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On the quantum separability of qubit registers

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I show that the bipartite separability of a pure qubit state hinges critically on the combinatorial structure of its computational-basis support. Boolean cube geometry is used to introduce a taxonomy that distinguishes support-guaranteed separability from cases in which entanglement depends on probability amplitudes. I provide closed-form support counts, identify forbidden configurations that enforce multipartite entanglement, and show how these results can enable fast entanglement diagnostics in quantum circuits. This framework offers immediate utility in classical simulation, entanglement-aware circuit design, and quantum error-correcting code analysis. This establishes support geometry as a practical and scalable tool for understanding entanglement in quantum information processing.

KEYWORDS
Boolean-cube geometry, combinatorial quantum information, quantum entanglement, quantum information processing, quantum separability

1 Introduction

Quantum entanglement is typically introduced through the *physical localization* of the quantum states in terms of quantum systems *being in* or *having* those states. However, this phenomenon, which is indeed fundamentally nonlocal (Bell, 1964; Aspect et al., 1982), is not even required to demonstrate nonlocality (Wang et al., 2025). Fundamental quantum mechanics have been constructed directly outside classical physics and even outside general classical thinking (Mugur-Schachter, 2008), and the mathematical concepts of quantum state entanglement and separability can be examined separately from the system's physical attributes. This bypasses the cybernetic problem (Gershenson, 2025) of defining a system and its boundaries, which is the root of various quantum *paradoxes*, of which Schrödinger's cat is perhaps the most prominent example, as it requires a box in which a cat is *localized*.

This study examines the bipartite separability conditions of pure qubit quantum registers as rays in Hilbert spaces devoid of any *spatial boundaries*. Although a quantum state does not need to be described by qubits, any m-level quantum state (pure or mixed) for $m > 2$ can be represented as a state on $\lceil \log_2(m) \rceil$ qubits via a substitution.

I consider the states separable only across certain bipartitions, as well as those separable only for certain values of their probability amplitudes, and I provide their distributions with respect to their support sizes. The results can be applied in quantum computational applications.

2 Methods

We can consider $\{0,1\}^n$ Boolean space, $n \in \mathbb{N}$, as a complete graph constructed upon n-cube, where a distinct index $j = 1, 2, \ldots, 2^n$ and a distinct address $a(j) = \{0,1\}^n$ can be assigned to each vertex (Łukaszyk, 2025a).

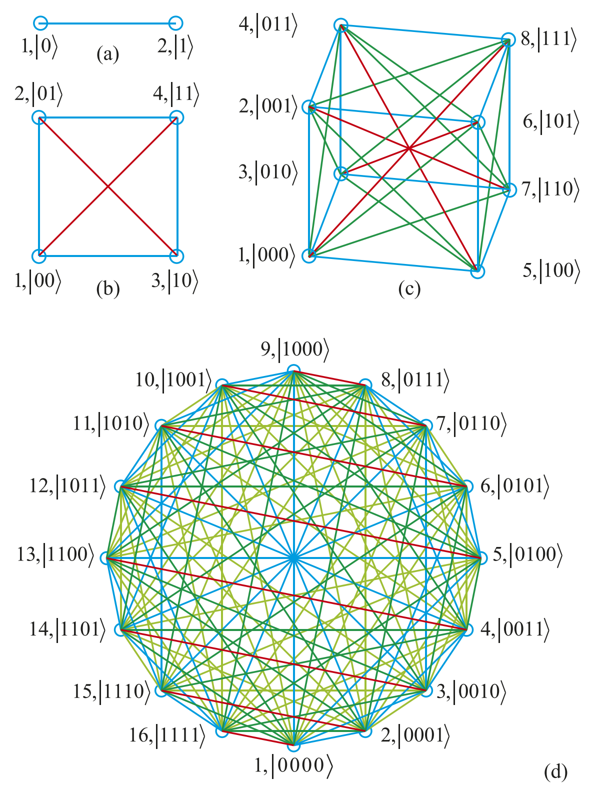

FIGURE 1
Maximum supports for **(a)** one-, **(b)** two-, **(c)** three-, and **(d)** four-qubit registers in the computational basis. Vertices and blue edges represent any-partition-separable states, green edges (partition-dependent-separable states), and red edges (states inseparable).

Consider a pure quantum register containing n qubits in the computational basis

$$|A\rangle = \sum_{j=1}^{2^n} \alpha_j |a(j)\rangle, \quad \sum_{j=1}^{2^n} |\alpha_j|^2 = 1, \tag{1}$$

where $\alpha_j \in \mathbb{C}$ are the probability amplitudes (PA; we sometimes write them as $\alpha_{a(j)}$), $|a(j)\rangle$ are the basis components (kets in Dirac notation), and $a(j)$ are their addresses. While the standard computational basis, with kets corresponding to the j^{th} vertices of n-cube (Duff, 2013), applies naturally to quantum circuits, quantum error correction codes, and so forth, it can also be considered a physical aspect of nature (Łukaszyk, 2025b).

The *support* of the state (Equation 1) $\text{supp}(|A\rangle)$ is the set of basis kets $|a(j)\rangle$ for which $\alpha_j \neq 0$, and the *support size* $(k = |\text{supp}(|A\rangle)|)$ is the cardinality of this set. The maximum support sizes for one to four qubits are shown in Figure 1. There are $2^{2^n} - 1$ distinct supports for n qubits starting from 2^n one-ket states (single vertices of an n-cube) and ending on a state containing all 2^n kets (all vertices of the n-cube).

A pure quantum state $|A\rangle$ is called *separable* if and only if the state can be written as a tensor product

$$|A\rangle = |B\rangle \otimes |C\rangle, \tag{2}$$

of at least two quantum states, which can be represented in the computational basis as similar qubit registers

$$\begin{aligned} |B\rangle &= \beta_1 |0_1 0_2 \ldots 0_{l-1} 0_l\rangle + \beta_2 |0_1 0_2 \ldots 0_{l-1} 1_l\rangle + \ldots \\ &\quad + \beta_{2^l - 1} |1_1 1_2 \ldots 1_{l-1} 0_l\rangle + \beta_{2^l} |1_1 1_2 \ldots 1_{l-1} 1_l\rangle, \\ |C\rangle &= \gamma_1 |0_1 0_2 \ldots 0_{m-1} 0_m\rangle + \gamma_2 |0_1 0_2 \ldots 0_{m-1} 1_m\rangle + \ldots \\ &\quad + \gamma_{2^m - 1} |1_1 1_2 \ldots 1_{m-1} 0_m\rangle + \gamma_{2^m} |1_1 1_2 \ldots 1_{m-1} 1_m\rangle, \end{aligned}$$

consisting respectively of $l > 0$ and $m > 0$ qubits, where $n = l + m$ and $\sum_{j=1}^{2^l} |\beta_j|^2 = \sum_{j=1}^{2^m} |\gamma_j|^2 = 1$.

