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Spacelike surface families interpolating common asymptotic curves in Minkowski 3-space

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In this paper, we propose a method for constructing families of spacelike surfaces in Minkowski 3-space $\mathbb{E}_1^3$ that share Bertrand curves as asymptotic curves. By using marching-scale functions, we derive the necessary conditions, provide flexible formulations, and establish a framework for constructing mutual spacelike Bertrand curves. Examples show how different functions generate surfaces interpolating the common asymptotic curves, offering new insights for geometric modeling and ruled surface theory.

KEYWORDS

Bertrand couple, tangent planes, iso-asymptotic curve, asymptotic, spacelike

1 Introduction

Among the various curves found on surfaces, asymptotic curves hold a special place due to their distinct geometric behavior. An asymptotic curve is one where the tangent vector continuously aligns with a direction that renders the normal curvature zero. In practical terms, this means the surface remains momentarily aligned with its tangent plane along the curve, a condition further reinforced by the persistent alignment of the curve's binormal vector with the surface normal. Consequently, the Gaussian curvature along such a curve is non-positive—that is, it is either zero or negative [1, 2]. Recent investigations have emphasized the importance of these curves, particularly in applications such as astronomy. For example, Hartman and Wintner [3] stressed that regularity conditions are critical for understanding asymptotic curves on surfaces with negative Gaussian curvature, whereas Kitagawa [4] showed that a flat torus embedded in a unit 3-sphere features cyclic asymptotic curves. Further studies by Garcia and Sotomayor [5] have shed light on the intrinsic properties of asymptotic curves in Euclidean spaces, and Garcia et al. [6] demonstrated that networks of these curves can maintain topological stability under small perturbations. Moreover, asymptotic curves play a significant role in diverse fields such as architecture and computer-aided design. In astrophysics, for instance, the analysis of Lyapunov orbits—which is crucial for understanding stellar escape trajectories—relies heavily on the geometry of asymptotic curves. Contopoulos [7] examined how unstable orbits can escape along these curves, whereas Efthymiopoulos et al. [8] observed that chaotic trajectories within fractal-like sets often mirror the behavior of unstable asymptotic curves. In the realm of free-form architecture, Flöry and Pottmann [9] developed a geometric modeling framework that uses strips of ruled surfaces, constructed by aligning rulings with asymptotic curves, and refined these initial designs to match specific target shapes.

In a related development, the concept of a surface family featuring a unique characteristic curve was first introduced by Wang et al. [10] in Euclidean 3-space ($\mathbb{E}^3$), where they constructed a family of surfaces sharing a common geodesic. This pioneering work has inspired numerous subsequent studies on surface families with shared key curves in both Euclidean and non-Euclidean contexts (e.g., [11–32]). In curve theory, establishing a robust correspondence between curves remains a fundamental challenge. One well-known example is the Bertrand pair, which is a classical case where two curves are in bijective correspondence and share the same principal normals [1, 2]. These Bertrand curves also serve as models for offset curves, which are integral to computer-aided design ($\mathcal{CAD}$) and manufacturing ($\mathcal{CAM}$) processes (see [33–35]). Despite these advances, no prior research has focused on constructing spacelike surfaces in $\mathbb{E}_1^3$ that incorporate Bertrand curves as asymptotic curves. In this study, we seek to bridge that gap by exploring how Bertrand curves can be utilized as asymptotic curves to generate families of spacelike surfaces in $\mathbb{E}_1^3$.

In this study, we introduce a method for constructing families of spacelike surfaces in Minkowski 3-space that share a common Bertrand curve as an asymptotic curve. By utilizing spacelike Bertrand curves and aligning the tangent planes of the surfaces with the osculating planes of the curves, parametric equations of the surfaces are developed using marching-scale functions. Necessary and sufficient conditions are derived to guarantee the asymptotic nature of the common curve on each surface. Several flexible formulations are proposed, and examples illustrate the generation of spacelike surfaces through different choices of marching-scale functions. The significance of this work lies in providing a systematic and versatile framework for generating geometrically meaningful surfaces, enhancing the theoretical understanding of Lorentzian geometry, and offering practical tools for applications in computer-aided geometric design, relativity, and broader differential geometry contexts.

2 Preliminaries

In this section, we provide a concise overview of the fundamentals of curves and surfaces in Minkowski 3-space $\mathbb{E}_1^3$ [14, 15]. Consider vectors $\mathbf{w} = (w_1, w_2, w_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ in $\mathbb{E}_1^3$. Their Lorentzian inner product is defined by

$$< \mathbf{w}, \mathbf{z} >= w_1 z_1 + w_2 z_2 - w_3 z_3.$$

We also define the cross product of $\mathbf{w}$ and $\mathbf{z}$ as

$$\mathbf{w} \times \mathbf{z} = (w_2 z_3 - w_3 z_2), (w_3 z_1 - w_1 z_3), -(w_1 z_2 - w_2 z_1).$$

As the Lorentzian inner product is an indefinite metric, any vector $\mathbf{w} \in \mathbb{E}_1^3$ can be classified by its causal character. Specifically, $\mathbf{w}$ is called spacelike ($\mathcal{SL}$) if $\langle \mathbf{w}, \mathbf{w} \rangle > 0$ or $\mathbf{w} = \mathbf{0}$, timelike ($\mathcal{TL}$) if $\langle \mathbf{w}, \mathbf{w} \rangle < 0$, and lightlike or null if $\langle \mathbf{w}, \mathbf{w} \rangle = 0$ while $\mathbf{w} \neq \mathbf{0}$. The norm of $\mathbf{w}$ is defined as $\|\mathbf{w}\| = \sqrt{|\langle \mathbf{w}, \mathbf{w} \rangle|}$. Accordingly, the hyperbolic unit sphere and the Lorentzian (de Sitter) unit sphere are given by

$$\mathcal{H}_+^2 = \left\{ \mathbf{w} \in \mathbb{E}_1^3 \mid \|\mathbf{w}\|^2 := w_1^2 + w_2^2 - w_3^2 = -1 \right\},$$

and

$$\mathcal{S}_1^2 = \left\{ \mathbf{w} \in \mathcal{E}_1^3 \mid \|\mathbf{w}\|^2 := w_1^2 + w_2^2 - w_3^2 = 1 \right\}.$$

Consider a unit-speed $\mathcal{SL}$-curve $\boldsymbol{\varphi}(w)$ in $\mathbb{E}_1^3$ that possesses a $\mathcal{TL}$-binormal vector. Its curvature and torsion are denoted by $\kappa(w)$ and $\tau(w)$, respectively. Let $\{\boldsymbol{\lambda}_1(w), \boldsymbol{\lambda}_2(w), \boldsymbol{\lambda}_3(w)\}$ be the corresponding Serret–Frenet frame along $\boldsymbol{\varphi}(w)$, where $\boldsymbol{\lambda}_1(w)$, $\boldsymbol{\lambda}_2(w)$, and $\boldsymbol{\lambda}_3(w)$ represent the unit tangent, principal normal, and binormal vectors, respectively. The frame vectors satisfy the following normalization conditions under the Lorentzian inner product:

$$< \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1 >=< \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_2 >= - < \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_3 >= 1,$$
$$\boldsymbol{\lambda}_1 \times \boldsymbol{\lambda}_2 = -\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_1 \times \boldsymbol{\lambda}_3 = -\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_2 \times \boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_1.$$

The derivatives of the frame vectors with respect to the arc-length parameter w are then expressed as

$$\begin{pmatrix} \boldsymbol{\lambda}_1' \\ \boldsymbol{\lambda}_2' \\ \boldsymbol{\lambda}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(w) & 0 \\ -\kappa(w) & 0 & \tau(w) \\ 0 & \tau(w) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \boldsymbol{\lambda}_3 \end{pmatrix}, \quad (2.1)$$

with the prime indicating differentiation with respect to w. The subspaces spanned by $Sp\{\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3\}$, $Sp\{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2\}$, and $Sp\{\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_1\}$ are referred to as the normal plane, osculating plane, and rectifying plane, respectively.

