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Null Bertrand partner *D*-curves on spacelike surfaces

Tanju Kahraman*

Department of Mathematics, Manisa Celal Bayar University, Manisa, Türkiye

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*CORRESPONDENCE
Tanju Kahraman,
tanju.kahraman@cbu.edu.tr

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In this paper, by using the Darboux frame of null curves, we define null Bertrand partner *D*-curves and present the relations between curvatures of these curves in Minkowski 3-space E_1^3. In addition, we obtain some special results. Finally, by considering surface construction methods, we provide examples for null Bertrand partner *D*-curves in E_1^3.

KEYWORDS

null curve, Bertrand, string, spacelike surfaces, partner curves

1 Introduction

The associated curve of a given curve is a fascinating subject of differential geometry. So, finding such a curve is an interesting problem. Many geometers have investigated this problem in different spaces. The well-known examples of associated curves are Bertrand and Mannheim curves in the Euclidean 3-space. A Bertrand curve is a curve that shares its principal normal vectors with another curve and is characterized by the property that $\lambda\kappa + \mu\tau = 1$, where λ, μ are constants [1]. Similarly, Mannheim curves are special curves for which the principal normal of one of the curves is linearly dependent on the binormal vector of the other curve.

Considering the curves on surfaces is more interesting and provides an idea for defining new types of associated curves on surfaces. We note that a new type of Bertrand curve has been defined on surfaces and is called the Bertrand partner *D*-curves [2, 3]. In this definition, the authors have considered the Darboux frames of surface curves and obtained some characterizations of those curves.

Moreover, studying a concept of Euclidean space within Minkowski space is particularly interesting since the curves of this space are related to physics and the theory of relativity. A timelike curve corresponds to the path of an observer moving slower than the speed of light, a null curve corresponds to the observer moving at the speed of light, and a spacelike curve corresponds to an observer moving faster than light [4]. Particularly, null curves have extra importance since the classical relativistic string is a surface or world-sheet in Minkowski space, which satisfies the Lorentzian analog of the minimal surface equation [5]. Moreover, string equations are useful tools for simplifying the wave equation and a few additional simple equations. For instance, the solution of a two-dimensional (2D) wave equation shows that strings are related to null curve pairs, and if the string is open, it is related to a single null curve [5, 6].

In this paper, we define null Bertrand partner *D*-curves lying on spacelike surfaces and present characterizations for these associated null curves. We obtain relations between curvatures of null Bertrand partner *D*-curves. Finally, we provide some examples for null Bertrand partner *D*-curves in Minkowski 3-space E_1^3.

2 Preliminaries

The Minkowski 3-space E_1^3 is the real vector space IR^3 provided with the standard flat metric given by $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $v = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, and null (light-like) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ can locally be spacelike, timelike, or null (light-like) if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike, or null (light-like), respectively. For any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3, the Lorentz vector product of x and y is defined as follows:

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2),$$

where $\delta_{ij} = \begin{cases} 1 \; i = j, \\ 0 \; i \neq j, \end{cases}$ $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = -e_2$ [7, 8].

$\{l, n, u\}$ is used to denote the moving frame along the null curve $\alpha(s)$ in E_1^3. For an arbitrary null curve $\alpha(s)$, the following Frenet formulas are given

$$\begin{bmatrix} l' \\ n' \\ u' \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_1 \\ 0 & 0 & -k_2 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} l \\ n \\ u \end{bmatrix},$$

where $\langle l, l\rangle = \langle n, n\rangle = \langle l, u\rangle = \langle u, n\rangle = 0$, $\langle u, u\rangle = \langle l, n\rangle = 1$ and "$'$" denotes the derivative with respect to the arc length parameter s [9].

Similar to the curves, a surface in E_1^3 can be timelike or spacelike. A surface $S : U \subset \mathbb{R}^2 \to E_1^3$ is called a timelike (spacelike) surface if the induced metric on the surface is a Lorentz metric (positive definite Riemannian metric); i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector, where U is an open set in $\mathbb{R}^2$ [10].