Otherwise, $|A\rangle$ is an *entangled* state, and the degree of its entanglement can be measured using various methods. One of them employs the entanglement (von Neumann) entropy (here in bits)

$$\mathcal{S}_A = -\sum_j \lambda_j \log_2(\lambda_j), \quad 0 \leq \mathcal{S}_A \leq \min(l, m), \tag{3}$$

where $\lambda_j \in \mathbb{R}_{\geq 0}$, $\sum_j \lambda_j = 1$ are the eigenvalues of the reduced density matrix

$$\rho_B = \text{Tr}_C(\rho_A) \quad \text{or} \quad \rho_C = \text{Tr}_B(\rho_A). \tag{4}$$

obtained by tracing out a specified set of, respectively, m or l qubits from the density matrix ρ_A of the state $|A\rangle$. If the state $|A\rangle$ is separable, $\mathcal{S}_A = 0$. Otherwise, if $(\mathcal{S}_A > 0)$, the state is inseparable, and for $\mathcal{S}_A = \min(l, m)$ it is maximally entangled. The eigenvalues of the density matrix (Equation 4) can also be used to express the state using the Schmidt decomposition as

$$|A\rangle = \sum_j \sqrt{\lambda_j} |b_j\rangle \otimes |c_j\rangle, \tag{5}$$

where $|b_j\rangle$ and $|c_j\rangle$ are orthonormal bases of states $|B\rangle$ and $|C\rangle$. Hence, the state (Equation 5) is separable if $\lambda_j = 1$ for some j and then $|b_j\rangle = |B\rangle$, $|c_j\rangle = |C\rangle$. The entanglement between the states $|B\rangle$ and $|C\rangle$ is invariant under a unitary operation $U_B \otimes U_C$ acting on the individual states $|B\rangle$ and $|C\rangle$; entanglement entropy (Equation 3) remains constant after such an operation.

States that can be written as tensor products of s other pure states $(|A\rangle = |B_1\rangle \otimes |B_2\rangle \otimes \cdots \otimes |B_s\rangle)$ are called *s-separable* (Dür and Cirac, 2000). However, in this study, we focus on the bipartite separability condition (Equation 2) of pure n-qubit quantum registers $|A_{n,k}\rangle$ across at least one bipartition and classify states according to the number and type of bipartitions across which they are separable, considering either equal or arbitrary, normalized PAs.

We explicitly exclude mixed states from the analysis as they lack unique computational-basis support and define entanglement via convexity rather than support geometry, thereby requiring methods designed for mixed states, such as the Peres–Horodecki criterion or entanglement witnesses. Furthermore, mixed states depend on the notion of classical probability.

3 Results

Lemma 1. The number of bipartitions a quantum register containing n qubits can be separable across is $2^c - 1$, where $0 \leq c \leq n - 1$.

Proof. The total number of subsets of a set containing n qubits is 2^n, including the empty set and this set itself. We have to exclude these two inseparable cases and also consider the symmetry of partitioning. Hence, the upper bound on the number of unique

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TABLE 1 The number of APS (blue) and PDS (three bipartitions: light green; one bipartition: dark green; no bipartitions: red) states of a quantum register containing $1 \le n \le 4$ qubits with arbitrary PAs with respect to the support size of the state (see text for details).

	Support size of a quantum state (k)																											
n	1	2	2	2	2	3	3	3	4	4	4	5	5	6	6	7	7	8	8	9	10	11	12	13	14	15	16	Σ
1	2	1																										3
2	4	4			2			4			1																	15
3	8	12		12	4		24	32		6	64		56		28		8		1									255
4	16	32	48	32	8	96	256	208	24	512	1,284	448	3,920	224	7,784	64	11,376	8	12,862	11,440	8,008	4,368	1,820	560	120	16	1	65,535

TABLE 2 The number of CMB_c supports given by Formula 6 and the number of PDS (no bipartitions: red; one bipartition: dark green; three bipartitions: light green) and APS (blue) states of a quantum register containing $1 \le n \le 4$ qubits (see text for details).

	$\sum_{k=2}^{2^n}\lvert CMBc(n,k)\rvert$				Arbitrary PAs			
n	$c = 0$	$c = 1$	$c = 2$	$c = 3$	PDS_0	PDS_1	PDS_2	APS
1	1	0	0	0				3
2	7	4	0	0	7			8
3	193	42	12	0	193	42		20
4	63,775	1,544	168	32	63,775	1,544	168	48

bipartitions for $c = n-1$ is given by $(2^n - 2)/2 = 2^{n-1} - 1$. An entanglement across one bipartition reduces the cardinality of the set to $n-1$ qubits that can be similarly partitioned across $(2^{n-1} - 2)/2 = 2^{n-2} - 1$ bipartitions. Finally, an entanglement across all bipartitions can be thought of as a set containing only one element and is thus inseparable.

For example, the register of four qubits can be separable at most across $2^{4-1} - 1 = 7$ bipartitions as {1|234}, {2|134}, {3|124}, {4|123}, {12|34}, {13|24}, and {14|23}, where "|" denotes the bipartition. However, an entanglement across one bipartition, say 1|4: = V, reduces the cardinality of this set to three {2|3V}, {3|2V}, and {V|23}.

In the following definition, we can give meaning to the free parameter c we introduced in Lemma 1.

Definition 1. We call a support a common-bit (CMB_c) support, where $0 \le c \le n-1$ is the largest integer, such that there exist c coordinates in which all $k \ge 2$ kets in the support have the same bit value.

Geometrically, a CMB_c support is an $(n-c)$-dimensional coordinate face of the Boolean hypercube, which under the Segre embedding (Cirici et al., 2021) corresponds to a coordinate-fixed face, where c coordinates are constant.

States that are separable across all bipartitions for all PAs are called *fully separable* (Dür et al., 1999). We redefine them in the context of bipartitions.

Definition 2. We call the state $|A\rangle$ any-partition-separable (APS) if it is a one-ket state or a state with CMB_{n-1} support.

For example, the following state with CMB_3 support

$$
\begin{aligned}
|A_{4,2}\rangle &= \alpha_1|0_1 0_2 0_3 0_4\rangle + \alpha_2|0_1 0_2 0_3 1_4\rangle = \\
&= |0_1\rangle\otimes\ (\alpha_1|0_2 0_3 0_4\rangle + \alpha_2|0_2 0_3 1_4\rangle) = |B_1\rangle\otimes|C_{234}\rangle = \\
&= |0_2\rangle\otimes\ (\alpha_1|0_1 0_3 0_4\rangle + \alpha_2|0_1 0_3 1_4\rangle) = |B_2\rangle\otimes|C_{134}\rangle = \\
&= |0_3\rangle\otimes\ (\alpha_1|0_1 0_2 0_4\rangle + \alpha_2|0_1 0_2 1_4\rangle) = |B_3\rangle\otimes|C_{124}\rangle = \\
&= (\alpha_1|0_4\rangle + \alpha_2|1_4\rangle)\otimes|0_1 0_2 0_3\rangle = |B_4\rangle\otimes|C_{123}\rangle = \\
&= |0_1 0_2\rangle\otimes\ (\alpha_1|0_3 0_4\rangle + \alpha_2|0_3 1_4\rangle) = |B_{12}\rangle\otimes|C_{34}\rangle = \\
&= |0_1 0_3\rangle\otimes\ (\alpha_1|0_2 0_4\rangle + \alpha_2|0_2 1_4\rangle) = |B_{13}\rangle\otimes|C_{24}\rangle = \\
&= |0_2 0_3\rangle\otimes\ (\alpha_1|0_1 0_4\rangle + \alpha_2|0_1 1_4\rangle) = |B_{23}\rangle\otimes|C_{14}\rangle
\end{aligned}
$$

is an APS state as it is separable across all seven bipartitions for all PAs α_1, α_2. As an APS state admits a superposition of, at most, one of its qubits, only states spanned over the vertices and 1-edges of n cube are APS states. If two or more qubits vary across the support, then some bipartition encounters differing values on both sides, forcing

TABLE 3 Collections $\mathcal{C}$ of subsets $\mathcal{S}$ of $\mathcal{U} \in \{1,2\}$ with full union and empty intersection and two-qubit states inseparable for all PAs.

