solved in a self-consistent way.

We interpret our results as an encouragement for the investigation of dually weighted tensor models for quantum gravity: Already with the rather simple and elementary techniques used in this contribution we were able to investigate qualitatively the double scaling limit of the dually weighted matrix model; analogous dually weighted tensor models such as the one proposed in [22] can thus be treated with the FRGE in a similar fashion. Our results indicate some practical advice for these future investigations:

- 1) The one-loop structure of the FRGE can only couple effective operators that differ by an index-dependent contraction between two adjacent tensors, not more. Therefore, to study the influence of an operator with index-dependence in more than one contraction one needs to include sufficient “intermediary” operators in the truncation.
- 2) For analytical investigations it is possible to work with abstract rigidity structures, that are defined through its properties, such as $\text{Tr}(C^m) = \delta_{m,2}$, however for numerical investigations one needs a projection onto the truncation and an explicit evaluation of the operator traces appearing on the RHS of the flow equation. This evaluation of the RHS requires an exact (or at least approximate) numerical representation of the abstract structure encoded in the rigidity matrix C . One might thus prefer the investigations of models for which one has one of these numerical representations at ones disposal.
- 3) If our present observations about the double scaling limit are transferable then one sees that the existence of a fixed point with certain characteristics can be found in rather small truncations. However, the critical exponents found in these truncations can be expected to differ significantly from the exact values (which of course should be accessible through larger truncations and optimized renormalization schemes).

These general observations can serve as a guide of what to expect in higher dimensions, where a modification of the propagator can be used to suppress the Feynman diagrams

that lead to most significant deviations from foliated spacetimes. Unfortunately, our observations do not have an immediate implication for the existence of a physically viable continuum limit in higher dimensions.

A RECIPE FOR CALCULATIONS

A detailed description of the recipe to do FRGE-calculations in matrix and tensor models has been presented in [24]. We essentially followed the recipe outlined there, but had to make some adjustments due to the appearance of the rigidity matrix C , which we present in the following.

A.1 Theory Space and Truncations

The theory space upon which we set up the flow equation must include the action proposed in [1] and thus include the rigidity matrix C . This action is invariant under the one-sided transformation $A \rightarrow AO$, for all matrices $O^T O = 1$. These actions can be expanded in terms of traces of products of $A^T A$ and C . It is useful to organize the trace operators systematically with increasing number of fields $A^T A$. For a fixed number of fields we observe that $\text{Tr}((A^T A)^{n_1} C \cdots (A^T A)^{n_k} C)$ defines a sequence of k integers $(n_1, \dots, n_k)$. Using cyclicity of the trace, we rotate the trace such that we obtain the highest number in the base ($\max\{n_1, \dots, n_k\} + 1$) number system, when $(n_1, \dots, n_k)$ are taken as the digits of a number in this system. Since we derive the beta functions in a vertex expansion, it is useful to choose truncations that contain only up to a fixed number of fields. Moreover, as discussed in the text, the structure of the beta functions decouples the bare action from many operators in such a truncation. It is therefore useful to consider truncations that contain only the operators that do not decouple.

A.2 Canonical Dimension

The canonical dimension of the operators can be derived from the requirement that the beta functions possess a ϵ -expansion, since one could not use them to investigate the continuum limit if it were otherwise. The initial condition for this is that the couplings that appear in the bare action possess the same scaling as prescribed by the bare action proposed by Benedetti and Henson, where the regulator is chosen in such a way that it possesses the large N -scaling of the kinetic term. The vertex expansion of the beta functions then provides a set of inequalities that determines the scaling of the operators. The difference with the pure matrix model case is that the appearance of the rigidity matrix C appears on both sides of the flow equation, so that its influence on the scaling arguments has to be taken into account.

A.3 Projection and Extraction of Beta Functions

The most important adjustment to the recipe provided in [28] concerns the projection onto the truncation and the derivation of beta functions. The vertex expansion, the same as described in [28], provides a lot of structural insight into the beta functions, because it shows how operator traces can be converted into traces

over products of $A^T A$ and C with the insertion of at most one regulator-dependent factor. These traces are not of the form of the operators that occur in the truncation, so one needs to find a projection onto the truncation. This is usually done by evaluating both sides of the FRGE on a family of field configurations that is large enough to distinguish all operators in the truncation, while one chooses them in such a way that the calculation is computationally feasible.

We have provided several families of field configurations that one can use to project onto the truncation, but this is not enough

to evaluate both sides of the flow equation due to the appearance of the rigidity matrix C in the traces. In order to evaluate the traces, one needs an explicit expression of C in the presence of the regulator terms.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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