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Measures and operators associated with Parseval distribution frames

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Continuing the study by Tschinke et al. (2019), we examine further aspects of distribution frames (namely, Gel'fand and Parseval), particularly regarding those that are more relevant for applications in quantum physics. Parseval distribution frames are, in particular, closely related to coherent states. Thus, POV measures, Naimark dilations, and operators defined by Parseval distribution frames are the main subjects of this paper. The main results are Theorems 2.2 and 3.1. Theorem 2.2 gives a sufficient conditions for the existence of such distribution coherent states for positive operator valued measures. Theorem 3.1 establishes conditions under which the distribution coherent states can be identified with the projections of some Gel'fand distribution basis in a larger Hilbert space (in Naimark's sense).

KEYWORDS

Parseval distribution frames, POV measures, Naimark dilations, operators, rigged Hilbert space

1 Introduction and preliminaries

Since a long time now, the language of rigged Hilbert space [1–4] has been used in the mathematical description of quantum mechanical systems for giving room to objects of common use in daily practice that can hardly be cast in the traditional approach with Hilbert space (e.g., [5–10, 25]). The case of the eigenvectors of the free Hamiltonian $\frac{p^2}{2m}$, where p denotes the linear momentum operator, is one of the simplest examples of such cases.

The theory of frames, discrete and continuous, plays an interesting role in quantum mechanics in at least two situations. The first one, which is closely related to the appearance of the so-called non-Hermitian Hamiltonians, has put on the stage families (mostly discrete) of non-orthogonal vectors (often eigenvectors of nonsymmetric operators) that constitute, in favorable cases, Riesz bases of the Hilbert space; they are generally obtained by modifying an orthonormal basis $\{e_n\}$ through the action of a bounded operator G with bounded inverse (the so-called metric operator). The second one is connected to the theory of coherent states, which are, often, continuous frames that are supposed to constitute a resolution of identity. In the language of frames, this property is denoted as (continuous) Parseval frames [11].

In the paper [12], in view of a more general treatment, the notion of distribution frames was introduced together with a family of relatives Riesz distribution frames, Parseval distribution frames, and Gel'fand distribution bases. They are all present in a rigged Hilbert space, and Gel'fand distribution bases are shown to be the natural generalization to the new environment of the familiar notion of orthonormal basis of Hilbert spaces. The generalized eigenvectors (in the sense of the Gel'fand–Maurin theorem [13, 14]) of $\frac{p^2}{2m}$ provide an instance of a Gel'fand distribution basis (a generalized eigenvalue expansion for unbounded normal operators can also be found in [15]). Given a self-adjoint operator A in a Hilbert space $\mathcal{H}$, its spectral behavior, when expressed in terms of generalized eigenvectors, can be studied using the formalism of Gel'fand distribution bases, as in [16].

In this paper, we focus our attention mostly on Parseval distribution frames; similar to the Gel'fand ones, they are resolutions of the identity in the sense that they satisfy a Parseval-like equality, but they are not necessarily μ-independent; in other words, they can be over-complete. An over-complete resolution of the identity is one of the characteristic features of coherent states that have been the subject of an enormous (and still increasing!) amount of literature; refer to [17] for a systematic treatment. Usually, coherent states are represented as vectors of some Hilbert space, but there are cases where more general objects (non-square integrable functions or even distributions) should be considered (e.g., [18], Section 5.1.3). In our opinion, these considerations motivate an approach that goes beyond Hilbert space; for this reason, after discussing some basic aspects of Bessel and Parseval distribution frames, here, we examine in details some aspects of the theory that are more related to possible applications, even if we maintain the analysis at a quite abstract level. To be more precise, we consider positive operator-valued (POV) measures defined by distribution maps (Section 2) and study the possibility of introducing Naimark dilations for rigged Hilbert spaces with the aim of showing that certain Parseval distribution frames can be obtained as projections of Gel'fand distribution maps (Section 3). Finally (Section 4), we examine operators defined by Parseval frames by means of certain sufficiently regular functions by some mathematical expressions that closely resemble the quantization procedure defined by coherent states.

The basic notions needed for the understanding of this paper are given here. A more detailed discussion can be found in [12, 19].

A rigged Hilbert space, or Gel'fand triplet, is a triple of spaces,

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times [t^\times],$$

where $\mathcal{D}$ is a dense subspace of a Hilbert space $\mathcal{H}$ (which is supposed to be infinite-dimensional and separable) and, at once, a locally convex space with topology t; throughout this paper, we will suppose that $\mathcal{D}[t]$ is a Fréchet and reflexive space. We denote by $\mathcal{D}^\times$ the conjugate dual of $\mathcal{D}$, which is endowed with the strong dual topology $t^\times$. We indicate by $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ the space of continuous linear maps from $\mathcal{D}[t]$ to $\mathcal{D}^\times[t^\times]$. In $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$, an involution $A \mapsto A^\dagger$ is defined by the equality $\langle Af|g\rangle = \overline{\langle A^\dagger g|f\rangle}$, where $f,g \in \mathcal{D}$.

Let μ be a Radon measure on the Borel sets of a locally compact space X. A distribution map is a μ-weakly measurable function $\omega: x \in X \to \omega_x \in \mathcal{D}^\times$.

The map ω is called a Bessel distribution map if there exists a continuous seminorm p on $\mathcal{D}[t]$ such that

$$\int_X |\langle f|\omega_x\rangle|^2 d\mu \le p(f)^2, \quad \forall f \in \mathcal{D};$$

in particular, ω is called bounded Bessel if $B > 0$ such that

$$\int_X |\langle f|\omega_x\rangle|^2 d\mu \le B\|f\|^2, \quad \forall f \in \mathcal{D}. \tag{1}$$

Finally, ω is a Parseval distribution frame if

$$\int_X |\langle f|\omega_x\rangle|^2 d\mu = \|f\|^2, \quad \forall f \in \mathcal{D}.$$

A Gel'fand distribution basis is a μ-independent Parseval one; that is, if ξ is measurable and $\int_X \xi(x)\langle g|\omega_x\rangle d\mu = 0$, $\forall g \in \mathcal{D}$, then $\xi = 0$ μ-a.e.