	$\mathcal{C}$	State	$\varnothing$	{1}	{2}	{1, 2}
1	$\{\varnothing, \{1\}, \{2\}, \{1,2\}\}$	$\lvert A_{4,4}\rangle =$	$\alpha_1\lvert 00\rangle$	$+\alpha_3\lvert 10\rangle$	$+\alpha_2\lvert 01\rangle$	$+\alpha_4\lvert 11\rangle$
2	$\{\varnothing, \{1\}, \{2\}\}$	$\lvert A_{4,3}\rangle =$	$\alpha_1\lvert 00\rangle$	$+\alpha_3\lvert 10\rangle$	$+\alpha_2\lvert 01\rangle$	
3	$\{\varnothing, \{2\}, \{1,2\}\}$	$\lvert A_{4,3}\rangle =$	$\alpha_1\lvert 00\rangle$	$+\alpha_3\lvert 10\rangle$		$+\alpha_4\lvert 11\rangle$
4	$\{\varnothing, \{1\}, \{1,2\}\}$	$\lvert A_{4,3}\rangle =$	$\alpha_1\lvert 00\rangle$		$+\alpha_2\lvert 01\rangle$	$+\alpha_4\lvert 11\rangle$
5	$\{\{1\}, \{2\}, \{1,2\}\}$	$\lvert A_{4,3}\rangle =$		$+\alpha_3\lvert 10\rangle$	$+\alpha_2\lvert 01\rangle$	$+\alpha_4\lvert 11\rangle$
6	$\{\varnothing, \{1,2\}\}$	$\lvert A_{4,2}\rangle =$	$\alpha_1\lvert 00\rangle$			$+\alpha_4\lvert 11\rangle$
7	$\{\{1\}, \{2\}\}$	$\lvert A_{4,2}\rangle =$		$+\alpha_3\lvert 10\rangle$	$+\alpha_2\lvert 01\rangle$	

amplitude-dependent constraints. As n-cube has $\binom{n}{m}2^{n-m}$ m faces, a quantum register has

$$
|APS(n)| = \binom{n}{0}2^{n-0} + \binom{n}{1}2^{n-1} = 2^{n-1}(n+2)
$$

APS states (vertices and blue edges in Figure 1).

The APS states introduce another definition.

Definition 3. We call state $|A\rangle$ a partition-dependent separable (PDS_c) state if it has a CMB_c support, where $0 \le c \le n-2$.

A state with a CMB_c support is separable across $2^c - 1$ bipartitions for all PAs. Therefore, a state with a CMB_0 support is inseparable for all PAs.

Theorem 1. The number of CMB_c supports of an n-qubit quantum register having the support size $k \ge 2$ is given by

$$
|CMBc(n,k)| = \binom{n}{c}2^c \sum_{m=0}^{n-c-1} (-1)^m \binom{n-c}{m} 2^m \binom{2^{n-c-m}}{k}. \tag{6}
$$

Proof. Consider a set containing all 2^n distinct binary strings of length n. $1 \le k \le 2^n$ bitstrings from this set can serve as a support of an n-qubit quantum register. All one-element supports (vertices) are the APS states. The remaining supports can have $0 \le c \le n-1$ common bit(s) in the same position(s). There are $\binom{n}{c}$ ways to choose a subset of c positions from n positions and 2^c ways to assign bits to these positions.

The remaining $n-c-1$ positions must ensure that, in none of them, do all k strings agree. The k strings are identical in the c positions, so choosing them is equivalent to selecting k distinct vectors from Boolean space $\{0,1\}^{n-c}$ such that no coordinate in these vectors is constant for all k vectors. The total number of such k subsets is $\binom{2^{n-c}}{k}$, but we need to subtract cases where at least one coordinate is constant, which can be done using the inclusion–exclusion principle. Hence, we sum over m, the number of coordinates forced to be constant. There are $\binom{n-c}{m}$ ways to choose a subset of m coordinates from $n-c$ coordinates and 2^m ways to assign bits to these coordinates. The effective space size becomes 2^{n-c-m}, so the count for those is $\binom{2^{n-c-m}}{k}$, with sign

<table>
<tr><td colspan="2" rowspan="2">APS</td><td>$ADS_{j,n-2}$</td><td>$ADS_{j,n-3}$</td><td>...</td><td>$ADS_{j,0}$</td></tr>
<tr><td>PDS_{n-2}</td><td>PDS_{n-3}</td><td>...</td><td>PDS_0</td></tr>
<tr><td>k=1</td><td>CMB_{n-1}</td><td>CMB_{n-2}</td><td>CMB_{n-3}</td><td>...</td><td>CMB_0</td></tr>
</table>

FIGURE 2
Taxonomy of the quantum states and supports introduced in this study.

$(-1)^m$. The inclusion–exclusion provides the sum in Formula 6, where $\binom{n}{k} = 0$ for $0 \le n < k$ if the binomial coefficient is defined in terms of a falling factorial. Formula 6 counts the number of ways $1 \le k \le 2^{n-c}$ bitstrings can be selected from the set of all 2^n bitstrings of length n, so that each of these strings has only $0 \le c \le n-1$ bit(s) in common in the same position(s), completing the proof.

For example, for $n = 3$ (cube), $k = 2$ (all possible edges) and $c = 1$ (only the face diagonals), Formula 6 takes the form

$$|\mathrm{CMB}_1(3,2)| = \binom{3}{1} 2^1 \sum_{m=0}^{3-1-1} (-1)^m \binom{3-1}{m} 2^m \binom{2^{3-1-m}}{2} = $$
$$= 6(1\cdot 1\cdot 1\cdot 6 - 1\cdot 2\cdot 2\cdot 1) = 6(6-4) = 12,$$

summing all six edges on each face of a cube (including face diagonals), subtracting four blue 1-edges having two common bits in the same position, and multiplying the result by six faces of the cube to count the states $\alpha_1|000\rangle + \alpha_4|011\rangle$, etc. (green edges in Figure 1c).