Definition 2.1: ([1, 2]) Let $\boldsymbol{\varphi}(w)$ and $\widehat{\boldsymbol{\varphi}}(w)$ be two curves in $\mathbb{E}_1^3$, with the respective principal normal vectors $\boldsymbol{\lambda}_2(w)$ and $\widehat{\boldsymbol{\lambda}}_2(w)$. The curves form a Bertrand pair if, at corresponding points, their principal normals are linearly dependent. Equivalently, there exists a constant f such that

$$\widehat{\boldsymbol{\varphi}}(w) = \boldsymbol{\varphi}(w) + f \boldsymbol{\lambda}_2(w). \quad (2.2)$$

Here, f is a fixed scalar.

We denote a surface $\mathfrak{R}$ by its parametrization

$$\mathfrak{R}: \boldsymbol{v}(w,t) = (v_1(w,t), v_2(w,t), v_3(w,t)), \quad (w,t) \in \mathbb{D} \subseteq \mathbb{R}^2.$$

Here, if we let $v_j(w,t) = \frac{\partial v}{\partial j}$, for $j = w$ and t, the surface's normal vector is defined as

$$\mathbf{n}(w,t) = \boldsymbol{v}_w \wedge \boldsymbol{v}_t,$$

with the property that $\langle \mathbf{n}, \boldsymbol{v}_w \rangle = \langle \mathbf{n}, \boldsymbol{v}_t \rangle = 0$.

Definition 2.2: ([1, 2]) A curve on a surface is called asymptotic if, at every point on the curve, the surface normal is aligned with the curve's binormal vector.

A curve $\boldsymbol{\varphi}(w)$ on a surface $\boldsymbol{v}(w,t)$ is called an iso-parametric curve if one of its parameters remains constant. In other words, there exists a fixed value t_0 such that $\boldsymbol{\varphi}(w) = \boldsymbol{v}(w, t_0)$, or similarly, a constant w_0 with $\boldsymbol{\varphi}(t) = \boldsymbol{v}(w_0, t)$. When a given curve $\boldsymbol{\varphi}(w)$ serves both as an asymptotic curve and as an iso-parametric curve on $\boldsymbol{v}(w,t)$, we refer to it as an iso-asymptotic of the surface.

Definition 2.3: A surface in Minkowski 3-space $\mathbb{E}_1^3$ is termed a $\mathcal{TL}$ surface if its induced metric is Lorentzian, and it is called an $\mathcal{SL}$ surface if the induced metric is a positive-definite Riemannian metric. Equivalently, a $\mathcal{SL}$ surface has a timelike normal vector, whereas a $\mathcal{TL}$ surface possesses a $\mathcal{SL}$-normal vector.

3 Main results

In this section, we present a novel method for constructing families of $\mathcal{SL}$ surfaces in $\mathbb{E}_1^3$ that share Bertrand curves as common asymptotic curves. Our approach leverages $\mathcal{SL}$-Bertrand curves in such a way that the tangent planes of the resulting $\mathcal{SL}$ surfaces align with the osculating planes of these $\mathcal{SL}$-Bertrand curves. To achieve this, we assume that the curves $\boldsymbol{\varphi}(w)$ and $\widehat{\boldsymbol{\varphi}}(w)$ are $\mathcal{SL}$-Bertrand curves with $\mathcal{TL}$-binormal vectors, as described in Equations 2.1, 2.2. Based on this assumption, one can define a surface family by

$$\mathfrak{R}:\boldsymbol{v}(w,t) = \boldsymbol{\varphi}(w) + \mathfrak{x}(w,t)\boldsymbol{\lambda}_1(w) + \mathfrak{y}(w,t)\boldsymbol{\lambda}_2(w);\ \ 0 \le t \le T, 0 \le v \le L, \tag{3.1}$$

which constitutes an $\mathcal{SL}$-surface family sharing the curve $\boldsymbol{\varphi}(w)$ as mutual curve in common. Similarly, the surface

$$\widehat{\mathfrak{R}}:\widehat{\boldsymbol{v}}(w,t) = \widehat{\boldsymbol{\varphi}}(w) + \mathfrak{x}(w,t)\widehat{\boldsymbol{\lambda}}_1(w) + \mathfrak{y}(w,t)\widehat{\boldsymbol{\lambda}}_2(w);\ \ 0 \le t \le T, 0 \le w \le L \tag{3.2}$$

also forms an $\mathcal{SL}$-surface family with $\widehat{\boldsymbol{\varphi}}(w)$ as a common curve. In these constructions, the functions $\mathfrak{x}(w,t)$ and $\mathfrak{y}(w,t)$ (which belong to C^1) are termed marching-scale functions, with the condition $\mathfrak{y}(w,t_0) \neq 0$ for some $t_0 \in [0,T]$. To ensure that $\widehat{\boldsymbol{\varphi}}(w)$ is an asymptotic curve on $\widehat{\mathfrak{R}}$, we must determine the appropriate conditions for the marching-scale functions. Computing the partial derivatives of $\widehat{\boldsymbol{v}}(w,t)$ gives

$$\left.\begin{array}{l}\widehat{\boldsymbol{v}}_w(w,t) = \left(1+\mathfrak{x}_w-\widehat{\kappa}\mathfrak{y}\right)\widehat{\boldsymbol{\lambda}}_1 + \left(\mathfrak{x}\widehat{\kappa}+\mathfrak{y}_w\right)\widehat{\boldsymbol{\lambda}}_2 + \mathfrak{y}\widehat{\tau}\widehat{\boldsymbol{\lambda}}_3,\\ \widehat{\boldsymbol{v}}_t(w,t) = \mathfrak{x}_t\widehat{\boldsymbol{\lambda}}_1 + \mathfrak{y}_t\widehat{\boldsymbol{\lambda}}_2,\end{array}\right\}, \tag{3.3}$$

where subscripts denote partial differentiation with respect to w and t, respectively. The surface normal is then defined by

$$\widehat{\mathbf{n}}(w,t) = -\mathfrak{y}\widehat{\tau}\mathfrak{y}_t\boldsymbol{\lambda}_1 + \mathfrak{x}_t\widehat{\tau}\mathfrak{y}\widehat{\boldsymbol{\lambda}}_2 + \left[\left(\mathfrak{x}\mathfrak{x}_t-\mathfrak{y}\mathfrak{y}_t\right)\widehat{\kappa} - \left(1+\mathfrak{x}_w-\mathfrak{y}\widehat{\kappa}\right)\mathfrak{y}_t\right]\widehat{\boldsymbol{\lambda}}_3. \tag{3.4}$$