A Riesz distribution basis ω is the image of a Gel'fand basis through a continuous operator $R: \mathcal{D}^\times \to \mathcal{D}^\times$ with continuous inverse. For details, we refer to [12].

A Gel'fand distribution basis presents two interesting features. On the one hand, it defines a POV measure on the σ-algebra of Borel sets. On the other hand, they can be used to construct sort-of scalar operators A through some appropriate function α; formally,

$$\langle Af|g\rangle = \int_X \alpha(x)\langle f|\zeta_x\rangle\langle\zeta_x|g\rangle d\mu,$$

on a suitable domain.

Here, we summarize the basic definitions, referring to [12] for details.

Let ω be a bounded Bessel distribution map. Then, the sesquilinear form defined by

$$\Omega(f,g) = \int_X \langle f|\omega_x\rangle\langle\omega_x|g\rangle d\mu$$

is well defined on $\mathcal{D}\times\mathcal{D}$. Moreover, it is $\|\cdot\|$-bounded; thus, it has a bounded extension $\widehat{\Omega}$ to $\mathcal{H}$. Hence, there exists a bounded operator $\hat{S}_\omega$ in $\mathcal{H}$ such that

$$\widehat{\Omega}(f,g) = \langle \hat{S}_\omega f|g\rangle, \quad \forall f,g \in \mathcal{H}. \tag{2}$$

As

$$\langle \hat{S}_\omega f|g\rangle = \int_X \langle f|\omega_x\rangle\langle\omega_x|g\rangle d\mu, \quad \forall f.g \in \mathcal{D}.$$

As in [15, Definition 3.6], we state that a distribution map ω is a distribution frame if there exist $A, B > 0$ such that

$$A\|f\|^2 \le \int_X |\langle f|\omega_x\rangle|^2 d\mu \le B\|f\|^2, \quad \forall f \in \mathcal{D}.$$

A distribution frame ω is clearly a bounded Bessel map. Thus, for the operator $\hat{S}_\omega$ defined in Equation 2, we have

$$A\|f\| \le \|\hat{S}_\omega f\| \le B\|f\|, \quad \forall f \in \mathcal{H}.$$

This inequality, together with the fact that $\hat{S}_\omega$ is symmetric, implies that $\hat{S}_\omega$ has a bounded inverse $\hat{S}_\omega^{-1}$ everywhere, as defined in $\mathcal{H}$.

If ω is a bounded Bessel distribution map and $\xi \in L^2(X,\mu)$, the conjugate linear functional on $\mathcal{D}$, defined by

$$\Lambda_\omega^\xi(g) := \int_X \xi(x)\langle\omega_x|g\rangle d\mu,$$

is bounded. Therefore, there exists a unique vector $h_\xi \in \mathcal{H}$ such that

$$\tilde{\Lambda}_\omega^\xi(g) = \langle h_\xi|g\rangle, \quad \forall g \in \mathcal{H}.$$

Therefore, we can define a linear map $D_\omega: L^2(X,\mu) \to \mathcal{H}$, which will be called the synthesis operator, by

$$D_\omega\xi = h_\xi, \quad \xi \in L^2(X,\mu).$$

Then, D_ω is bounded from $L^2(X,\mu)$ to $\mathcal{H}$. Hence, it has a bounded adjoint $C_\omega := D_\omega^*$ called the analysis operator, which acts as follows:

$$C_\omega: f \in \mathcal{D} \to \xi_f \in L^2(X,\mu), \text{where } \xi_f(x) = \langle f|\omega_x\rangle, \ x \in X.$$

The synthesis operator D_ω takes values in $\mathcal{H}$, and it is bounded and $\|D_\omega\| \le B^{1/2}$.

Remark 1.1: Let us describe the action of C_ω. If $g \in \mathcal{H}$ and $\{g_n\}$ is a sequence of elements of $\mathcal{D}$, norm converging to g, then the sequence $\{\eta_n\}$, where $\eta_n = \langle g_n|\omega_\cdot\rangle$, is convergent in $L^2(X,\mu)$ as ω is Bessel bounded. Put $\eta := \|\cdot\|_2 - \lim_{n\to\infty}\eta_n$. The function η does not depend on the choice of the sequence $\{g_n\}$ approximating g in $\mathcal{H}$. Then, for D_ω^*, we have

$$\langle D_\omega \xi|g\rangle = \lim_{n\to\infty}\int_X \xi(x)\langle\omega_x|g_n\rangle\, d\mu = \int_X \xi(x)\overline{\eta(x)}d\mu.$$

Hence, $D_\omega^* g = \eta$. Notably, the function $\eta \in L^2(X,\mu)$ depends linearly on g. In [12, 16], a linear functional was formally defined as $\check{\omega}_x$ by

$$\text{if}\quad g\in\mathcal{H};\ g_n \to g,\quad \langle g|\check{\omega}_x\rangle := \lim_{n\to\infty}\langle g_n|\omega_x\rangle \quad \textit{pointwise}.$$

This should be read only as a notation shorthand because $\check{\omega}_x$ is not well defined. In this note, we prefer to adopt a different notation and directly use the operator C_ω.

A μ-weakly measurable function $\omega: X \to \mathcal{D}^\times$ is called a distribution basis for $\mathcal{D}$ if, for every $f \in \mathcal{D}$, there exists a unique measurable function ξ_f such that

$$\langle f|g\rangle = \int_X \xi_f(x)\langle\omega_x|g\rangle\, d\mu, \quad \forall g\in\mathcal{D},$$

and, for every $x \in X$, the linear functional $f \in \mathcal{D} \to \xi_f(x) \in \mathbb{C}$ is continuous on $\mathcal{D}[t]$.