The distributions of $|\mathrm{PDS}_c(n,k)|$ states are listed in Table 1 as functions of the support size k and summed in Table 2 along with the values given by Formula 6 for $1 \le n \le 4$ and $0 \le c \le n-1$. They were numerically cross-validated for $n \le 4$ by calculating the eigenvalues λ_j of the reduced density matrices (Equation 4) for each of $2^{2^n} - 1$ quantum states corresponding to distinct supports for each of $2^{n-1} - 1$ possible bipartitions, assuming equal or arbitrary PAs. The Schmidt decomposition (Equation 5) certifies that a given state is separable along a given bipartition if $\lambda_j \approx 1$ for some j. For example, the PDS_1 state

$$|A_{3,4}\rangle = \sqrt{\frac{1}{3}}|100\rangle + \sqrt{\frac{1}{6}}|101\rangle + \sqrt{\frac{1}{6}}|110\rangle + \sqrt{\frac{1}{3}}|111\rangle \qquad (7)$$

has the following eigenvalues of the reduced density matrix $\rho_B = \mathrm{Tr}_C(\rho_{A_{3,4}})$ (Equation 4)

$$\lambda_{\{1|23\}} = \{0, 1\},$$
$$\lambda_{\{2|13\}} = \lambda_{\{3|12\}} = \left\{\frac{3-2\sqrt{2}}{6}, \frac{3+2\sqrt{2}}{6}\right\},$$

and is thus separable only across the partition {1|23}, while for the remaining two partitions, it has a fractional, weak entanglement entropy (Equation 3) $\mathcal{S}_A \approx 0.1873$.

Lemma 2. The number of states that are inseparable for all normalized PAs grows super-exponentially as a function of n, and for $n \ge 2$ corresponds to the number of ways to choose a collection $\mathcal{C}$ of subsets $\mathcal{S}$ of $\mathcal{U} \in \{1, 2, \ldots, n\}$ such that $\cup\{\mathcal{S} \in \mathcal{C}\} = \mathcal{U}$ and $\cap\{\mathcal{S} \in \mathcal{C}\} = \varnothing$ (OEIS sequence A131288).

Proof. The number of such supports is given by summing all $|\mathrm{CMB}_0(n,k)|$ factors given by Formula 6 over $k \ge 2$. Thus

$$\sum_{k=2}^{2^n} |\mathrm{CMBc}(n,k)| = \sum_{k=2}^{2^n} \binom{n}{0} 2^0 \sum_{m=0}^{n-0-1} (-1)^m \binom{n-0}{m} 2^m \binom{2^{n-0-m}}{k} =$$
$$= \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} 2^m \sum_{k=2}^{2^n} \binom{2^{n-m}}{k} =$$
$$= \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} 2^m \left[\sum_{k=0}^{2^n} \binom{2^{n-m}}{k} - \binom{2^{n-m}}{0} - \binom{2^{n-m}}{1}\right]$$
$$= \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} 2^m \left(2^{2^{n-m}} - 1 - 2^{n-m}\right),$$

simplifies to the formula of OEIS sequence A131288 $\{1_1, 7_2, 193_3, 63, 775_4, \ldots\}$ for $n \ge 1$.

For example, for $n = 2$, the set $\mathcal{U} \in \{1, 2\}$ has power set $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$, and there are seven ways to choose such a collection $\mathcal{C}$ of subsets $\mathcal{S}$ of $\mathcal{U}$ with full union and empty intersection corresponding to the states inseparable across all bipartitions for all PA values (Table 3).

Lemma 3. The maximum support size of a CMB_c support is $k_{\max} = 2^{n-c}$.

Proof. Since c out of n bits are the same in all basis kets of the support, the remaining $n - c$ bits must be diversified in all the kets, and the maximum number of such sequences is 2^{n-c}. CMB_c supports correspond to Hamming-weight-constrained subcubes.

Lemma 4. The number of CMB_c supporting having the maximum support size $k_{\max}$ is

$$|\mathrm{CMB}_c(n, k_{\max})| = \binom{n}{c} 2^c.$$

Proof. This follows from substituting $k_{\max}$ into Equation 6. For example, there are $2n$ CMB_1 supports having such a maximum support size, defined by $(n-1)$-dimensional facets of n cube, as they have the largest support size for n that can be partitioned across the same bipartition.

For example, the maximum support size for the PDS_1 state of three qubits is $2^{3-1} = 4$, and there are six states of the form

$$|A_{3,4}\rangle = \alpha_1|0_1 0_2 0_3\rangle + \alpha_2|0_1 0_2 1_3\rangle + \alpha_3|0_1 1_2 0_3\rangle + \alpha_4|0_1 1_2 1_3\rangle =$$
$$= |0_1\rangle \otimes (\alpha_1|0_2 0_3\rangle + \alpha_2|0_2 1_3\rangle + \alpha_3|1_2 0_3\rangle + \alpha_4|1_2 1_3\rangle)$$

in this case separable only across the {1|23} bipartition for all normalized PAs.

Certain states are separable only for specific PAs. Therefore, we introduce the last definition.

Definition 4. We call the state $|A\rangle$ an amplitude-dependent separable ($\mathrm{ADS}_{j,c}$) state, where $0 \le c < j \le n-1$, if its PAs can be arranged in a rank (Equation 8) 1 $k_B \times k_C$ matrix.

The PAs' α matrix is an outer product of one column and one row matrices of PAs of the states $|B\rangle$ and $|C\rangle$ of the tensor product (Equation 2)

max No. of qubits of the state B	the support size of the state B	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
4	16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	336	352	368	384	400	416	432	448	464	480	496	512	528	544	560	576	592	608	624	640	656	672
	15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360	375	390	405	420	435	450	465	480	495	510	525	540	555	570	585	600	615	630
	14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	294	308	322	336	350	364	378	392	406	420	434	448	462	476	490	504	518	532	546	560	574	588
	13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325	338	351	364	377	390	403	416	429	442	455	468	481	494	507	520	533	546
	12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288	300	312	324	336	348	360	372	384	396	408	420	432	444	456	468	480	492	504
	11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275	286	297	308	319	330	341	352	363	374	385	396	407	418	429	440	451	462
	10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250	260	270	280	290	300	310	320	330	340	350	360	370	380	390	400	410	420
	9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225	234	243	252	261	270	279	288	297	306	315	324	333	342	351	360	369	378
3	8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192	200	208	216	224	232	240	248	256	264	272	280	288	296	304	312	320	328	336
	7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175	182	189	196	203	210	217	224	231	238	245	252	259	266	273	280	287	294
	6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150	156	162	168	174	180	186	192	198	204	210	216	222	228	234	240	246	252
	5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175	180	185	190	195	200	205	210
2	4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100	104	108	112	116	120	124	128	132	136	140	144	148	152	156	160	164	168
	3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87	90	93	96	99	102	105	108	111	114	117	120	123	126
1	2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62	64	66	68	70	72	74	76	78	80	82	84
	1	APS	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
	the support size of the state C	1	2	3	4	5	6	7	8								16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
	max No. of qubits of the state C		1		2				3								4																5										

FIGURE 3
Possible support sizes k (products of the support sizes k_B in the left column and k_C in the bottom row) of $ADS_{c,j}$ states for $2 \leq n \leq 7$ qubits, indicated with different colors. Forbidden sizes in boxes (OEIS sequence A390536) are those where no bipartition can factor the support into $k_B \cdot k_C$ satisfying Equation 9 (see text for details).