As $\widehat{\boldsymbol{\varphi}}(w)$ is an iso-parametric curve on $\widehat{\mathfrak{R}}$, there exists a parameter value $t = t_0 \in [0,T]$ such that $\widehat{\boldsymbol{v}}(w,t_0) = \widehat{\boldsymbol{\varphi}}(w)$. This implies that

$$\mathfrak{x}(w,t_0) = \mathfrak{y}(w,t_0) = 0, \mathfrak{x}_w(w,t_0) = \mathfrak{y}_w(w,t_0) = 0. \tag{3.5}$$

Evaluating the normal vector at $t = t_0$ then simplifies to

$$\widehat{\mathbf{n}}(w,t_0) = -\mathfrak{y}_t(w,t_0)\widehat{\boldsymbol{\lambda}}_3(w). \tag{3.6}$$

This result shows that the surface normal along $\widehat{\boldsymbol{\varphi}}(w)$ is parallel to the binormal vector, ensuring that $\widehat{\boldsymbol{\varphi}}(w)$ is indeed an $\mathcal{SL}$-asymptotic curve on $\widehat{\mathfrak{R}}$. From the relations in Equations 3.1–3.6, we consequently derive the following theorem.

Theorem 3.1: A curve $\widehat{\boldsymbol{\varphi}}(w)$ is an iso-asymptotic (i.e., an asymptotic) curve on the $\mathcal{SL}$-surface family $\widehat{\mathfrak{R}}$ if and only if

$$\left.\begin{array}{l}\mathfrak{x}(w,t_0) = \mathfrak{y}(w,t_0) = 0,\\ \mathfrak{y}_t(w,t_0) \neq 0,\ 0 \le t_0 \le T, \quad 0 \le w \le L.\end{array}\right\} \tag{3.7}$$

for some $t_0 \in [0,T]$ and for all $w \in [0,L]$.

For simplification and practical evaluation, we assume that the marching-scale functions $\mathfrak{x}(w,t)$ and $\mathfrak{y}(w,t)$ can be separated as follows:

$$\mathfrak{x}(w,t) = l(w)\mathfrak{X}(t),$$
$$\mathfrak{y}(w,t) = m(w)\mathfrak{Y}(t),$$

where $l(w)$, $m(w)$, $\mathfrak{X}(t)$, and $\mathfrak{Y}(t)$ are C^1 functions that are not identically zero. From Theorem 3.1, we then obtain the following:

Corollary 3.1: The curve $\widehat{\boldsymbol{\varphi}}(w)$ is an asymptotic curve on the $\mathcal{SL}$-surface family $\widehat{\mathfrak{R}}$ if and only if

$$\left.\begin{array}{l}\mathfrak{X}(t_0) = \mathfrak{Y}(t_0) = 0, \quad l(w) = const. \neq 0, m(w) = const. \neq 0,\\ \dfrac{d\mathfrak{Y}(t_0)}{dt} = const. \neq 0, 0 \le t_0 \le T, \quad 0 \le w \le L,\end{array}\right\} \tag{3.8}$$

with $t_0 \in [0,T]$ and $w \in [0,L]$.

To construct surfaces $\mathfrak{R}$ and $\widehat{\mathfrak{R}}$ that interpolate $\boldsymbol{\varphi}(w)$ and $\widehat{\boldsymbol{\varphi}}(w)$ as common asymptotic $\mathcal{SL}$-Bertrand curves, we first define the marching-scale functions as in Equation 3.7 and then substitute these into Equations 3.1, 3.2 to obtain the parametrization. Moreover, for practical flexibility, the functions $\mathfrak{x}(w,t)$ and $\mathfrak{y}(w,t)$ can be expressed in alternative forms while still allowing sufficient freedom to ensure that the surfaces $\mathfrak{R}$ and $\widehat{\mathfrak{R}}$ interpolate $\boldsymbol{\varphi}(w)$ and $\widehat{\boldsymbol{\varphi}}(w)$ as mutual asymptotic curves. Therefore, we assume that $\mathfrak{x}(w,t)$ and $\mathfrak{y}(w,t)$ may also be given in two additional formats:

1. We assume that the marching-scale functions are defined by

$$\begin{cases}\mathfrak{x}(w,t) = \sum\limits_{k=1}^{p} a_{1k} l(w)^k \mathfrak{X}(t)^k,\\ \mathfrak{y}(w,t) = \sum\limits_{k=1}^{p} b_{1k} m(w)^k \mathfrak{Y}(t)^k,\end{cases}$$

where $l(w)$, $m(w)$, $\mathfrak{X}(t)$, and $\mathfrak{Y}(t)$ belong to C^1; the coefficients a_{1k} and b_{1k} are real numbers for $k = 1,2,\ldots,p$; and neither $l(w)$ nor $m(w)$ is identically zero. In this setting, the sufficient conditions to ensure that the pair of curves $\{\boldsymbol{\varphi}(w), \widehat{\boldsymbol{\varphi}}(w)\}$ are common asymptotic curves on the surfaces $\{\mathfrak{R}, \widehat{\mathfrak{R}}\}$ are given by

$$\begin{cases}\mathfrak{X}(t_0) = \mathfrak{Y}(t_0) = 0,\\ b_{11} \neq 0, m(w) \neq 0, \text{and } \dfrac{d\mathfrak{Y}(t_0)}{dt} = const. \neq 0.\end{cases} \tag{3.9}$$

2. Alternatively, we assume the functions take a compositional form:

$$\begin{cases}\mathfrak{x}(w,t) = f\left(\sum\limits_{k=1}^{p} a_{1k} l^k(w)\mathfrak{X}^k(t)\right),\\ \mathfrak{y}(w,t) = g\left(\sum\limits_{k=1}^{p} b_{1k} m^k(w)\mathfrak{Y}^k(t)\right),\end{cases}$$

then the corresponding conditions become

$$\begin{cases}\mathfrak{X}(t_0) = \mathfrak{Y}(t_0) = f(0) = g(0) = 0,\\ b_{11} \neq 0, \dfrac{d\mathfrak{Y}(t_0)}{dt} = const \neq 0,\ m(w) \neq 0, g'(0) \neq 0,\end{cases} \tag{3.10}$$

where $l(w), m(w)$, and $\mathfrak{X}(t), \mathfrak{Y}(t) \in C^1$, with $a_{ij}, b_{ij} \in \mathbb{R}$ for $i = 1,2$, and $j = 1,2,\ldots,p$, $l(w)$, and $m(w)$ are not identically zero. As there are no