Let S be a spacelike surface in E_1^3 defined on an open set $U \subset \mathbb{R}^2$, and let us consider a null curve $\alpha(s)$ on S with Frenet frame $\{T, N, B\}$. Since $\alpha(s)$ lies on S, there exists another frame along $\alpha(s)$, which is called the Darboux frame of $\alpha(s)$ and is denoted by $\{T, U, V\}$. In this frame, T is the unit tangent of the curve, U is the unit normal of the surface S along $\alpha(s)$, and V is the unique vector obtained by

$$V = \frac{1}{\langle X, T\rangle} \left\{ X - \frac{\langle X, X\rangle}{2\langle X, T\rangle} T \right\}, \; X \in T_{\alpha(t)}S, \; \langle X, T\rangle \neq 0,$$

where

$$\langle T, T\rangle = \langle V, V\rangle = \langle T, U\rangle = \langle V, U\rangle = 0, \; \langle T, V\rangle = \langle U, U\rangle = 1. \quad (1)$$

Therefore, the Darboux formula of the moving frame is

$$\begin{bmatrix} T' \\ V' \\ U' \end{bmatrix} = \begin{bmatrix} k_g & 0 & k_n \\ 0 & -k_g & \tau_g \\ -\tau_g & -k_n & 0 \end{bmatrix} \begin{bmatrix} T \\ V \\ U \end{bmatrix}. \quad (2)$$

In these formulas, k_g, k_n, and τ_g are called the geodesic curvature, the normal curvature, and the geodesic torsion, respectively. Henceforth, we use "quote" to denote the derivative with respect to the arc length parameter of $\alpha(s)$ [9, 11].

3 Null Bertrand partner D-curves on spacelike surfaces in E_1^3

In this section, by considering the Darboux frame of null curves, we define null Bertrand partner D-curves and provide the characterizations of these curves in E_1^3.

Definition 1: *Let S_1 and S_2 be oriented spacelike surfaces in E_1^3, and let us consider the unit-speed null curves $\alpha_1(s_1)$ and $\alpha_2(s_2)$ lying fully on S_1 and S_2, respectively. The Darboux frames of null curves $\alpha_1(s_1)$ and $\alpha_2(s_2)$ are denoted by $\{T_1, U_1, V_1\}$ and $\{T_2, U_2, V_2\}$, respectively. If there exists a corresponding relationship between the curves α and α_1 such that at the corresponding points of the curves, the Darboux frame element U_1 of α_1 coincides with the Darboux frame element U_2 of α_2, then α_1 is called a null Bertrand D-curve and α_2 is called a null Bertrand partner D-curve of α_1. Then, the pair $\{\alpha_1, \alpha_2\}$ is said to be a null Bertrand D-pair.*

Theorem 1: *Let S_1 and S_2 be oriented spacelike surfaces in E_1^3, and let null curves $\alpha_1(s_1)$ and $\alpha_2(s_2)$ with non-zero normal curvatures k_{n_1} and k_{n_2} lie on S_1 and S_2, respectively. Then, $\alpha_1(s_1)$ and $\alpha_2(s_2)$ are null Bertrand partner D-curves if and only if the following equality holds:*

$$k_{n_2}^2 \left(\frac{ds}{ds_1} \right)^4 = k_{n_1}^2. \quad (3)$$

Proof. Suppose that the pair $\{\alpha_1, \alpha_2\}$ is a null Bertrand D-pair. The Darboux frames of $\alpha_1(s_1)$ and $\alpha_2(s_2)$ are denoted by $\{T_1, U_1, V_1\}$ and $\{T_2, U_2, V_2\}$, respectively. Then, by the definition, we can assume that

$$\alpha_2(s_2) = \alpha_1(s_1) + \lambda(s_1) U_1(s_1), \quad (4)$$

for some smooth function $\lambda(s_1)$. By taking the derivative of Equation 4 with respect to s_1 and applying the Darboux Formula 2, we obtain

$$T_2 \frac{ds_2}{ds_1} = \left(1 - \lambda\tau_{g_1}\right) T_1 - \lambda k_{n_1} V_1 + \lambda' U_1. \quad (5)$$