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k_C} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k_C} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{k_B1} & \alpha_{k_B2} & \cdots & \alpha_{k_Bk_C} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_{k_B} \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{k_C} \end{bmatrix}. \tag{8}$$

States and supports of Definitions 1-4 are listed in Figure 2. Unlike APS and PDS_c states, the separability of the $ADS_{j,c}$ is not a property of the support alone: the support merely permits separability, which is realized only when the PA matrix factors as a rank-1 outer product.

Lemma 5. An n-qubit state with support size k can be an ADS state only if there exists a bipartition of the $n = l + m$ qubits $(l, m \geq 1)$ and k is a composite number satisfying

$$k = k_B \cdot k_C \leq 2^{\lceil \log_2(k_B) \rceil + \lceil \log_2(k_C) \rceil}, \tag{9}$$

where $k_B, k_C \geq 2$ are the support sizes of these qubits.

Proof. The smallest composite number is 4, so two qubits achieve the minimum support size of an ADS state, $(k = 4)$. Larger support sizes must be either even $(2 \times \ldots)$ to provide separability across at least one bipartition or composite numbers satisfying the inequalities $k_B \leq 2^l$ and $k_C \leq 2^m$ for $n = l + m$.

Lemma 6. A state having a support size $2^{n-1} < k \leq 2^n$ that is prime or violates the inequality (Equation 9) is unconditionally entangled; there is no bipartition allowing its separability even for amplitude-dependent tuning.

For example, even though $3 \cdot 5 = 15 < 16 = 2^4$, $\lceil \log_2(3) \rceil + \lceil \log_2(5) \rceil = 2 + 3 = 5 > 4 = \lceil \log_2(15) \rceil$: the support size of two qubits is at most 4, while five kets are required for the second state in the product $15 = 3 \cdot 5$. Such states correspond to *genuinely multipartite entangled* (Palazuelos and Vicente, 2022) or *fully inseparable* (Dür et al., 1999) states, such as Bell, Greenberger–Horne–Zeilinger (GHZ), or W-states.

Lemma 7. An $ADS_{j,c}$ state is separable across $2^j - 2^c$ bipartitions (OEIS sequence A023758).

Proof. We have to exclude PDS_c states separable across $2^c - 1$ bipartitions for all PAs (if any, i.e., for $c > 0$), from the larger set containing also the states separable across $2^j - 1$ bipartitions both for all and for specific PAs to find $(2^j - 1) - (2^c - 1) = 2^j - 2^c$ PAs' specific bipartitions. In other words, $j - c$ indexes the *depth* of amplitude-dependent separability.

By Lemma 7, PDS_c and APS states are mutually exclusive. A support of a state has an inherent *PDS_c* classification, defining its separability for arbitrary PAs. That same state may also become separable across additional partitions (that were previously entangled across) if its PAs are fine-tuned.

It is always possible to adjust the PAs of the α matrix of an $ADS_{j,c}$ state so that it has rank 1. In particular, equal PAs make the α matrix constant, which factors as an outer product of two all-1 vectors β and γ scaled by α, and any such outer-product matrix is rank 1.

Possible support sizes k of ADS states for $2 \leq n \leq 7$ are shown in Figure 3 along with forbidden sizes violating the inequality (Equation 9). Figure 3 also shows support sizes providing separability across different numbers of bipartitions. For example, a four-qubit state with $k = 12$ (gray zone) can be a $l = 1$, $m = 3$, $k = 12 = 2 \cdot 6$ $ADS_{1,0}$ state separable across one bipartition or a $l = m = 2$, $k = 12 = 3 \cdot 4$ $ADS_{2,0}$ state separable across three bipartitions.

For example, the PAs of the state

$$\begin{aligned} |A_{4,10}\rangle = {} & \alpha_{0000}|0000\rangle + \alpha_{0001}|0001\rangle + \alpha_{0010}|0010\rangle + \alpha_{0011}|0011\rangle \\ & + \alpha_{0100}|0100\rangle + \alpha_{0101}|0101\rangle + \alpha_{0110}|0110\rangle + \alpha_{0111}|0111\rangle \\ & + \alpha_{1010}|1010\rangle + \alpha_{1011}|1011\rangle \end{aligned}$$

can be written as an outer product

$$\begin{bmatrix} \alpha_{0000} & \alpha_{0001} \\ \alpha_{0010} & \alpha_{0011} \\ \alpha_{0100} & \alpha_{0101} \\ \alpha_{0110} & \alpha_{0111} \\ \alpha_{1010} & \alpha_{1011} \end{bmatrix} = \begin{bmatrix} \beta_{000} \\ \beta_{001} \\ \beta_{010} \\ \beta_{011} \\ \beta_{101} \end{bmatrix} \begin{bmatrix} \gamma_0 & \gamma_1 \end{bmatrix} \tag{10}$$

as it is an $ADS_{1,0}$ state separable across the bipartition {123|4} as

$$|A_{4,10}\rangle = (\beta_{000}|000\rangle + \beta_{001}|001\rangle + \beta_{010}|010\rangle + \\ +\beta_{011}|011\rangle + \beta_{101}|101\rangle) \otimes (\gamma_0|0\rangle + \gamma_1|1\rangle)$$

if and only if

$$\frac{\alpha_{0000}}{\alpha_{0001}} = \frac{\alpha_{0010}}{\alpha_{0011}} = \frac{\alpha_{0100}}{\alpha_{0101}} = \frac{\alpha_{0110}}{\alpha_{0111}} = \frac{\alpha_{1010}}{\alpha_{1011}} = \frac{\gamma_0}{\gamma_1},$$

that is, when the columns of the α matrix (Equation 10) are linearly dependent or, equivalently, if its rank is 1. Similarly, the PAs of the state

$$|A_{4,9}\rangle = \alpha_{0101}|0101\rangle + \alpha_{0110}|0110\rangle + \alpha_{0111}|0111\rangle + \\ \alpha_{1001}|1001\rangle + \alpha_{1010}|1010\rangle + \alpha_{1011}|1011\rangle + \\ \alpha_{1101}|1101\rangle + \alpha_{1110}|1110\rangle + \alpha_{1111}|1111\rangle,$$

can be written as an outer product

$$\begin{bmatrix} \alpha_{0101} & \alpha_{0110} & \alpha_{0111} \\ \alpha_{1001} & \alpha_{1010} & \alpha_{1011} \\ \alpha_{1101} & \alpha_{1110} & \alpha_{1111} \end{bmatrix} = \begin{bmatrix} \beta_{01} \\ \beta_{10} \\ \beta_{11} \end{bmatrix} \begin{bmatrix} \gamma_{01} & \gamma_{10} & \gamma_{11} \end{bmatrix} \tag{11}$$