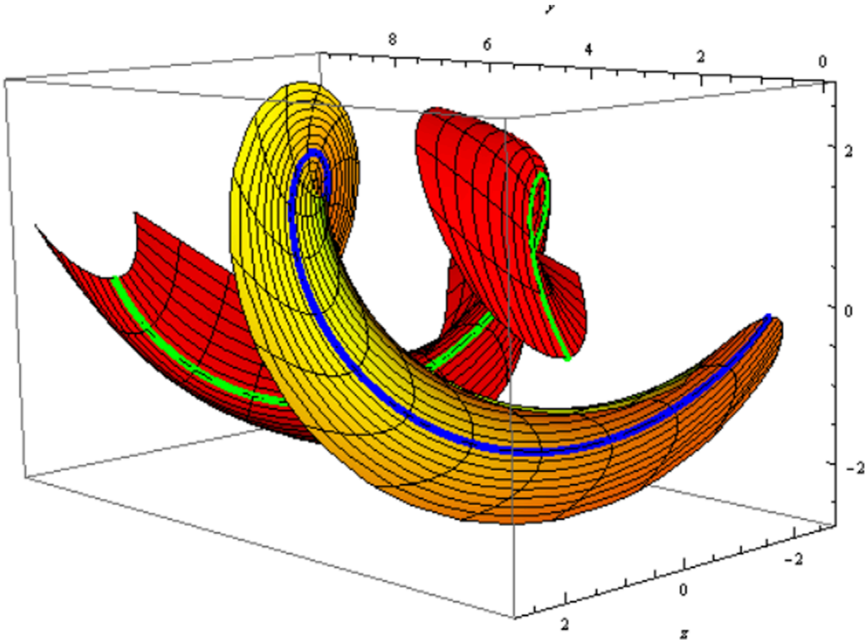

FIGURE 1
$\mathfrak{R} \cup \widehat{\mathfrak{R}}$ with $\mathfrak{x}(w,t) = 1 - \cos t$ and $\mathfrak{y}(w,t) = \sin t$.

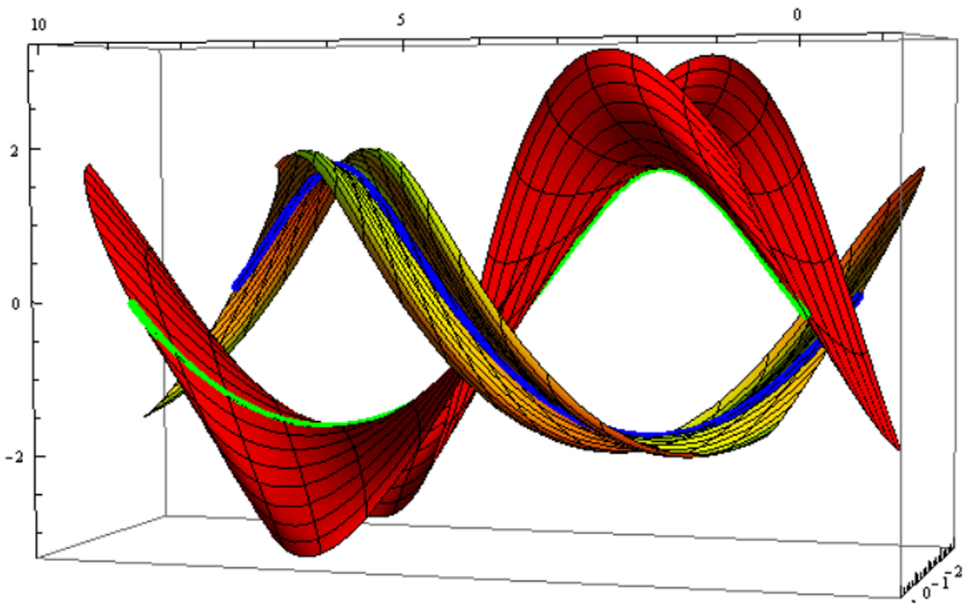

FIGURE 2
$\mathfrak{R} \cup \widehat{\mathfrak{R}}$ with $\mathfrak{x}(w,t) = \sin t + \Sigma_{k=2}^{4} a_{1k} \sin^k t$ and $\mathfrak{y}(w,t) = (1 - \cos t) + \Sigma_{k=2}^{4} b_{1k} (1 - \cos t)^k$.

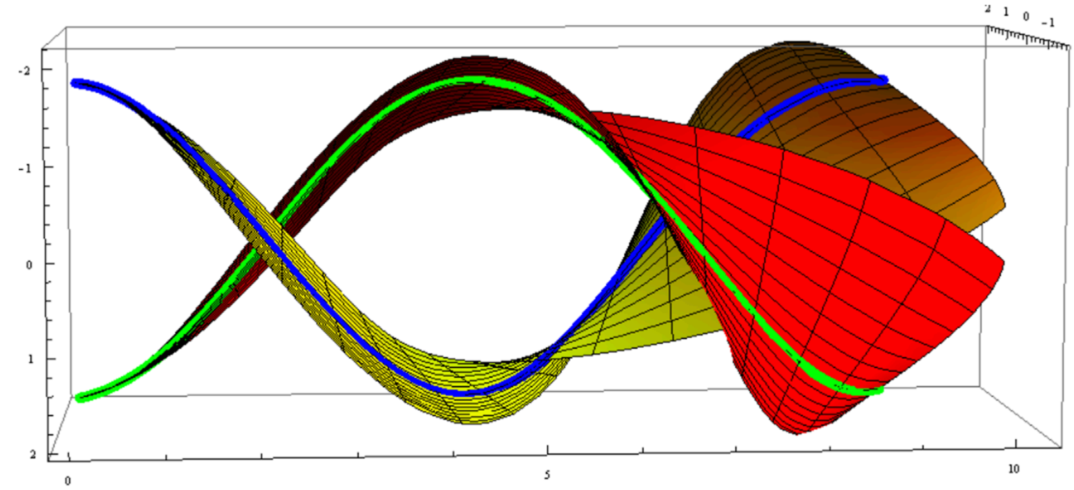

FIGURE 3
$\mathfrak{R}\cup\widehat{\mathfrak{R}}$ with $\mathfrak{x}(w,t)=\sin(\Sigma_{k=1}^{4}w^{k}t^{k})$ and $\mathfrak{y}(w,t)=\Sigma_{k=1}^{4}w^{k}t^{k}$.