If ω is a distribution basis, by the definition itself, there exists a unique μ-weakly measurable map $\theta: X \to \mathcal{D}^\times$ such that $\xi_f(x) = \langle f|\theta_x\rangle$, $\forall f \in \mathcal{D}$. Hence, the following identity holds:

$$\langle f|g\rangle = \int_X \langle f|\theta_x\rangle\langle\omega_x|g\rangle\, d\mu, \quad \forall f,g\in\mathcal{D}.$$

We consider θ the dual map of ω.

Furthermore, considering the complex conjugate of the above expression, we obtain the following:

$$f = \int_X \langle f|\omega_x\rangle\theta_x d\mu, \quad \forall f\in\mathcal{D}.$$

Then, if θ is μ-independent, θ is also a distribution basis.

2 POV measures associated with distribution frames

Let ω be a weakly measurable distribution map. For every $f \in \mathcal{D}$, the integral

$$\Delta \mapsto \int_\Delta |\langle f|\omega_x\rangle|^2 d\mu$$

defines a positive measure on Σ, which is finite if ω is Bessel. In this case,

$$\Omega_\Delta(f,g) = \int_\Delta \langle f|\omega_x\rangle\langle\omega_x|g\rangle\, d\mu, \quad f,g\in\mathcal{D}$$

defines a jointly continuous positive sesquilinear form on $\mathcal{D}\times\mathcal{D}$. The set of all jointly continuous, positive sesquilinear forms on $\mathcal{D}\times\mathcal{D}$ will be denoted by $\mathcal{P}(\mathcal{D})$. In this case, there exists a positive operator $T(\Delta) \in \mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ (i.e., $\langle Tf|f\rangle \ge 0$, $\forall f \in \mathcal{D}$) such that

$$\int_\Delta \langle f|\omega_x\rangle\langle\omega_x|g\rangle\, d\mu = \langle T(\Delta)f|g\rangle, \quad \forall f,g\in\mathcal{D}.$$

In addition, the map

$$\Delta\in\Sigma \to T(\Delta)\in\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$$

defines a POV measure on Σ. In particular, if ω is bounded Bessel, for each $\Delta \in \Sigma$, $T(\Delta)$ is a bounded operator that can be extended to the whole Hilbert space $\mathcal{H}$. In this case, one can find a Naimark dilation (i.e., a larger Hilbert space, having $\mathcal{H}$ as closed subspace) and a projection valued (PV) measure that reduces to $T(\Delta)$ on $\mathcal{H}$.

On the other hand, given a POV measure T, one can pose the question of whether it is possible to find a distribution map ω such that

$$\langle T(\Delta)f|g\rangle = \int_\Delta \langle f|\omega_x\rangle\langle\omega_x|g\rangle\, d\mu, \quad \forall\Delta\in\Sigma;\ f,g\in\mathcal{D}.$$

It is quite natural to look at the right-hand side of the previous equality as the basic ingredients of a type of spectral resolution of an operator: generalized projections, to be more precise. If ω is a distribution map, for each $x \in X$, the map

$$(f,g)\in\mathcal{D}\times\mathcal{D} \to \langle f|\omega_x\rangle\langle\omega_x|g\rangle \tag{3}$$

is a jointly continuous sesquilinear form. Hence, for each $x \in X$, there exists an operator $P_{\omega_x} \in \mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ such that

$$\tau_{\omega_x}(f,g) = \langle P_{\omega_x}f|g\rangle, \quad \forall f,g\in\mathcal{D}. \tag{4}$$

From Equation 3, 4, it follows that

$$P_{\omega_x}f = \langle f|\omega_x\rangle\omega_x, \quad \forall f\in\mathcal{D}.$$

The operator P_{ω_x} is symmetric ($P_{\omega_x} = P_{\omega_x}^\dagger$) and positive ($\langle P_{\omega_x}f|f\rangle \ge 0$, for every $f \in \mathcal{D}$). Even though P_ω does not satisfy $P_\omega = P_\omega^2$ (which is meaningless), we can reasonably consider P_ω a generalized one-dimensional projection.

As a first step, given a POV measure, as above, we want to find a map $\tau_x: \mathcal{D}\times\mathcal{D} \to L^1(X,\mu)$ such that

$$\langle T(\Delta)f|g\rangle = \int_\Delta \tau_x(f,g)\, d\mu, \quad \forall f,g\in\mathcal{D};\ \forall\Delta\in\Sigma.$$

For this to be possible, it is necessary and sufficient that the POV measure T be absolutely continuous with respect to μ in a weak sense; that is, for every $f,g \in \mathcal{D}$, the complex measure $\Delta \to \langle T(\Delta)f|g\rangle$ is absolutely continuous with respect to μ. In this case, the Radon–Nikodym theorem guarantees the existence of a μ-measurable function $\tau: x \in X \to \tau_x \in \mathcal{S}(\mathcal{D})$, the space of all sesquilinear forms on $\mathcal{D}\times\mathcal{D}$, such that

$$\tau(f,g) = \frac{dT}{d\mu}(f,g), \quad \forall f,g\in\mathcal{D}.$$

In fact, for μ almost all $x \in X$, τ_x is a sesquilinear form on $\mathcal{D}$.

Next, we want to show that under certain assumptions, there exists a weakly measurable function $\omega: X \to \mathcal{D}^\times$ such that

$$\tau_\cdot(f,g) = \langle f|\omega_\cdot\rangle\langle\omega_\cdot|g\rangle, \quad \forall f,g\in\mathcal{D}.$$

Let us first introduce some notations. If $\sigma \in \mathcal{P}(\mathcal{D})$, we put

$$N(\sigma) = \{f \in \mathcal{D}: \sigma(f,f) = 0\}.$$

The Cauchy–Schwarz inequality and the continuity of σ imply that $N(\sigma)$ is a closed subspace of $\mathcal{D}[t]$.