and it is also the $ADS_{1,0}$ state separable only across the bipartition {12|34} as

$$|A_{4,9}\rangle = (\beta_{01}|01\rangle + \beta_{10}|10\rangle + \beta_{11}|11\rangle) \\ \otimes (\gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle)$$

again, if all the rows and columns of the α matrix (Equation 11) are linearly independent (its rank is 1). The state (omitting kets here for clarity)

$$|A_{4,4}\rangle = \alpha_{0000} + \alpha_{0001} + \alpha_{0110} + \alpha_{0111} = \\ = (\beta_{000} + \beta_{011}) \otimes (\gamma_0 + \gamma_1) = \\ = \left(\beta_{0_2 0_3} + \beta_{1_2 1_3}\right) \otimes \left(\gamma_{0_1 0_4} + \gamma_{0_1 1_4}\right)$$

is a PDS_1 support separable across the bipartition {1|234}, and it is also a $ADS_{2,1}$ state separable across two bipartitions {123|4} and {23|14} if the α matrix

$$\begin{bmatrix} \alpha_{0000} & \alpha_{0001} \\ \alpha_{0110} & \alpha_{0111} \end{bmatrix} = \begin{bmatrix} \beta_{0_1 0_2 0_3} \\ \beta_{0_1 1_2 1_3} \end{bmatrix} \begin{bmatrix} \gamma_{0_4} & \gamma_{1_4} \end{bmatrix} = \begin{bmatrix} \beta_{0_2 0_3} \\ \beta_{1_2 1_3} \end{bmatrix} \begin{bmatrix} \gamma_{0_1 0_4} & \gamma_{0_1 1_4} \end{bmatrix}$$

has a rank of 1; that is if

$$\alpha_{0000}\alpha_{0111} = \alpha_{0110}\alpha_{0001}.$$

Any CMB_c support having the maximum support size 2^{n-c} has basis kets with bits differing only in the same $u := n - c \geq 2$ positions (i.e., spanning 2^u vertices of u-cube). In this case, we can define $\alpha_{\hat{x}}$ as the PA corresponding to the ket being the bitwise complement of the relevant u positions of the ket associated with a PA α_x, and the separability condition for the PAs α_x and $\alpha_{\hat{x}}$ is

$$\alpha_x \alpha_{\hat{x}} = \text{const} \neq 0, \quad \forall x \in \{0,1\}^u. \tag{12}$$

In other words, the products of the PAs associated with the vertices defining all the 2^{u-1} longest diagonals of the u-cube must be equal. In particular,

$$\begin{array}{lll} \alpha_0 & = \alpha_1 & \text{if } u = 1, \\ \alpha_{00}\alpha_{11} & = \alpha_{01}\alpha_{10} & \text{if } u = 2, \\ \alpha_{000}\alpha_{111} & = \alpha_{001}\alpha_{110} = \alpha_{010}\alpha_{101} = \alpha_{100}\alpha_{011} & \text{if } u = 3, \\ \alpha_{0000}\alpha_{1111} & = \alpha_{0001}\alpha_{1110} = \alpha_{0010}\alpha_{1101} = \alpha_{0011}\alpha_{1100} = & \\ \alpha_{0100}\alpha_{1011} & = \alpha_{0101}\alpha_{1010} = \alpha_{0110}\alpha_{1001} = \alpha_{0111}\alpha_{1000} & \text{if } u = 4. \\ \ldots & & \end{array} \tag{13}$$

For example, the state

$$|A_{4,4}\rangle = \alpha_{1100}|1_1 1_2 0_3 0_4\rangle + \alpha_{1101}|1_1 1_2 0_3 1_4\rangle + \\ \alpha_{1110}|1_1 1_2 1_3 0_4\rangle + \alpha_{1111}|1_1 1_2 1_3 1_4\rangle = \\ = |B_1\rangle \otimes |C_{234}\rangle = |B_2\rangle \otimes |C_{134}\rangle = |B_{12}\rangle \otimes |C_{34}\rangle$$

has two common bits in the same positions 1 and 2, so it is a CMB_2 support separable across $2^2 - 1 = 3$ bipartitions {1|234}, {2|134}, or {12|34} for all PAs. But if

$$\frac{\alpha_{1101}}{\alpha_{1100}} = \frac{\alpha_{1111}}{\alpha_{1101}} = \frac{\gamma_{10}}{\gamma_{11}} = \frac{\gamma_{111}}{\gamma_{110}}.$$

it is also an $ADS_{3,2}$ state separable across four bipartitions {3|124}, {4|123}, {13|24}, and {14|23}, separable as

$$|A\rangle = \\ = (\beta_0|0_3\rangle + \beta_1|1_3\rangle) \otimes (\gamma_{110}|1_1 1_2 0_4\rangle + \gamma_{111}|1_1 1_2 1_4\rangle) = \\ = (\beta_0|0_4\rangle + \beta_1|1_4\rangle) \otimes (\gamma_{110}|1_1 1_2 0_3\rangle + \gamma_{111}|1_1 1_2 1_3\rangle) = \\ = (\beta_{10}|1_1 0_3\rangle + \beta_{11}|1_1 1_3\rangle) \otimes (\gamma_{10}|1_2 0_4\rangle + \gamma_{11}|1_2 1_4\rangle) = \\ = (\beta_{10}|1_1 0_4\rangle + \beta_{11}|1_1 1_4\rangle) \otimes (\gamma_{10}|1_2 0_3\rangle + \gamma_{11}|1_2 1_3\rangle).$$

For the same reasons, the state (Equation 7) is separable across all three bipartitions after swapping PAs α_{110} with α_{111}.

For example, the following state supported on four basis kets with two qubits spanned over four vertices of the 2-cube

$$|A_{4,4}\rangle = \alpha_{0_1 0_2}|0_1 1 0_2 1\rangle + \alpha_{0_1 1_2}|0_1 1 1_2 1\rangle + \\ + \alpha_{1_1 0_2}|1_1 1 0_2 1\rangle + \alpha_{1_1 1_2}|1_1 1 1_2 1\rangle = \\ = \left(\beta_{0_1}|0_1 1\rangle + \beta_{1_1}|1_1 1\rangle\right) \otimes \left(\gamma_{0_2}|0_2 1\rangle + \gamma_{1_2}|1_2 1\rangle\right)$$

has a CMB_2 support separable across three bipartitions {2|134}, {13|24}, and {4|123}. However, if $\alpha_{0_1 0_2}\alpha_{1_1 1_2} = \alpha_{0_1 1_2}\alpha_{1_1 0_2}$, it is also an $ADS_{3,2}$ state separable across four bipartitions {1|234}, {12|34}, {3|124}, and {14|23}.