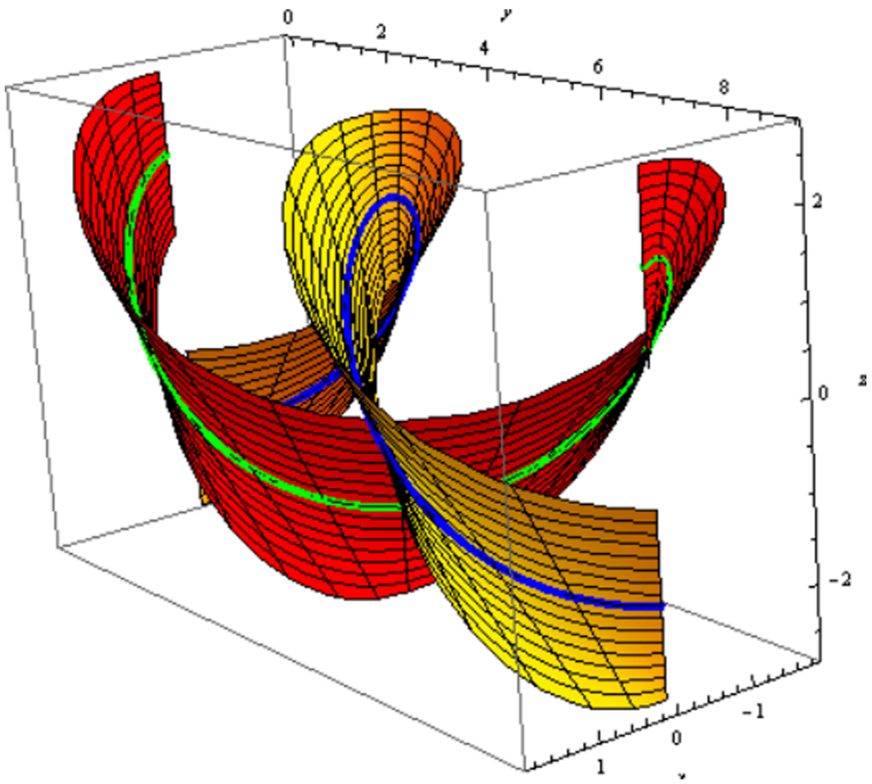

FIGURE 4
$\mathfrak{R}\cup\widehat{\mathfrak{R}}$with $\gamma(w)=\sin w$and $\beta(w)=\cos w$.

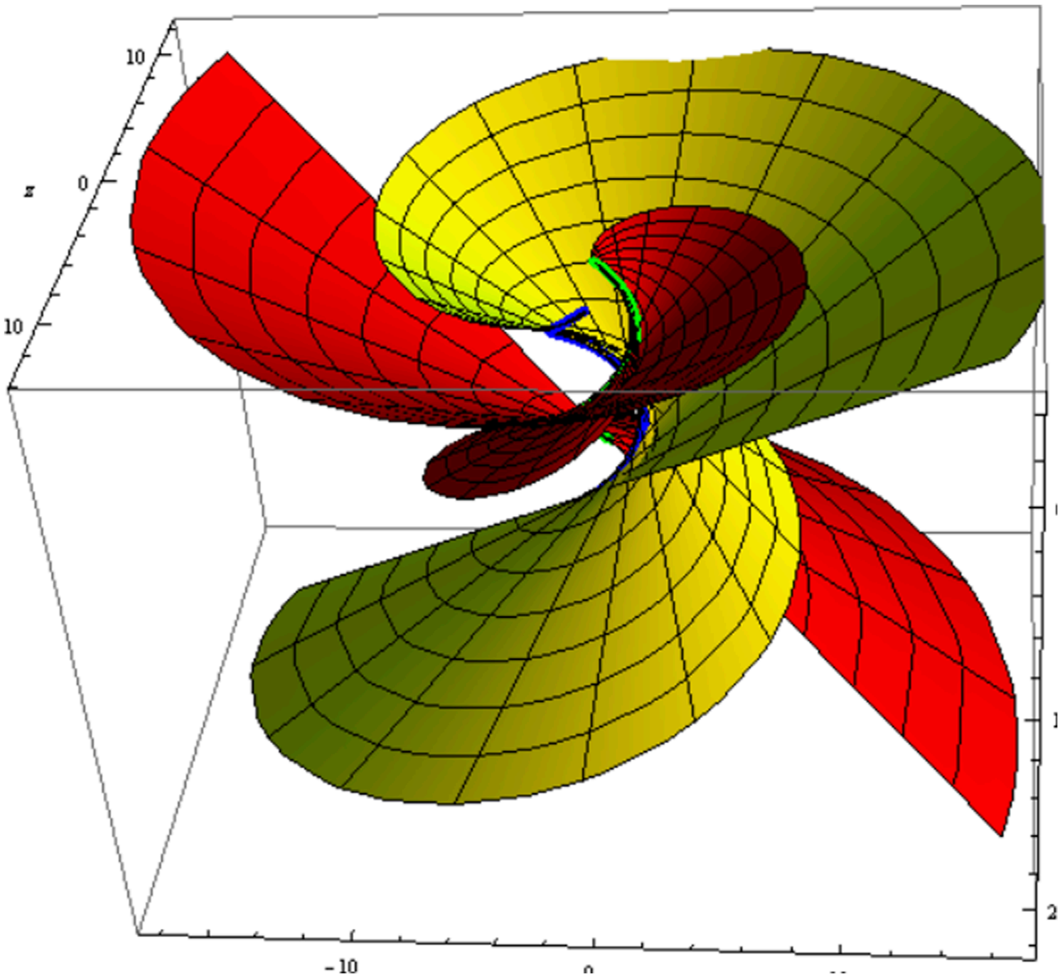

FIGURE 5
$\mathfrak{R}\cup\widehat{\mathfrak{R}}$ with $\gamma(w)=\beta(w)=w$.

additional restrictions imposed by the given curve in Equations 3.8, 3.9 or Equation 3.10, one can always construct the pair of surfaces $\{\widehat{\mathfrak{R}},\ \mathfrak{R}\}$ with $\{\boldsymbol{\varphi}(w),\ \widehat{\boldsymbol{\varphi}}(w)\}$ as their mutual asymptotic curves by suitably choosing these flexible marching-scale functions.

Example 3.1: We consider the curve

$$\boldsymbol{\varphi}(w)=\left(\sqrt{3}\sin w,\ \sqrt{2}w,\ \sqrt{3}\cos w\right),\ 0\le w\le 2\pi.$$

Then one may define the associated frame vectors as

$$\left.\begin{array}{c}\boldsymbol{\lambda}_1(w)=\left(\sqrt{3}\cos w,\ \sqrt{2},-\sqrt{3}\sin w\right),\\ \boldsymbol{\lambda}_2(w)=(-\sin w,0,-\cos w),\\ \boldsymbol{\lambda}_3(w)=\left(-\sqrt{2}\cos w,-\sqrt{3},\ \sqrt{2}\sin w\right).\end{array}\right\}$$

With these, the $\mathcal{SL}$-surface family $\mathfrak{R}$ is given by

$$\mathfrak{R}\ :\ \boldsymbol{v}(w,t)=\left(\sqrt{3}\sin w,\sqrt{2}w,\sqrt{3}\cos w\right)+(\mathfrak{x}(w,t),\mathfrak{y}(w,t),0)\times\begin{pmatrix}\sqrt{3}\cos w & \sqrt{2} & \sqrt{3}\sin w\\ -\sin w & 0 & -\cos w\\ -\sqrt{2}\cos w & -\sqrt{3} & \sqrt{2}\sin w\end{pmatrix},$$

where $0\le\ w\le 2\pi$.