Lemma 2.1: Let $\sigma \in \mathcal{P}(\mathcal{D})$ and $\sigma \not\equiv 0$. The following statements are equivalent:

i. There exist $\eta \in \mathcal{D}^\times$ (unique up to a factor z with $|z| = 1$) and $u \in \mathcal{D}$ such that

$$\sigma(f,g) = \langle f|\eta\rangle \langle \eta|g\rangle, \quad \forall f,g \in \mathcal{D} \text{ and } \sigma(u,u) = 1.$$

i. $N(\sigma)$ is a proper closed maximal subspace of $\mathcal{D}$.

Proof: we assume that for some $\eta \in \mathcal{D}^\times$, $\sigma(f,g) = \langle f|\eta\rangle \langle \eta|g\rangle$, for every $f,g \in \mathcal{D}$. Then, $f \in N(\sigma)$, if, and only if, $\langle f|\eta\rangle = 0$; i.e., $N(\sigma) =$ Ker η, and the latter is a closed maximal subspace of $\mathcal{D}$, as is known. Moreover, as $\sigma \not\equiv 0$, we can find $u \in \mathcal{D}$ such that $\sigma(u,u) = 1$.

Next, we assume that $\eta' \in \mathcal{D}^\times$ also satisfies $\sigma(f,g) = \langle f|\eta'\rangle \langle \eta'|g\rangle$, for every $f,g \in \mathcal{D}$. Then, $\sigma(f,f) = |\langle f|\eta\rangle|^2 = |\langle f|\eta'\rangle|^2$, for every $f \in \mathcal{D}$. This implies that $\eta' = z\eta$, with $|z| = 1$.

Conversely, we assume that $N(\sigma)$ is a proper closed maximal subspace of $\mathcal{D}$ and $u \notin N(\sigma)$. We can assume $\sigma(u,u) = 1$. Then, due to the maximality of $N(\sigma)$, every element $f \in \mathcal{D}$ can be written as $f = \lambda u + n$, with $n \in N(\sigma)$. Define $\langle \eta|f\rangle = \overline{\lambda}$. Then, $\eta \in \mathcal{D}^\times$ as Ker $\eta = N(\sigma)$ is closed. Then, if $f = \lambda u + n$, $g = \mu u + n'$ are elements of $\mathcal{D}$,

$$\sigma(f,g) = \sigma(\lambda u + n, \mu u + n') = \lambda\overline{\mu}\sigma(u,u) = \langle f|\eta\rangle \langle \eta|g\rangle.$$

Let us come back to the function τ.

Theorem 2.2: Let $x \in X \to \tau_x \in \mathcal{P}(\mathcal{D})$ be obtained as the Radon–Nikodym derivative of the POV measure T. We assume that

i. codim $N(\tau_x) = 1$ for μ almost every $x \in X$;
ii. there exists $u \in \mathcal{D}$ such that $\mu(\{x \in X: \tau_x(u,u) \neq 1\}) = 0$

Then, there exists a weakly measurable distribution map $\omega: x \in X \to \omega_x \in \mathcal{D}^\times$ such that for μ almost every $x \in X$,

$$\tau_x(f,g) = \langle f|\omega_x\rangle \langle \omega_x|g\rangle, \quad \forall f,g \in \mathcal{D}.$$

Proof: by (i) and (ii) for almost every $x \in X$, the conditions of Lemma 2.1 are fulfilled. Then, for these $x \in X$, the set $\{\omega \in \mathcal{D}^\times: \tau_x(f,f) = |\langle f|\omega\rangle|^2, \ \forall f \in \mathcal{D}\}$ is non-empty, so, we can define a function $x \mapsto \omega_x$ by picking one element in each of these sets. As $\tau_x(f,g) = \langle f|\eta_x\rangle \langle \eta_x|g\rangle$, for all $f,g \in \mathcal{D}$, the function ω defined in this way is weakly measurable. The statement then follows by observing that the condition codim $N(\tau_x) = 1$ is equivalent to stating that $N(\tau_x)$ is closed and maximal.

3 Naimark dilations of rigged Hilbert spaces

Naimark dilations are powerful tools in operator theory, and they are also relevant in other contexts. In [20], this technique has been adopted for certain aspects of frame theory: in particular, the authors show that a Parseval frame is the projection of an orthonormal basis in a larger Hilbert space. Our problem is now to try and extend this result to distribution frames. Let us start with some preliminary remarks.

Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a RHS, with $\mathcal{D}[t]$ a Fréchet and reflexive space, and let $\mathcal{K}$ be another Hilbert space containing $\mathcal{H}$ as a closed subspace. Then $\mathcal{K} = \mathcal{H} \oplus \mathcal{M}$, where $\mathcal{M}$ denotes the orthogonal complement of $\mathcal{H}$ in $\mathcal{K}$. Let us consider the space $\mathcal{E} = \mathcal{D} \oplus \mathcal{M}$ endowed with the topology defined by the semi-norms

$$\rho_n(f \oplus \phi) = p_n(f) + \|\phi\|, \quad f \in \mathcal{D}, \ \phi \in \mathcal{M},$$

where $\{p_n\}$ is a countable family of semi-norms defining the topology of $\mathcal{D}$. Clearly, $\mathcal{E}$ is Fréchet.

We claim that $(\mathcal{D} \oplus \mathcal{M})^\times = \mathcal{D}^\times \oplus \mathcal{M}$ so that

$$\mathcal{D} \oplus \mathcal{M}[t_\oplus] \subset \mathcal{K} \subset \mathcal{D}^\times \oplus \mathcal{M}[t_\oplus^\times]$$

is a RHS, which we call the Naimark dilation of $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. On the one hand, if $F_\oplus \in (\mathcal{D} \oplus \mathcal{M})^\times$, then $F_0(f) := F_\oplus(f \oplus 0)$ defines a continuous conjugate linear functional on $\mathcal{D}$, and $F_1(m) := F_\oplus(0 \oplus m)$ defines a bounded conjugate linear functional on $\mathcal{M}$, so, there exists $m' \in \mathcal{M}$ such that $F_1(m) = \langle m'|m\rangle$, for every $m \in \mathcal{M}$. Therefore, $(\mathcal{D} \oplus \mathcal{M})^\times \subset \mathcal{D}^\times \oplus \mathcal{M}$. The converse inclusion is obvious.