Since the direction of U_1 coincides with the direction of U_2, the inner product of Equation 5 with U_1 yields

$$\lambda'(s_1) = 0. \quad (6)$$

Thus, λ is a non-zero constant. Now, equality Equation 5 can be written as

$$T_2 \frac{ds_2}{ds_1} = \left(1 - \lambda\tau_{g_1}\right) T_1 - \lambda k_{n_1} V_1. \quad (7)$$

Taking the inner product of Equation 7 with itself, we obtain

$$0 = 2k_{n_1}\left(1 - \lambda\tau_{g_1}\right). \quad (8)$$

From Equation 8, we obtain

$$\tau_{g_1} = \frac{1}{\lambda}. \quad (9)$$

Therefore, Equation 7 can be written as follows:

$$T_2 \frac{ds_2}{ds_1} = -\lambda k_{n_1} V_1. \quad (10)$$

By taking the derivative of Equation 10, we obtain

$$\left(k_{g_2}\left(\frac{ds_2}{ds_1}\right)^2+\frac{d^2s_2}{ds_1^2}\right)T_2+k_{n_2}\left(\frac{ds_2}{ds_1}\right)^2U_2=-k_{n_1}U_1+\left(-\lambda k_{n_1}'+\lambda k_{n_1}k_{g_1}\right)V_1, \tag{11}$$

and taking the inner product of Equation 11 with itself, we obtain

$$k_{n_2}^2\left(\frac{ds_2}{ds_1}\right)^4=k_{n_1}^2, \tag{12}$$

which yields Equation 3.

Conversely, we assume that Equation 3 holds. For a non-zero constant λ, we define a curve as

$$\alpha_2(s_2)=\alpha_1(s_1)+\lambda U_1(s_1). \tag{13}$$

We will prove that α_1 is a null Bertrand D-curve and that α_2 is the null Bertrand partner D-curve of α_1. By taking the derivative of Equation 13 with respect to s_1 twice, we obtain

$$T_2\frac{ds_2}{ds_1}=\left(1-\lambda\tau_{g_1}\right)T_1-\lambda k_{n_1}V_1, \tag{14}$$

and

$$\begin{aligned}\left(k_{g_2}\left(\frac{ds_2}{ds_1}\right)^2+\frac{d^2s_2}{ds_1^2}\right)T_2+k_{n_2}\left(\frac{ds_2}{ds_1}\right)^2U_2=&\left(-\lambda\tau_{g_1}'+k_{g_1}-\lambda k_{g_1}\tau_{g_1}\right)T_1\\&+\left(-\lambda k_{n_1}'+\lambda k_{n_1}k_{g_1}\right)V_1\\&+\left(k_{n_1}-2\lambda k_{n_1}\tau_{g_1}\right)U_1\end{aligned}, \tag{15}$$

respectively. Taking the cross-product of Equation 15 and Equation 14, we obtain

$$k_{n_2}\left(\frac{ds_2}{ds_1}\right)^3V_2=\lambda k_{n_1}^2\left(1-2\lambda\tau_{g_1}\right)T_1. \tag{16}$$

Without loss of generality, taking the inner product of Equation 14 with itself yields $\tau_{g_1}=\frac{1}{\lambda}$. Thus, Equation 14 can be written as

$$T_2\frac{ds_2}{ds_1}=-\lambda k_{n_1}V_1. \tag{17}$$

Finally, the cross-product of Equation 16 and Equation 17 shows that the Darboux frame element U_1 of α_1 coincides with the Darboux frame element U_2 of α_2 at the corresponding points of the curves; i.e., the curves α_1 and α_2 are null Bertrand D-pair curves.

Theorem 2 has the following corollaries.

Corollary 1: *The distance between the corresponding points of null Bertrand curves is constant and is given by* $\lambda=\frac{1}{\tau_{g_1}}$.

Corollary 2: *Let the pair* $\{\alpha_1,\alpha_2\}$ *be a null Bertrand D-pair. Then, the geodesic torsion of* α_1 *is a non-zero constant and is given by* $\tau_{g_1}=\frac{1}{\lambda}$.

Corollary 3: *There is no null Bertrand D-curve* α *that is a principal line; i.e.,* $\tau_g=0$.