A three-qubit register (we omit kets here for clarity)

$$|A_{4,8}\rangle = \alpha_{000}\cdots + \alpha_{001}\cdots + \alpha_{010}\cdots + \alpha_{011}\ldots \\ +\alpha_{100}\cdots + \alpha_{101}\cdots + \alpha_{110}\cdots + \alpha_{111}\cdots = \\ (\beta_0 + \beta_1) \otimes (\gamma_{00} + \gamma_{01} + \gamma_{10} + \gamma_{11})$$

can be an $ADS_{1,0}$ state separable across one bipartition ({1|23}) if the α matrix

$$\begin{bmatrix} \alpha_{0_1 0_2 0_3} & \alpha_{0_1 0_2 1_3} & \alpha_{0_1 1_2 0_3} & \alpha_{0_1 1_2 1_3} \\ \alpha_{1_1 0_2 0_3} & \alpha_{1_1 0_2 1_3} & \alpha_{1_1 1_2 0_3} & \alpha_{1_1 1_2 1_3} \end{bmatrix} = \begin{bmatrix} \beta_{0_1} \\ \beta_{1_1} \end{bmatrix} \begin{bmatrix} \gamma_{0_2 0_3} & \gamma_{0_2 1_3} & \gamma_{1_2 0_3} & \gamma_{1_2 1_3} \end{bmatrix}$$

is rank 1, which is equivalent to

$$\begin{array}{lll} \alpha_{000}\alpha_{101} = \alpha_{001}\alpha_{100} & \wedge & \alpha_{000}\alpha_{110} = \alpha_{010}\alpha_{100} \quad \wedge \\ \alpha_{000}\alpha_{111} = \alpha_{011}\alpha_{100} & \wedge & \alpha_{001}\alpha_{110} = \alpha_{010}\alpha_{101} \quad \wedge \\ \alpha_{001}\alpha_{111} = \alpha_{011}\alpha_{101} & \wedge & \alpha_{010}\alpha_{111} = \alpha_{011}\alpha_{110}, \end{array}$$

across one bipartition ({2|13}) if the α matrix

$$\begin{bmatrix} \alpha_{0_1 0_2 0_3} & \alpha_{0_1 0_2 1_3} & \alpha_{1_1 0_2 0_3} & \alpha_{1_1 0_2 1_3} \\ \alpha_{0_1 1_2 0_3} & \alpha_{0_1 1_2 1_3} & \alpha_{1_1 0_2 0_3} & \alpha_{1_1 1_2 1_3} \end{bmatrix} = \begin{bmatrix} \beta_{0_2} \\ \beta_{1_2} \end{bmatrix} \begin{bmatrix} \gamma_{0_1 0_3} & \gamma_{0_1 1_3} & \gamma_{1_1 0_3} & \gamma_{1_1 1_3} \end{bmatrix}$$

is rank 1, and across one bipartition ({3|12}) if the α matrix

$$\begin{bmatrix} \alpha_{0_1 0_2 0_3} & \alpha_{0_1 1_2 0_3} & \alpha_{1_1 0_2 0_3} & \alpha_{1_1 1_2 0_3} \\ \alpha_{0_1 0_2 1_3} & \alpha_{0_1 1_2 1_3} & \alpha_{1_1 0_2 1_3} & \alpha_{1_1 1_2 1_3} \end{bmatrix} = \begin{bmatrix} \beta_{0_3} \\ \beta_{1_3} \end{bmatrix} \begin{bmatrix} \gamma_{0_1 0_2} & \gamma_{0_1 1_2} & \gamma_{1_1 0_2} & \gamma_{1_1 1_2} \end{bmatrix}$$

is rank 1. It can also be the $ADS_{2,0}$ state separable across all $2^2 - 1 = 3$ bipartitions if $\alpha_{000}\alpha_{111} = \alpha_{001}\alpha_{110} = \alpha_{010}\alpha_{101} = \alpha_{011}\alpha_{100}$—that is, if the products for all four main 3-cube diagonals are equal. The four-qubit register

$$
\begin{aligned}
|A\rangle = \alpha_{0000}\cdots + \alpha_{0001}\cdots + \alpha_{0010}\cdots + \alpha_{0011}\ldots + \alpha_{0100}\cdots + \alpha_{0101}\cdots \\
+ \alpha_{0110}\cdots + \alpha_{0111}\ldots + \alpha_{1000}\cdots + \alpha_{1001}\cdots + \alpha_{1010}\cdots \\
+ \alpha_{1011}\ldots + \alpha_{1100}\cdots + \alpha_{1101}\cdots + \alpha_{1110}\cdots + \alpha_{1111}\cdots = \\
= (\beta_0 + \beta_1)\otimes(\gamma_{000} + \gamma_{001} + \gamma_{010} + \gamma_{011} + \gamma_{100} + \gamma_{101} + \gamma_{110} + \gamma_{111}) \\
= (\beta_{00} + \beta_{01} + \beta_{10} + \beta_{11})\otimes(\gamma_{00} + \gamma_{01} + \gamma_{10} + \gamma_{11})
\end{aligned}
$$

can be an $ADS_{1,0}$ state separable across one of four possible bipartitions ({1|234} is shown below) if the α matrix is factorable as

$$
\begin{bmatrix} \alpha_{0000} & \alpha_{0001} & \alpha_{0010} & \alpha_{0011} & \alpha_{0100} & \alpha_{0101} & \alpha_{0110} & \alpha_{0111} \\ \alpha_{1000} & \alpha_{1001} & \alpha_{1010} & \alpha_{1011} & \alpha_{1100} & \alpha_{1101} & \alpha_{1110} & \alpha_{1111} \end{bmatrix} =
\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \begin{bmatrix} \gamma_{000} & \gamma_{001} & \gamma_{010} & \gamma_{011} & \gamma_{100} & \gamma_{101} & \gamma_{110} & \gamma_{111} \end{bmatrix}
$$

an $ADS_{1,0}$ state separable across one of three bipartitions ({12|34} is shown below) if the α matrix is factorable as

$$
\begin{bmatrix} \alpha_{0000} & \alpha_{0001} & \alpha_{0010} & \alpha_{0011} \\ \alpha_{0100} & \alpha_{0101} & \alpha_{0110} & \alpha_{0111} \\ \alpha_{1000} & \alpha_{1001} & \alpha_{1010} & \alpha_{1011} \\ \alpha_{1100} & \alpha_{1101} & \alpha_{1110} & \alpha_{1111} \end{bmatrix} = \begin{bmatrix} \beta_{00} \\ \beta_{01} \\ \beta_{10} \\ \beta_{11} \end{bmatrix} \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{11} \end{bmatrix}
$$

and an $ADS_{3,0}$ state separable across all seven bipartitions if all PAs are equal.

A 2-cube (square) is the smallest n-cube that provides support size for the $ADS_{1,1}$ state. Even though one qubit (1-cube, segment) is always separable, the separability condition (Equation 12) can be extended to the case $(j = 1)$ where it implies the equality of two PAs (Equation 13). This is a specific case of the condition of equal superposition of a qubit: the $ADS_{1,0}$ state

$$|A_{1,2}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

with a vanishing relative phase between the basis kets. On the complex plane, $\alpha_0 = \alpha_1$ represents the same point on the circle of radius $1/\sqrt{2}$, while $|\alpha_0|^2 = |\alpha_1|^2 = 1/2$ represents all points of this circle.