Let ω be a Parseval distribution frame; that is,

$$\int_X |\langle f|\omega_x\rangle|^2 d\mu = \|f\|^2, \quad \forall f \in \mathcal{D}.$$

In this case, the analysis operator C_ω,

$$C_\omega: f \in \mathcal{D} \to \langle f|\omega_\cdot\rangle \in L^2(X,\mu),$$

is an isometry; hence, the closure of $C_\omega \mathcal{D}$ can be identified with a closed (generally, proper) subspace of $L^2(X,\mu)$.

Let us put $\mathcal{D}_\# = C_\omega \mathcal{D}$ and $\mathcal{H}_\# = C_\omega \mathcal{H}$. It is clear that $\mathcal{D}_\#$ is a dense subspace of $\mathcal{H}_\#$. If the topology of $\mathcal{D}$ is defined by the family of semi-norms $\{p_n\}_{n\in\mathbb{N}}$, it is natural to define a topology on $\mathcal{D}_\#$ by means of the semi-norms $\{p_n^C\}_{n\in\mathbb{N}}$ defined by

$$p_n^C(\phi) = p_n(C_\omega^{-1}\phi), \quad \phi \in \mathcal{D}_\#.$$

Let $\mathcal{D}_\#^\times$ denote the conjugate dual of $\mathcal{D}_\#$. In this way, we constructed a rigged Hilbert space whose central Hilbert space is a closed subspace of $L^2(X,\mu)$.

Let $\mathcal{M} := \mathcal{H}_\#^\perp \subset L^2(X,\mu)$, and consider the rigged Hilbert space constructed as above. $\mathbb{P}$ is used to denote the orthogonal projection of $L^2(X,\mu)$ onto $\mathcal{H}_\#$. Then, $\mathbb{P}$ maps $\mathcal{D}_\#$ onto itself as $\phi \in \mathcal{D}_\#$ if, and only if, $\phi = C_\omega f$ for some $f \in \mathcal{D}$; then, $\mathbb{P}C_\omega f = C_\omega f$ by the definition of $\mathbb{P}$, and so $\mathbb{P}\phi \in \mathcal{D}_\#$. Moreover, we have, if $\phi = \mathbb{P}f \in \mathcal{D}_\#$,

$$p_n^C(\mathbb{P}\phi) = p_n(C_\omega^{-1}\mathbb{P}\phi) = p_n(C_\omega^{-1}\mathbb{P}C_\omega f) = p_n(f).$$

Hence, $\mathbb{P}$ is continuous from $\mathcal{D}_\#$ to itself. Therefore, there exist $\mathbb{P}^\times: \mathcal{D}_\#^\times \to \mathcal{D}_\#^\times$ such that

$$\langle \mathbb{P}\phi|\Phi\rangle = \langle \phi|\mathbb{P}^\times\Phi\rangle, \forall \phi \in \mathcal{D}_\#, \Phi \in \mathcal{D}_\#^\times.$$

Clearly, $\mathbb{P}^\times$ extends $\mathbb{P}$ to $\mathcal{D}_\#^\times$.

Let us now consider the rigged Hilbert space

$$\mathcal{D}_\# \subset \mathcal{H}_\# \subset \mathcal{D}_\#^\times.$$

The definition of the topology of $\mathcal{D}_\#$ implies that C_ω is continuous from $\mathcal{D}$ to $\mathcal{D}_\#$, and it is also one-to-one. Hence, there exists $C_\omega^\times : \mathcal{D}_\#^\times \to \mathcal{D}^\times$ such that

$$\langle C_\omega f | \Phi \rangle = \langle f | C_\omega^\times \Phi \rangle, \quad \forall f \in \mathcal{D}, \Phi \in \mathcal{D}_\#^\times. \tag{5}$$

We state that $C_\omega^\times \mathcal{D}_\#^\times = \mathcal{D}^\times$. Indeed, Equation 5 implies that $C_\omega^\times \mathcal{D}_\#^\times \subset \mathcal{D}^\times$. On the other hand, as C_ω^{-1} is also continuous if $F \in \mathcal{D}^\times$, the functional $H(\phi) = \langle F | C_\omega^{-1} \phi \rangle$ is in $\mathcal{D}_\#^\times$ and

$$\langle F | C_\omega^{-1} \phi \rangle = \langle (C_\omega^{-1})^\times F | \phi \rangle, \quad \forall \phi \in \mathcal{D}_\#.$$

The equality $F = C_\omega^\times (C_\omega^{-1})^\times F$ implies the statement.

Let $\zeta : x \in X \to \zeta_x \in \mathcal{D}_\#^\times$ be a Gel'fand distribution basis. Then, $\omega_x := \mathbb{P}^\times \zeta_x$ is a Parseval distribution frame. Indeed,

$$\int |\langle f | \omega_x \rangle|^2 d\mu = \int |\langle f | \mathbb{P}^\times \zeta_x \rangle|^2 d\mu = \int |\langle \mathbb{P} f | \zeta_x \rangle|^2 d\mu = \|\mathbb{P} f\|^2 = \|f^2\|.$$

We want to state the converse; that is, given a Parseval distribution frame ω, does there exist a Gel'fand distribution basis ζ in a larger rigged Hilbert space such that ω is the projection of ζ?

Let $C(X)$ denote the space of continuous functions on X, endowed with the locally convex topology τ_0 defined by the semi-norms $\varphi \mapsto p_K(\phi) = \sup_{x \in K} |\varphi(x)|$, $K \subset X$, and K compact.

Theorem 3.1: Let ω be a Parseval distribution frame. It is assumed that C_ω maps $\mathcal{D}$ into $C(X)$ and that C_ω is continuous from $\mathcal{D}[t]$ to $C(X)[\tau_0]$. Moreover, it is assumed that the evaluation map δ_x on $C(X)$ defined by $\langle \varphi | \delta_x \rangle = \varphi(x)$ is continuous on $\mathcal{D}_\# = C_\omega \mathcal{D}$ with its own topology. Then, ω can be identified with the projection $\mathbb{P}\delta_x$ of the Gel'fand distribution basis δ.