Now, if we set $f=2\sqrt{3}$ in Equation 2.1, then the curve is modified as

$$\widehat{\boldsymbol{\varphi}}(w)\coloneqq\boldsymbol{\varphi}(w)+2\sqrt{3}\boldsymbol{\lambda}_2(w)=\left(-\sqrt{3}\sin w,\ \sqrt{2}w,-\sqrt{3}\cos w\right).$$

For this modified curve, the corresponding frame vectors become

$$\left.\begin{array}{c}\widehat{\boldsymbol{\lambda}}_1(w)=\left(-\sqrt{3}\cos w,\ \sqrt{2},\ \sqrt{3}\sin w\right),\\ \widehat{\boldsymbol{\lambda}}_2(w)=(\sin w,0,\cos w),\\ \widehat{\boldsymbol{\lambda}}_3(w)=\left(\sqrt{2}\cos w,-\sqrt{3},-\sqrt{2}\sin w\right).\end{array}\right\}$$

Thus, the $\mathcal{SL}$-surface family $\widehat{\mathfrak{R}}$ is now parameterized by

$$\widehat{\mathfrak{R}}\ :\ \widehat{\boldsymbol{v}}(w,t)=\left(-\sqrt{3}\sin w,\sqrt{2}w,-\sqrt{3}\cos w\right)+(\mathfrak{x}(w,t),\mathfrak{y}(w,t),0)\times\begin{pmatrix}-\sqrt{3}\cos w & \sqrt{2} & \sqrt{3}\sin w\\ \sin w & 0 & \cos w\\ \sqrt{2}\cos w & -\sqrt{3} & -\sqrt{2}\sin w\end{pmatrix}.$$

In each case, the functions $\mathfrak{x}(w,t)$ and $\mathfrak{y}(w,t)$ (the marching-scale functions) provide the necessary degrees of freedom to generate the family of $\mathcal{SL}$ surfaces, with $\boldsymbol{\varphi}(w)$ serving as the common asymptotic curve.

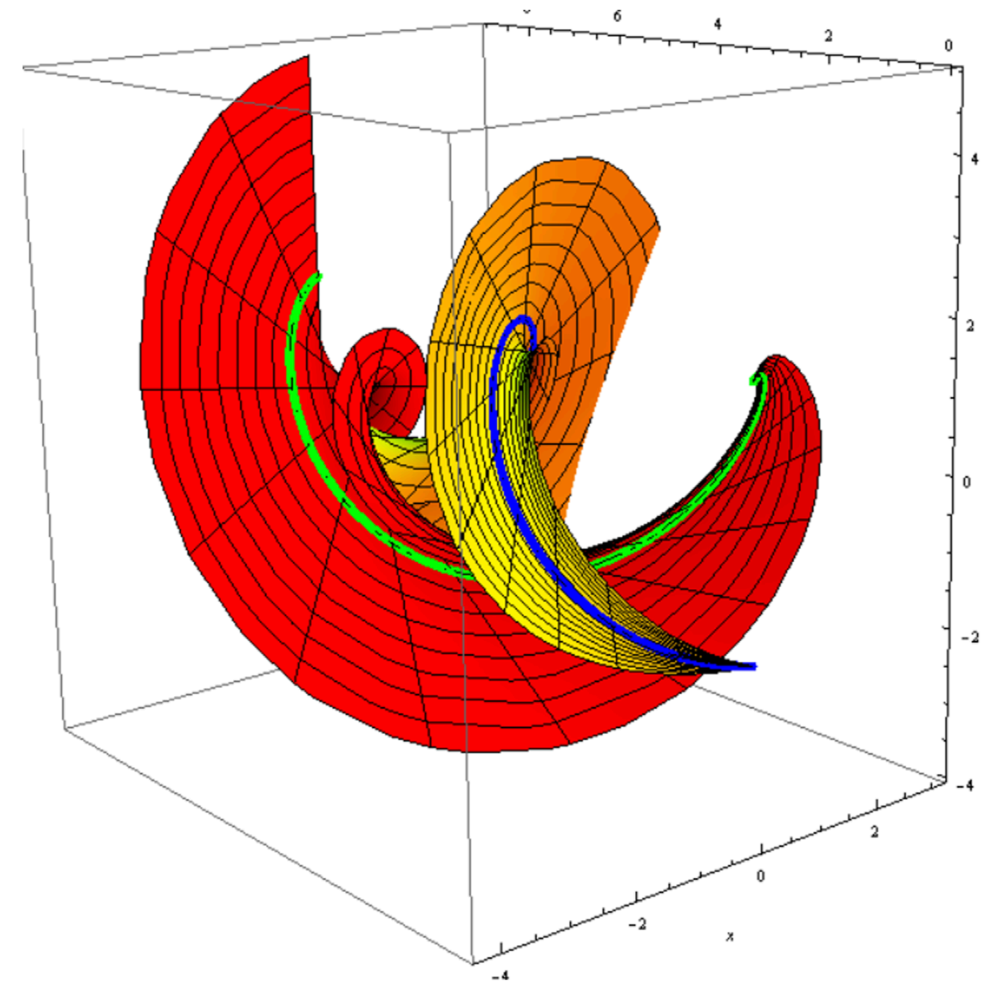

FIGURE 6
$\mathfrak{R}\cup\widehat{\mathfrak{R}}$ $\gamma(w)=1-\cos w$, $\beta(w)=w$.

1. In the case, where $\mathfrak{x}(w,t)=1-\cot t$, $\mathfrak{y}(w,t)=\sin t$, $t_0=0$, and $-1\leq t\leq 1$, the conditions in Equation 3.8 are satisfied. Consequently, the constructed pair of surfaces $\{\mathfrak{R}, \widehat{\mathfrak{R}}\}$ (Figure 1) features $\boldsymbol{\varphi}(w)$ on the surface $\mathfrak{R}$, whereas the green curve represents $\widehat{\boldsymbol{\varphi}}(w)$ on the surface $\widehat{\mathfrak{R}}$.
2. If we define

$$\left.\begin{aligned}\mathfrak{x}(w,t)&=\sin t+\sum_{k=2}^{4}a_{1k}\sin^k t,\\ \mathfrak{y}(w,t)&=(1-\cos t)+\sum_{k=2}^{4}b_{1k}(1-\cos t)^k,\end{aligned}\right\}$$

for $0\leq t\leq 2\pi$ and $0\leq w\leq 2\pi$, with $t_0=0$, and coefficients a_{1k}, $b_{1k}\in\mathbb{R}$, then the conditions given in Equation 3.9 are fulfilled. For example, if we choose $a_{1k}=b_{1k}=0.001$, the corresponding pair of surfaces $\{\mathfrak{R},\widehat{\mathfrak{R}}\}$ is illustrated in Figure 2. In this figure, the blue curve represents $\widehat{\boldsymbol{\varphi}}(w)$ on $\widehat{\mathfrak{R}}$, whereas the green curve represents $\boldsymbol{\varphi}(w)$ on $\mathfrak{R}$.