4 Applications

In general, determining across which bipartition (if any) a state is separable is computationally exponential. The computational complexity for finding all eigenvalues of a Hermitian matrix of size M is $O(M^3)$—in our case, $M = 2^{\lfloor n/2 \rfloor} \approx 2^{n/2}$ for the maximum number of qubits of the state $|A\rangle$ or $|B\rangle$ in the tensor product (Equation 2), yielding $O(8^{n/2})$. Calculating the reduced density matrix also requires $O(2^{n+\lfloor n/2 \rfloor}) \approx O(8^{n/2})$ operations. This yields an exponential complexity of $(2^{n-1}-1)O(8^{n/2}) = O(\sqrt{32}^n) = O(2^{2.5n})$ for all $2^{n-1}-1$ bipartitions to check. On the other hand, determining the distribution of bits in the same positions across k basis kets of an n-qubit state required to classify a support of a state as PDS_c has a computational complexity of only $O(nk)$. The speedup $(2^{2.5n}/nk)$ thus grows exponentially with n for all $k \le 2^n$ as a complexity ratio. It is particularly useful, however, in the sparse-support regime $k \ll 2^{n/2}$, where the support geometry is informative, as is typical for many structured, post-selected, or shallow-circuit states.

The support structure alone provides immediate separability bounds, allowing one to bypass expensive density-matrix computations. In particular, $c \neq 0$ is a direct parameter of an *entanglement confinement*, as the state always has the form of

$$|A_{n,k}\rangle = |B_{n-c,k_B}\rangle \otimes |C^{\text{classical}}_{c,k_C}\rangle \tag{14}$$

containing c classical (i.e., non-superposed) qubits up to relabeling.

Furthermore, states having support sizes of Lemma 6 cannot be realized by any separable bipartition even if PAs are equal. As a consequence, a uniform superposition over any 15 computational basis states in four qubits, for example, must be entangled across every possible bipartition—a genuinely multipartite entangled state (Palazuelos and Vicente, 2022). This provides an extremely simple way to construct states with guaranteed full multipartite entanglement by choosing any support of a forbidden size (e.g., with equal PAs).

The support taxonomy can be used as a preprocessing tool:

1. if the state is of APS or PDSc type, its separability is known without Schmidt analysis;
2. if the state support size is forbidden (Lemma 6), its unconditional entanglement is known without Schmidt analysis;
3. if the state is of ADS type, only then is Schmidt analysis required.

Hence, the support taxonomy does not replace Schmidt ranks; rather, it restricts their necessity to ADS states.

Further exemplary applications of the introduced taxonomy are provided below.

4.1 Tracking entanglement spreading in quantum circuits

Consider the standard textbook circuit that creates an n-qubit GHZ state which starts with the $|0_1 0_2 \ldots 0_n\rangle$ APS state, applies the Hadamard gate to the first qubit to obtain the CMB_{n-1} support (still APS state), and then applies a chain of CNOT gates to arrive at the final CMB_0 state, as shown below.

$$
\begin{aligned}
|A_{n,1}\rangle &= |0_1 0_2 \ldots 0_n\rangle \overset{\text{H}}{\rightarrow} \\
|A_{n,2}\rangle &= \frac{1}{\sqrt{2}}(|0_1\rangle + |1_1\rangle)\otimes|0_2 0_3 \ldots 0_n\rangle \overset{\text{CNOT}}{\rightarrow} \\
|A_{n,2}\rangle &= \frac{1}{\sqrt{2}}(|0_1 0_2\rangle + |1_1 1_2\rangle)\otimes|0_3 1_4 \ldots 0_n\rangle \overset{\text{CNOT}}{\rightarrow} \\
|A_{n,2}\rangle &= \frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle + |1_1 1_2 1_3\rangle)\otimes|0_4 0_5 \ldots 0_n\rangle \overset{\text{CNOT}}{\rightarrow} \\
&\ldots \\
|A_{n,2}\rangle &= \frac{1}{\sqrt{2}}(|0_1 0_2 \ldots 0_n\rangle + |1_1 1_2 \ldots 1_n\rangle).
\end{aligned} \tag{15}
$$

Each operation reduces the number of common bits c by exactly one and delocalizes the entanglement to one additional qubit. The taxonomy proposed in this study, therefore, gives an exact, single-integer metric that tracks how entanglement spreads through the CNOT chain—something that would otherwise require expensive Schmidt decompositions across all $2^{n-1}-1$ bipartitions. This is particularly useful for analyzing fault-tolerance thresholds in GHZ preparation or distribution: if an error occurs when c is still large, the entanglement is still localized to a small block, so the error may be easier to correct locally. This is illustrated in Equation 15, where the fourth qubit which erroneously flipped after the first CNOT gate is corrected before the second one.

4.2 Classical simulation speedup for localized-entanglement states

Because high-c states are in the form (Formula 14), any quantum circuit that produces a state with large c can be classically simulated by discarding the c *classical* qubits and only simulating the remaining $n - c$ qubits. Even for arbitrary PAs, the fixed qubits contribute only a global factor and never entangle with anything. Consider, for example, a circuit that produces a ten-qubit state with $c = 7$. The state is then a fixed classical 7-bit string on seven qubits tensored with an entangled three-qubit state, and the unitary evolution can be simulated on just three qubits instead of ten, providing exponential savings. This feature applies directly to:

- Sparse-state or low-weight circuits;
- Post-selected computations;
- Certain variational ansätze that accidently stay in high-c supports;
- Debugging shallow circuits where support has not fully spread, and so forth.

The $O(nk)$ check for c is vastly cheaper than computing entanglement entropies across all bipartitions, making it a practical pre-filter for tensor-network or exact simulators: "if $c \geq$ some threshold, reduce to $n - c$ qubits".

In summary, the taxonomy turns the hypercube support into a powerful diagnostic tool that is both theoretically clean and practically cheap, especially for tracking entanglement dynamics in circuits, thus enabling aggressive classical simulation when entanglement stays localized and constructing fully guaranteed inseparable states. The taxonomy enables both the detection of entanglement properties in existing states and the design of states with desired entanglement characteristics.

5 Conclusion

Classifying quantum states using n-cube separability structures provides a fast ($O(nk)$ vs. $O(2^{2.5n})$) structural way to assess entanglement without full-state tomography or diagonalization. Other potential applications in quantum computing include quantum machine learning and data encoding, quantum circuit optimization, quantum communication and network protocols, and quantum-to-classical boundary studies.

Data availability statement

The datasets presented in this study can be found in the online repository https://github.com/szluk/n-qubes (accessed on 19 September 2025).

Author contributions

SŁ: Formal Analysis, Data curation, Writing – review and editing, Project administration, Validation, Methodology, Writing – original draft, Software, Investigation, Visualization, Supervision, Resources, Funding acquisition, Conceptualization.

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Conflict of interest

Author SŁ is the owner of Łukaszyk Patent Attorneys.

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Supplementary material

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/frqst.2026.1754112/full#supplementary-material

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