Proof: indeed, we have

$$\langle C_\omega f | \mathbb{P} \delta_x \rangle = \langle \mathbb{P} C_\omega f | \delta_x \rangle = \langle C_\omega f | \delta_x \rangle = (C_\omega f)(x) = \langle f | \omega_x \rangle.$$

By (13)

$$\langle C_\omega f | \mathbb{P} \delta_x \rangle = \langle f | C_\omega^\times \mathbb{P} \delta_x \rangle.$$

Hence, $\omega_x = C_\omega^\times \mathbb{P} \delta_x$.

Let us come back to the POV measure defined in the previous section. We adapt to our situation some known results concerning the POV measures defined by tight frames (e.g., [2, Section 3.2]). Let $\xi \in L^2(X, \mu)$ and Δ be a Borel subset of X. We define an operator $E(\Delta)$ with values in $L^2(X, \mu)$ by

$$(E(\Delta)\xi)(x) = \chi_\Delta(x) \xi(x), \quad \xi \in L^2(X, \mu).$$

This is clearly a PV measure.

Let $\xi, \eta \in C_\omega \mathcal{D} \subset L^2(X, \mu)$; then, there exist vectors $f, g \in \mathcal{D}$ such that $f = C_\omega^{-1} \mathbb{P} \xi$ and $g = C_\omega^{-1} \mathbb{P} \eta$.

$$\begin{aligned}
\langle \mathbb{P} E(\Delta) \mathbb{P} \xi | \eta \rangle_2 &= \langle E(\Delta) \mathbb{P} \xi | \mathbb{P} \eta \rangle_2 = \int_X \chi_\Delta(x) (C_\omega f)(x) \overline{(C_\omega g)(x)} d\mu \\
&= \int_\Delta (C_\omega f)(x) \overline{(C_\omega g)(x)} d\mu = \int_\Delta \langle f | \omega_x \rangle \langle \omega_x | g \rangle d\mu \\
&= \langle T(\Delta) f | g \rangle = \langle T(\Delta) C_\omega^{-1} \mathbb{P} \xi | C_\omega^{-1} \mathbb{P} \eta \rangle.
\end{aligned}$$

Thus,

$$(C_\omega^{-1})^\times T(\Delta) C_\omega^{-1} = \mathbb{P} E(\Delta) \mathbb{P}, \quad \forall \Delta \in \Sigma.$$

Hence, the POV measure T can be identified with the projection of a PV measure E on a larger rigged Hilbert space.

4 Parseval frames, coherent states, and quantization

Let ω be a Parseval distribution frame; this fact can be expressed equivalently as follows:

$$\langle f | g \rangle = \int_X \langle f | \omega_x \rangle \langle \omega_x | g \rangle d\mu, \quad f, g \in \mathcal{D}, \tag{6}$$

which, at least in the case when ω takes values in the Hilbert space, is called a resolution of the identity. This is a terminology more frequently used in Physics, particularly when dealing with coherent states that satisfy an equality corresponding to Equation 6 and some more conditions (in the classical formulation: saturation of the Heisenberg inequality, being eigenvectors of the annihilation operator, or being obtained by the action of the Weyl–Heisenberg group on some vacuum state). More general coherent states are often generated as orbits produced by a certain representation of a group (locally compact or Lie); these representations are supposed to be square-integrable. Non-square-integrable representations of groups can, however, also be envisaged (see [2, Ch.8] for a complete discussion). As already mentioned in the Introduction section, coherent states that are represented by non-square integrable functions or even by true distributions have also been considered in some applications. Thus, finally, it is not so exotic to take into account $\mathcal{D}^\times$-valued functions satisfying (15), that is, Parseval distribution frames.

The quantization procedure is an important aspect of coherent states. It is obtained by associating to a sufficiently regular function α defined on X with the operator A_α that, in our language, can be formally written as follows:

$$\langle A_\alpha f | g \rangle = \int_X \alpha(x) \langle f | \omega_x \rangle \langle \omega_x | g \rangle d\mu. \tag{7}$$

For discrete Parseval frames in Hilbert space, operators defined by obvious modifications of Equation 7 have been studied in [21, 22].

Finally, we remark that in the case of $\mathcal{H}$-valued maps, operators of type Equation 8 are closely related with the continuous frame multipliers considered by Balasz et al. in [23] (see also [24]).

Let us begin with an example.

Example 4.1: [15, Example 4.1] Let $\zeta : x \in X \to \zeta_x \in \mathcal{D}^\times$ be a Gel'fand distribution basis. Then, an operator A (type of diagonal operator) can be introduced as follows, starting from a (complex valued) measurable function α such that

$$\int_X |\alpha(x) \langle f | \zeta_x \rangle|^2 d\mu < \infty, \ \forall f \in \mathcal{D}.$$

Put

$$A f = \int_X \alpha(x) \langle f | \zeta_x \rangle \zeta_x d\mu, \quad f \in \mathcal{D}.$$

The assumptions imply that A maps $\mathcal{D}$ into $\mathcal{H}$ and it is a closable operator in $\mathcal{H}$. The domain of its closure $\overline{A}$ is

$$D(\overline{A}) = \left\{ f \in \mathcal{H} : \int_X |\alpha(x) (C_\zeta f)(x)|^2 d\mu < \infty \right\}.$$

The operator A is bounded if, and only if, $\alpha \in L^\infty(X, \mu)$. The spectrum $\sigma(\overline{A})$ is given by the closure of the essential range of α, that is, the set of $z \in \mathbb{C}$ such that

$$\mu\{x : |\alpha(x) - z| < \epsilon\} > 0, \quad \forall \epsilon > 0.$$

Moreover, if A and its adjoint A^* leave $\mathcal{D}$ invariant, for almost every $x \in X$, $\alpha(x)$ is a generalized eigenvalue of A, in the sense of Gel'fand: A has an extension to $\mathcal{D}^\times$, let us call it $\widehat{A}$, and, for almost every $x \in X$,

$$\left\langle \widehat{A}\zeta_x | g \right\rangle = \alpha(x) \langle \zeta_x | g \rangle, \quad \forall g \in \mathcal{D}.$$

A similar construction is possible by starting from a Riesz distribution map. For details, we refer to [12].