3. If we define

$$\mathfrak{x}(w,t)=\sin\left(\sum_{k=1}^{4}b_{1k}w^k t^k\right),\ \mathfrak{y}(w,t)=\sum_{k=1}^{4}b_{1k}w^k t^k,$$

where $0\leq w\leq 2\pi$, $-0.1\leq t\leq 0.1$, and set $t_0=0$, then the criteria outlined in Equation 3.10 are satisfied. Specifically, when choosing $b_{1k}=1$ for all k, the corresponding surfaces $\{\mathfrak{R},\widehat{\mathfrak{R}}\}$ are illustrated in Figure 3. In this visualization, the blue curve denotes $\widehat{\boldsymbol{\varphi}}(w)$ on $\widehat{\mathfrak{R}}$, whereas the green curve represents $\boldsymbol{\varphi}(w)$ on the corresponding surface $\mathfrak{R}$.

This example illustrates that one can extend this construction of $\mathcal{SL}$-surface families by selecting additional combinations or sets of curves to interpolate as desired.

3.1 $\mathcal{SL}$ ruled surfaces with $\mathcal{SL}$ -Bertrand curves

In this subsection, we analyze the structure of $\mathcal{SL}$-ruled surfaces that incorporate $\mathcal{SL}$-Bertrand curves as asymptotic curves. For the ease of interpretation, let $\widehat{\boldsymbol{\varphi}}(w)$ be a unit-speed $\mathcal{SL}$-curve with a $\mathcal{TL}$-binormal vector in $\mathbb{E}_1^3$. We consider $\widehat{\boldsymbol{v}}(w,t)$ as an $\mathcal{SL}$-ruled surface whose base curve is $\widehat{\boldsymbol{\varphi}}(w)$, and we assume that $\boldsymbol{\varphi}(w)$ is also an iso-parametric $\mathcal{SL}$ curve of $\boldsymbol{v}(w,t)$. Then, there exists a specific value t_0, such that $\widehat{\boldsymbol{v}}(w,t_0)=\widehat{\boldsymbol{\varphi}}(w)$. This leads to the following relation:

$$\widehat{\mathfrak{R}}:\widehat{\boldsymbol{v}}(w,t)-\widehat{\boldsymbol{v}}(w,t_0)=(t-t_0)\widehat{\mathbf{g}}(w),\ \text{with}\ 0\le w\le L, t,\ t_0\in[0,T],$$

where $\widehat{\mathbf{g}}(w)$ is a $\mathcal{SL}$-unit vector along the rulings. Using Equation 3.2, we obtain

$$(t-t_0)\widehat{\mathbf{g}}(w)=\mathfrak{x}(w,t)\widehat{\boldsymbol{\lambda}}_1(w)+\mathfrak{y}(w,t)\widehat{\boldsymbol{\lambda}}_2(w),\ 0\le w\le L,\ \text{with}\ t,\ t_0\in[0,T],$$

which represents a system of two equations involving the unknown functions $\mathfrak{x}(w,t)$ and $\mathfrak{y}(w,t)$. To express these explicitly, we use

$$\begin{aligned}\mathfrak{x}(w,t)&=(t-t_0)<\widehat{\mathbf{g}},\widehat{\boldsymbol{\lambda}}_1>=(t-t_0)\det\left(\widehat{\mathbf{g}},\widehat{\boldsymbol{\lambda}}_2,\widehat{\boldsymbol{\lambda}}_3\right),\\ \mathfrak{y}(w,t)&=(t-t_0)<\widehat{\mathbf{g}},\widehat{\boldsymbol{\lambda}}_2>=-(t-t_0)\det\left(\widehat{\mathbf{g}},\widehat{\boldsymbol{\lambda}}_1,\widehat{\boldsymbol{\lambda}}_3\right).\end{aligned}\tag{3.11}$$

Equation 3.11 gives the necessary and sufficient conditions for $\widehat{\mathfrak{R}}$ to be a $\mathcal{SL}$-ruled surface. According to Theorem 3.1, if the curve $\widehat{\boldsymbol{\varphi}}(w)$ is an asymptotic curve on $\widehat{\mathfrak{R}}$, then $\det(\widehat{\mathbf{g}},\widehat{\boldsymbol{\lambda}}_1,\widehat{\boldsymbol{\lambda}}_3)\neq 0$. Consequently, at any point along $\widehat{\boldsymbol{\varphi}}(w)$, the ruling direction $\widehat{\mathbf{g}}(w)$ belongs to the span of $\{\widehat{\boldsymbol{\lambda}}_1,\widehat{\boldsymbol{\lambda}}_2\}$. Furthermore, $\widehat{\mathbf{g}}(w)$ and $\widehat{\boldsymbol{\lambda}}_1(w)$ must not be collinear, leading to the expression

$$\widehat{\mathbf{g}}(w)=\gamma(w)\widehat{\boldsymbol{\lambda}}_1(w)+\beta(w)\widehat{\boldsymbol{\lambda}}_2(w),\ 0\le w\le L,$$

for some real functions $\gamma(w)$ and $\beta(w)\neq 0$. As a result, the family of iso-parametric $\mathcal{SL}$-ruled surfaces sharing the common $\mathcal{SL}$-asymptotic curve $\widehat{\boldsymbol{\varphi}}(w)$ can be expressed as

$$\widehat{\mathfrak{R}}:\widehat{\boldsymbol{v}}(w,t)=\widehat{\boldsymbol{\varphi}}(w)+t\left(\gamma(w)\widehat{\boldsymbol{\lambda}}_1(w)+\beta(w)\widehat{\boldsymbol{\lambda}}_2(w)\right),\ 0\le t\le T,\quad 0\le w\le L,\tag{3.12}$$

for certain real-valued functions $\gamma(w)$ and $\beta(w)\neq 0$. The unit normal to the surface $\widehat{\mathfrak{R}}$ is given by

$$\widehat{\mathbf{n}}(w,t)=t\beta\widehat{\tau}\left(\gamma\widehat{\boldsymbol{\lambda}}_2-\beta\widehat{\boldsymbol{\lambda}}_1\right)-\left[\beta+t\left(\gamma^2\widehat{\kappa}-\beta^2\widehat{\kappa}+\gamma\beta'+\beta\gamma'\right)\widehat{\boldsymbol{\lambda}}_3.\right.$$

Evaluating at $t=0$, which corresponds to the curve $\widehat{\boldsymbol{\varphi}}(w)$, we obtain

$$\widehat{\mathbf{n}}(w,0)=-\beta\widehat{\boldsymbol{\lambda}}_3.$$

Thus, $\widehat{\boldsymbol{\varphi}}(w)$ remains an asymptotic curve on $\widehat{\mathfrak{R}}$.