Theorem 2: *Let* $\alpha_1(s_1)$ *and* $\alpha_2(s_2)$ *be null Bertrand partner D-curves with non-zero normal curvatures* k_{n_1} *and* k_{n_2}*, respectively. Then,*

$$k_{n_1}=k_{n_2}\left(1+2\lambda\tau_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2. \tag{18}$$

Proof. Based on the definition, we can assume that

$$\alpha_1(s_1)=\alpha_2(s_2)-\lambda U_2(s_2) \tag{19}$$

for a non-zero constant λ. By taking the derivative of Equation 19 with respect to s_1 twice and applying the Darboux formulas, we obtain

$$T_1=\left(1+\lambda\tau_{g_2}\right)\frac{ds_2}{ds_1}T_2+\lambda k_{n_2}\frac{ds_2}{ds_1}V_2 \tag{20}$$

and

$$\begin{aligned}T_1'&=k_{g_1}T_1+k_{n_1}U_1\\&=\left[\left(\lambda\tau_{g_2}'+k_{g_2}+\lambda k_{g_2}\tau_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2+\left(1+\lambda\tau_{g_2}\right)\frac{d^2s_2}{ds_1^2}\right]T_2\\&\quad+\left[\left(\lambda k_{n_2}'-\lambda k_{n_2}k_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2+\lambda k_{n_2}\frac{d^2s_2}{ds_1^2}\right]V_2\\&\quad+k_{n_2}\left(1+2\lambda\tau_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2U_2\end{aligned}, \tag{21}$$

respectively. By substituting Equation 20 into Equation 21, we obtain

$$\begin{aligned}&k_{g_1}\left(1+\lambda\tau_{g_2}\right)\frac{ds_2}{ds_1}T_2+\lambda k_{g_1}k_{n_2}\frac{ds_2}{ds_1}V_2+k_{n_1}U_1\\&=\Bigg[\left(\lambda\tau_{g_2}'+k_{g_2}+\lambda k_{g_2}\tau_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2\\&\quad-k_{g_1}\left(1+\lambda\tau_{g_2}\right)\frac{ds_2}{ds_1}+\left(1+\lambda\tau_{g_2}\right)\frac{d^2s_2}{ds_1^2}\Bigg]T_2\\&\quad+\Bigg[\left(\lambda k_{n_2}'-\lambda k_{n_2}k_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2\\&\quad-\lambda k_{n_2}k_{g_1}\frac{ds_2}{ds_1}+\lambda k_{n_2}\frac{d^2s_2}{ds_1^2}\Bigg]V_2\\&\quad+k_{n_2}\left(1+2\lambda\tau_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2U_2\end{aligned}. \tag{22}$$

Since $\alpha_1(s_1)$ and $\alpha_2(s_2)$ are null Bertrand partner D–curves, we obtain the desired equation

$$k_{n_1}=k_{n_2}\left(1+2\lambda\tau_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2. \tag{23}$$

Theorem 3: *Let* α_1 *and* α_2 *be null Bertrand partner D–curves lying on surfaces* S_1 *and* S_2*, respectively. Then, the geodesic torsion of* α_2 *is constant and is given by* $\tau_{g_2}=\frac{-1}{\lambda}$.

Proof. From (Equation 3), we obtain

$$\left(\frac{ds_2}{ds_1}\right)^2=\mp\frac{k_{n_1}}{k_{n_2}}, \tag{24}$$

and by substituting Equation 24 into Equation 23, we obtain

$$\tau_{g_2}=\frac{-1}{\lambda}. \tag{25}$$

From corollary 4 and theorem 7, we have the following corollary.