Let us now consider a more general situation. It is assumed that ω is a distribution map and we are given a measurable function $\alpha: X \to \mathbb{C}$ such that the sesquilinear form

$$\Omega_\alpha(f,g) = \int_X \alpha(x) \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu \tag{8}$$

is defined for all $f,g \in \mathcal{D}$. Let us suppose that there exists a continuous semi-norm p' such that

$$|\Omega_\alpha(f,g)| = \left| \int_X \alpha(x) \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu \right| \le p'(f) p'(g), \quad \forall f,g \in \mathcal{D}.$$

Then, there exists an operator $\Lambda_\alpha \in \mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ such that

$$\Omega_\alpha(f,g) = \langle \Lambda_\alpha f|g\rangle. \quad \forall f,g \in \mathcal{D}.$$

Let us assume that

$$\int_X |\alpha(x) \langle f|\omega_x\rangle|^2 d\mu < \infty \quad \forall f \in \mathcal{D},$$

and that ω is bounded Bessel. In this case, using the inequality Equation 1, we obtain the following:

$$\begin{aligned} |\Omega_\alpha(f,g)| &= \left| \int_X \alpha(x) \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu \right| \\ &\le \left(\int_X |\alpha(x) \langle f|\omega_x\rangle|^2 d\mu \right)^{\frac{1}{2}} \left(\int_X |\langle g|\omega_x\rangle|^2 d\mu \right)^{\frac{1}{2}} = K_f B^{\frac{1}{2}} \|g\|. \end{aligned} \tag{9}$$

From Equation 9, it follows that

$$\Lambda_\alpha f = \int_X \alpha(x) \langle f|\omega_x\rangle \omega_x d\mu, \quad f \in \mathcal{D}$$

is a vector in $\mathcal{H}$; for this reason, it is more convenient to adopt the notation $A := \Lambda_\alpha$. As ω is a bounded Bessel distribution map, the operators D_ω and C_ω are bounded, so in particular,

$$\|Af\|^2 \le B \|\alpha \langle f|\omega_\cdot\rangle\|_2^2 = B \int_X |\alpha(x) \langle f|\omega_x\rangle|^2 d\mu, \quad \forall f \in \mathcal{D}.$$

It is then natural to choose

$$D(A) := \left\{ f \in \mathcal{H} : \int_X |\alpha(x) \langle f|\omega_x\rangle|^2 d\mu < \infty \right\}.$$

In this case, the analysis operator C_ω is bounded and admits a bounded extension to $\mathcal{H}$, which is denoted again as C_ω. We look for the adjoint A^* of A. As is well known, the set $D(A^*)$ is given for all $g \in \mathcal{H}$ such that there exists $g^* \in \mathcal{H}$, for which

$$\langle Af|g\rangle = \langle f|g^*\rangle, \quad \forall f \in \mathcal{D}.$$

We have $\mathcal{D} \subset D(A^*)$ as $\langle Af|g\rangle = \int \alpha(x) \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu$, by the definition of the sesquilinear form Ω_α in Equation 8, and clearly,

$$g^* = \int \overline{\alpha(x)} \langle g|\omega_x\rangle \omega_x d\mu.$$

We now prove that

$$D(A^*) = \left\{ g \in \mathcal{H} : \int |\overline{\alpha(x)} (C_\omega g)(x)|^2 d\mu < \infty \right\},$$

and

$$A^* g = \int_X \overline{\alpha(x)} (C_\omega g)(x) \, d\mu, \quad g \in \mathcal{H}.$$

Indeed, recalling that we have identified $\int_X \alpha(x) \langle f|\omega_x\rangle \omega_x d\mu$ with $Af \in \mathcal{H}$, we have, for $\{g_n\} \subset \mathcal{D}$, $g_n \to g \in D(A^*)$

$$\begin{aligned} \langle Af|g\rangle &= \left\langle \int_X \alpha(x) \langle f|\omega_x\rangle \omega_x d\mu \mid g \right\rangle = \left\langle \int_X \alpha(x) \langle f|\omega_x\rangle \omega_x d\mu \mid \lim_{n\to\infty} g_n \right\rangle \\ &= \lim_{n\to\infty} \left\langle \int_X \alpha(x) \langle f|\omega_x\rangle \omega_x d\mu \mid g_n \right\rangle = \lim_{n\to\infty} \int_X \alpha(x) \langle f|\omega_x\rangle \langle \omega_x|g_n\rangle \, d\mu \\ &= \int_X \alpha(x) \langle f|\omega_x\rangle (C_\omega g)(x) \, d\mu, \end{aligned}$$

by the continuity of the inner product of $L^2(X,\mu)$.

In a similar way, we prove that

$$D(A^{**}) = \left\{ f \in \mathcal{H} : \int |\alpha(x) (C_\omega f)(x)|^2 d\mu < \infty \right\}$$

$$A^{**} f = \int_X \alpha(x) (C_\omega f)(x) \, d\mu, \quad f \in \mathcal{H}.$$

This also explicitly proves the statement about $\bar{A}$ given in [15, Example 4.1].

All this also applies when ω is a Parseval frame, but in this case, something more can be said. In particular, we can characterize the boundedness of the operator A.