Theorem 3.2: The pair of surfaces$\{\mathfrak{R},\ \widehat{\mathfrak{R}}\}$ interpolates the curves $\{\boldsymbol{\varphi}(w),\ \widehat{\boldsymbol{\varphi}}(w)\}$ as mutual asymptotic $\mathcal{SL}$-Bertrand curves if and only if there exists $t_0\in[0,T]$ and functions $\gamma(w)$ and $\beta(w)\neq 0$ such that $\widehat{\mathfrak{R}}$ and $\mathfrak{R}$ are represented by Equation 3.12 and

$$\mathfrak{R}:\boldsymbol{v}(w,t)=\boldsymbol{\varphi}(w)+t(\gamma(w)\boldsymbol{\lambda}_1(w)+\beta(w)\boldsymbol{\lambda}_2),\ 0\le t\le T,\quad 0\le w\le L,\tag{3.13}$$

for some real-valued functions $\gamma(w)$ and $\beta(w)\neq 0$.

It is important to highlight in Equation 3.12 (respectively, Equation 3.13) that a $\mathcal{SL}$ asymptotic curve passes through every point on the curve $\widehat{\boldsymbol{\varphi}}(w)$ (respectively, $\boldsymbol{\varphi}(w)$). One of these curves is $\widehat{\boldsymbol{\varphi}}$ (rersp. $\boldsymbol{\varphi}(w)$) itself, whereas the other corresponds to a $\mathcal{SL}$-line aligned with the direction $\widehat{\mathbf{g}}(w)$ ((respectively, $\mathbf{g}(w)$), as described in Equation 3.12 (respectively, (Equation 3.13)).

Example 3.2: Building upon Example 3.1, we consider the following cases:

1. For $\gamma(w)=\sin w$ and $\beta(w)=\cos w$, the surfaces $\{\mathfrak{R},\ \widehat{\mathfrak{R}}\}$ interpolating $\{\boldsymbol{\varphi}(w),\ \widehat{\boldsymbol{\varphi}}(w)\}$ as mutual asymptotic $\mathcal{SL}$-Bertrand curves are given as follows (Figure 4):

$$\mathfrak{R}:\boldsymbol{v}(w,t)=\begin{pmatrix}\sqrt{3}\sin w+\frac{t}{2}\left(\sqrt{3}-1\right)\sin 2w\\ \sqrt{2}w+\sqrt{2}t\sin w\\ \sqrt{3}\cos w-t\left(\sqrt{3}\sin^2 w+\cos^2 w\right)\end{pmatrix},$$

and

$$\widehat{\mathfrak{R}}:\widehat{\boldsymbol{v}}(w,t)=\begin{pmatrix}-\sqrt{3}\sin w-\frac{t}{2}\left(\sqrt{3}-1\right)\sin 2w\\ \sqrt{2}w+\sqrt{2}t\sin w\\ -\sqrt{3}\cos w+t\left(\sqrt{3}\sin^2 w+\cos^2 w\right)\end{pmatrix},$$

where $-1\le\ t\le 1$ and $0\le\ w\le 2\pi$. In Figure 4, the blue curve represents $\widehat{\boldsymbol{\varphi}}(w)$ on $\widehat{\mathfrak{R}}$, whereas the green curve illustrates $\boldsymbol{\varphi}(w)$ on $\mathfrak{R}$.

2. If $\gamma(w)=\beta(w)=w$, then the surfaces $\{\mathfrak{R},\ \widehat{\mathfrak{R}}\}$ interpolating $\{\boldsymbol{\varphi}(w),\ \widehat{\boldsymbol{\varphi}}(w)\}$ as mutual asymptotic $\mathcal{SL}$-Bertrand curves are given by (Figure 5):

$$\mathfrak{R}:\boldsymbol{v}(w,t)=\begin{pmatrix}\sqrt{3}\sin w+tw\left(\sqrt{3}\cos w-\sin w\right)\\ \sqrt{2}w(1+t)\\ \sqrt{3}\cos w-tw\left(\sqrt{3}\sin w+\cos w\right)\end{pmatrix},$$

and

$$\widehat{\mathfrak{R}}:\widehat{\boldsymbol{v}}(w,t)=\begin{pmatrix}-\sqrt{3}\sin w-tw\left(\sqrt{3}\cos w-\sin w\right)\\ \sqrt{2}w(1+t)\\ -\sqrt{3}\cos w+tw\left(\sqrt{3}\sin w+\cos w\right)\end{pmatrix},$$

where $-1.5\le\ t\le 1.5$ and $0\le\ w\le 2\pi$. In Figure 5, the blue curve represents $\widehat{\boldsymbol{\varphi}}(w)$ on $\widehat{\mathfrak{R}}$, whereas the green curve corresponds to $\boldsymbol{\varphi}(w)$ on $\mathfrak{R}$.

3. If $\gamma(w)=1-\cos w$ and $\beta(w)=w$, then the surfaces $\{\mathfrak{R},\ \widehat{\mathfrak{R}}\}$ interpolating $\{\boldsymbol{\varphi}(w),\ \widehat{\boldsymbol{\varphi}}(w)\}$ as mutual asymptotic $\mathcal{SL}$-Bertrand curves are given by (Figure 6):

$$\mathfrak{R}:\boldsymbol{v}(w,t)=\begin{pmatrix}\sqrt{2}\sin w+t\left(\sqrt{3}\sin w-\sqrt{2}\right)\cos w\\ \sqrt{2}w+t\left(\sqrt{2}\sin w-\sqrt{3}\right)\\ \sqrt{3}\cos w-t\left(\sqrt{3}\sin w-\sqrt{2}\right)\sin w\end{pmatrix},$$

and

$$\widehat{\mathfrak{R}}:\widehat{\boldsymbol{v}}(w,t)=\begin{pmatrix}-\sqrt{2}\sin w-t\left(\sqrt{3}\sin w-\sqrt{2}\right)\cos w\\ \sqrt{2}w+t\left(\sqrt{2}\sin w-\sqrt{3}\right)\\ -\sqrt{3}\cos w+t\left(\sqrt{3}\sin sw-\sqrt{2}\right)\sin w\end{pmatrix},$$

where $-.5\le\ t\le .5$ and $0\le\ w\le 2\pi$. In Figure 6, the blue curve represents $\widehat{\boldsymbol{\varphi}}(w)$ on $\mathfrak{R}$, whereas the green curve corresponds to $\boldsymbol{\varphi}(w)$ on $\mathfrak{R}$.

4 Conclusion

In this study, we propose a method for constructing families of spacelike surfaces in Minkowski 3-space that share a common Bertrand curve as an asymptotic curve. By aligning surface tangent planes with the osculating planes of spacelike Bertrand curves and employing marching-scale functions, we establish a flexible parametrization framework and derive the necessary and sufficient conditions for asymptoticity. Examples highlight the versatility of the method, which enriches surface modeling in Lorentzian geometry and offers applications in differential geometry, relativity, and computer-aided design. Potential extensions include timelike or null surfaces, higher dimensions, and dynamic surface evolution.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

AA: Data curation, Software, Resources, Conceptualization, Funding acquisition, Writing – review and editing, Writing original draft, Formal analysis, Investigation. RA-B: Supervision, Writing – review and editing, Conceptualization, Software, Investigation, Methodology, Funding acquisition, Resources, Formal analysis, Project administration, Writing – original draft, Validation, Data curation, Visualization.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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