Corollary 4: *The relationship between the geodesic torsions of null Bertrand partner D–curves* α_1 *and* α_2 *is given by*

$$\tau_{g_1}=-\tau_{g_2}=\frac{1}{\lambda}.$$

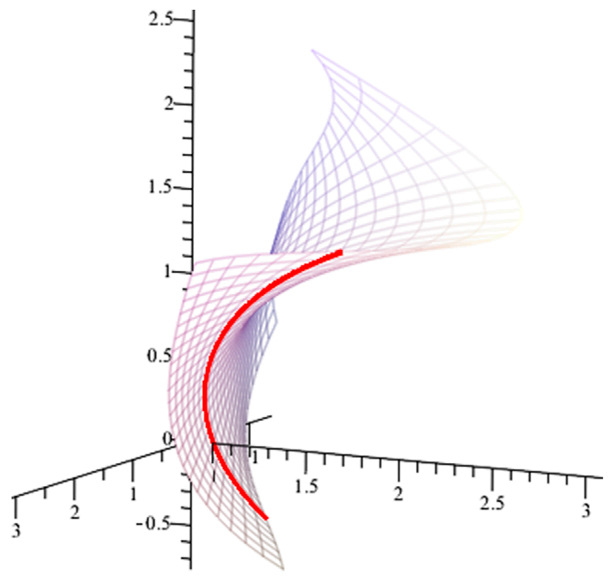

FIGURE 1
Surface $S_1(s,v)$ and geodesic null curve $\alpha_1(s)$.

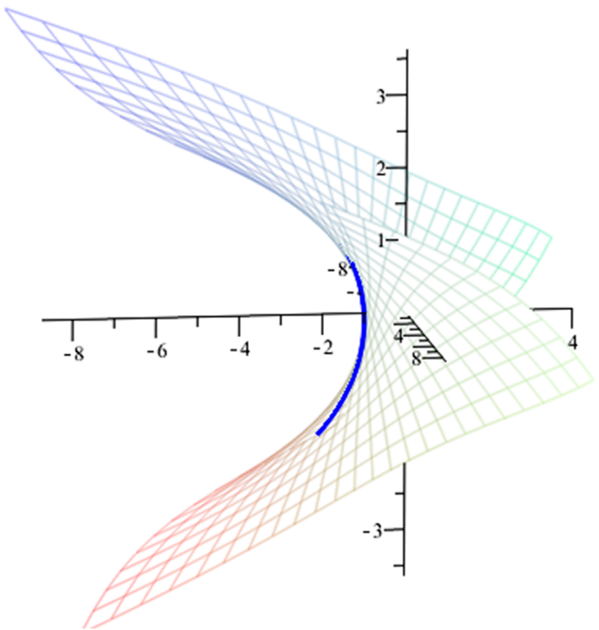

FIGURE 2
Surface $S_2(s,v)$ and geodesic null Bertrand partner D–curve $\alpha_2(s)$.

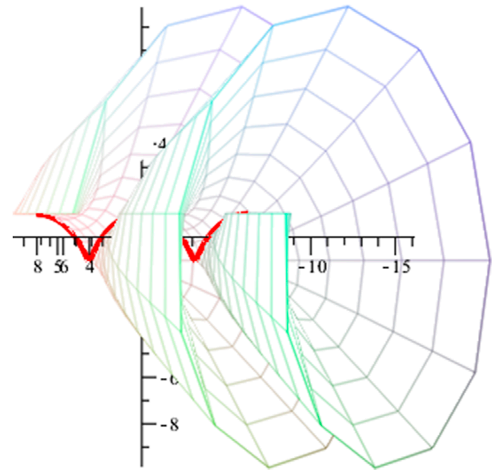

FIGURE 3
Surface $S_3(s,v)$ and null geodesic curve $\alpha_3(s)$.

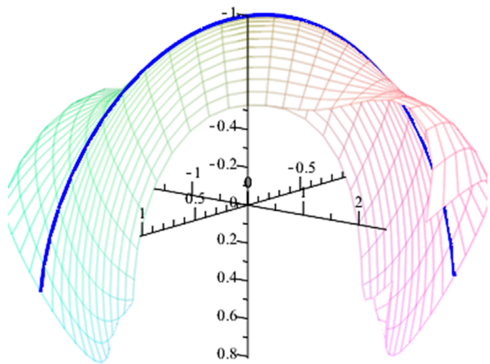

FIGURE 4
Surface $S_4(s,v)$ and null geodesic Bertrand partner D–curve $\alpha_4(s)$.