Proposition 4.2: Let ω be a Parseval frame, and A the operator is defined by

$$\begin{aligned} D(A) &= \left\{ f \in \mathcal{H} : \int_X |\alpha(x) \langle f|\omega_x\rangle|^2 d\mu < \infty \right\} \\ Af &= \int_X \alpha(x) \langle f|\omega_x\rangle \omega_x d\mu, \quad f \in D(A). \end{aligned}$$

Let us assume that

$$\|Af\|^2 = \int_x |\alpha(x) C_\omega f|^2 d\mu, \quad \forall f \in D(A).$$

Then, A is bounded if, and only if, $\alpha \in L^\infty(X,\mu)$.

Proof: the sufficiency is obvious. Let us assume that A is bounded, and let $\overline{A}$ be its closure (which is defined everywhere in $\mathcal{H}$ and bounded). Let us assume that $\alpha \notin L^\infty(\mathbb{R})$. Then, for every $n \in \mathbb{N}$, the set $E_n = \{x \in \mathbb{R} : |\alpha(x)| > n\}$ has positive measure. Let χ_n denote the characteristic function of E_n. As ω is a Parseval frame, C_ω is an isometry of $\mathcal{H}$ into $L^2(X,\mu)$. The density of $\mathcal{D}$ in $\mathcal{H}$ implies that $C_\omega \mathcal{H}$ is an infinite dimensional separable Hilbert space; hence, there exists a unitary operator V from $L^2(X,\mu)$ to $C_\omega \mathcal{H}$; then, we can find an element $f_n \in \mathcal{H}$ such that $C_\omega f_n = V\chi_n$ and $\|f_n\| = \|V\chi_n\|_2 = \|\chi_n\|_2 = \mu(E_n)^{1/2}$. Then,

$$\|\overline{A} f_n\|^2 = \int_{\mathbb{R}} |\alpha(x)|^2 |(C_\omega f_n)(x)|^2 d\mu > n^2 \|f_n\|^2,$$

is a contradiction.

Proposition 4.2 allows us to get some information on the spectrum of the operator A.

Let us first show that the operator defined through the function $\frac{1}{\alpha(x)-\lambda}$, when defined almost everywhere, is the natural candidate to produce the inverse of the operator defined by $\alpha(x) - \lambda$. It is assumed that the function $h(x) := (\alpha(x) - \lambda)^{-1}$ is well defined and essentially bounded. Then, if $f \in \mathcal{D}$,

$$|\langle f|g\rangle| = \left|\int_X \frac{\alpha(x)-\lambda}{\alpha(x)-\lambda} \langle f|\omega_x\rangle \langle \omega_x|g\rangle \, d\mu\right| \leq \|(\alpha-\lambda)^{-1}\|_\infty \|(\alpha-\lambda)\langle f|\omega_\cdot\rangle\|_2 \|\langle \omega_\cdot|g\rangle\|_2 = \|(\alpha-\lambda)^{-1}\|_\infty \|(\alpha-\lambda)\langle f|\omega_\cdot\rangle\|_2 \|g\|.$$

This implies that the vector $\int_X (\alpha(x)-\lambda)^{-1} \langle f|\omega_x\rangle \omega_x d\mu$ is in $D(A-\lambda I)$ and the following equality holds:

$$\langle f|g\rangle = \left\langle (A-\lambda I) \int_X \frac{1}{\alpha(x)-\lambda} \langle f|\omega_x\rangle \omega_x d\mu |g \right\rangle.$$

Then, if $\frac{1}{\alpha(x)-\lambda} \in L^\infty(X,\mu)$, the resolvent operator $(A-\lambda I)^{-1}$ is well defined and bounded, which implies that there exists $M > 0$ such that

$$\mu\{x \in \mathbb{R} : |\alpha(x) - \lambda| < M^{-1}\} = 0.$$

In other words, if $\lambda \notin \text{Im}_{ess}\ \alpha$, then $\lambda \in \rho(A_\alpha)$. Equivalently,

$$\sigma(A) \subset \{z \in \mathbb{C} : \forall \epsilon > 0 \quad \mu\{x \in \mathbb{R} : |\alpha(x) - z| < \epsilon\} > 0\}.$$

Example 4.3: (the case of Riesz distribution bases, [15, Example 4.2] revisited) Let ω be a Riesz distribution basis and θ its dual. Let α be a (complex valued) measurable function such that

$$\int_X |\alpha(x) \langle f|\theta_x\rangle|^2 d\mu < \infty, \ \forall f \in \mathcal{D}.$$

A linear operator H on $\mathcal{D}$ can then be defined by

$$Hf = \int_X \alpha(x) \langle f|\theta_x\rangle \omega_x d\mu.$$

In addition, in this case, one can see that $Hf \in \mathcal{H}$ so that $H:\mathcal{D} \to \mathcal{H}$. Indeed, let us consider the sesquilinear form on $\mathcal{D} \times \mathcal{D}$:

$$\Omega(f,g) = \int_X \alpha(x) \langle f|\omega_x\rangle \langle \theta_x|g\rangle \, d\mu.$$

Then, as in Equation 7,

$$|\Omega(f,g)| \leq \left(\int_X |\alpha(x)\langle f|\omega_x\rangle|^2 d\mu\right)^{\frac{1}{2}} \left(\int_X |\langle g|\theta_x\rangle|^2 d\mu\right)^{\frac{1}{2}} \leq K_f B^{\frac{1}{2}} \|g\| \quad \forall g \in \mathcal{D}.$$

Hence, $\int_X \alpha(x) \langle f|\omega_x\rangle \theta_x d\mu$ can be identified with a vector in $\mathcal{H}$. Regarding the adjoint H^*, in similar way as before, we obtain

$$D(H^*) = \left\{g \in \mathcal{H} : \int_X |\overline{\alpha(x)} (C_\theta g)(x)|^2 d\mu < \infty\right\}.$$

$$H^* g = \int_X \overline{\alpha(x)} (C_\theta g)(x) \, d\mu.$$

Here, C_θ is (the extension of) the analysis operator corresponding to θ.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

CT: Formal Analysis, Conceptualization, Methodology, Writing – review and editing, Investigation, Writing – original draft. FT: Writing – review and editing, Writing – original draft, Investigation, Formal Analysis, Conceptualization, Methodology.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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