Corollary 5: *Let α_1 and α_2 be null Bertrand partner D–curves. Then, the curvatures of α_1 and α_2 hold*

$$\left(\lambda k'_{n_2} - \lambda k_{n_2} k_{g_2}\right)\left(\frac{ds_2}{ds_1}\right)^2 = \lambda k_{n_2}\left(k_{g_1}\frac{ds_2}{ds_1} - \frac{d^2 s_2}{ds_1^2}\right). \tag{26}$$

Proof. It is proven based on Equation 22.

Corollary 6: *Let α_1 and α_2 be null Bertrand partner D–curves. Then, α_1 and α_2 are geodesic null Bertrand partner D–curves on S_1 and S_2 if and only if*

$$k'_{n_2}\left(\frac{ds_2}{ds_1}\right)^2 + \frac{d^2 s_2}{ds_1^2} k_{n_2} = 0.$$

Proof. The proof is clear from Equation 26.

4 Examples

In this section, we provide some examples of null Bertrand partner D–curves. For this purpose, we use a method related to the construction of spacelike surfaces [12].

Example 1: *Let us consider the null curve* $\alpha_1(s) = (\sinh s, \cosh s, s)$. *Then, by using the method proposed by* [12], *the spacelike surface* S_1 *containing* $\alpha_1(s)$ *as a geodesic is obtained as* $S_1(s,v) = (A_1(s,v), B_1(s,v), C_1(s,v))$, *where*

$$A_1(s,v) = \sinh s\,(1+\sin(sv)) - \frac{1}{2}\sinh v \cosh s,$$
$$B_1(s,v) = \cosh s\,(1+\sin(sv)) - \frac{1}{2}\sinh s \sinh v,$$
$$C_1(s,v) = s + \frac{1}{2}\sinh v,$$

where $-0.5 \le s, v \le 1.5$ (Figure 1). Then, by using the Darboux frame components, the Bertrand partner D–curve α_2 of α_1 is obtained as

$$\alpha_2(s) = (-\sinh s, -\cosh s, s).$$

Then, we can construct a spacelike surface $S_2(s,v)$ with null geodesic α_2 as $S_2(s,v) = (A_2(s,v), B_2(s,v), C_2(s,v))$, where

$$A_2(s,v) = -\sinh s(1+\sin(sv)) - \cosh s \sinh v,$$
$$B_2(s,v) = -\cosh s(1+\sin(sv)) - \sinh s \sinh v,$$
$$C_2(s,v) = s + \sinh v,$$

and $-1.5 \le s, v \le 1.5$ (Figure 2).

Example 2: *Let* $\alpha_3(s) = (s, \sin s, \cos s)$ *be a null curve. Similarly, by using the method described by* [12], *the surface* $S_3(s,v)$ *containing* $\alpha_3(s)$ *as a geodesic is constructed as* $S_3(s,v) = (A_3(s,v), B_3(s,v), C_3(s,v))$, *where*

$$A_3(s,v) = s + \sin v - v^2,$$
$$B_3(s,v) = \sin s + \sin v \cos s + v^2 \cos s,$$
$$C_3(s,v) = \cos s - \sin v \sin s - v^2 \sin s,$$

and $-2\pi \le s \le 2\pi$ and $-\pi \le v \le \pi$ (Figure 3). Then, by using the Darboux frame, the Bertrand partner D–curve α_4 of α_3 is obtained as

$$\alpha_4(s) = (s, -\sin s, -\cos s).$$

Now, the surface $S_4(s,v)$ containing α_4 as a geodesic is constructed as $S_4(s,v) = (A_4(s,v), B_4(s,v), C_4(s,v))$, where

$$A_4(s,v) = s - \frac{1}{2}\sin(s^2 v),$$
$$B_4(s,v) = -\sin s - \frac{1}{2}\cos s \sin(s^2 v) + \sin s \sin v,$$
$$C_4(s,v) = -\cos s + \frac{1}{2}\sin s \sin(s^2 v) + \cos s \sin v,$$

and $-2 \le s \le 2$ and $-1 \le v \le 1$ (Figure 4).

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

TK: Writing – original draft.

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