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History of mathematics as context for improving strategy flexibility in cubic equations

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Studies show that using original historical sources as a context for teaching problem-solving enhance students problem-solving skills and mathematical understanding. Despite, the extent to which it can improve students' strategy flexibility in problem-solving is not well reported. This paper present quasi-experimental study that sought to examine whether history-integrated lessons on cubic equations improves senior high school students' strategy flexibility. Participants were 128 Form 2 science students from two senior high schools in the Ashanti Region of Ghana, assigned to an experimental group or a control group. The experimental group was taught original historical approaches to solving cubic equations. Lessons emphasised the algebraic methods developed by early mathematicians such as del Ferro, Tartaglia, and Cardano. The control group, on the other hand, were taught the standard algebraic methods such as rational root test and factorisation. Afterwards (i) post-test data was collected using strategy flexibility test and was analysed to identify if there was statistically significant difference between the flexibility scores of the control and experimental groups; and (ii) qualitative data was collected from selected students using an interview to explain the reasons behind the difference observed within the post-test data. Findings showed that the experimental group had improved flexible and innovative first-step strategy use than the control group. Thematic analysis of the qualitative data revealed three themes that support the observed difference in strategy flexibility (i) expansion of strategy repertoire, where students reported that they now have knowledge of alternative methods for solving cubic equations, (ii) situational strategy selection where students reported of having improved ability to select strategies appropriate to a given equation structure, and (iii) motivational dispositions where students reported of becoming interested, enjoyed, engaged and confident. The results suggest that teaching students' non-linear equation-solving by history-integration can meaningfully enhance strategy flexibility. These findings align with the theory of adaptive expertise and findings of existing empirical studies in the history and pedagogy of mathematics. The study recommends that teachers should teach problem-solving in algebra using the historical context of concepts. The study is limited in the sense that participants were assigned to groups non-randomly, lessons were short in duration, and sample was not larger enough. Future research should employ more rigorous designs, larger samples, extended lesson periods, and follow-up assessments to evaluate long-term retention and transfer of strategy flexibility.

KEYWORDS

adaptive expertise, cubic equations, history of mathematics, algebra, strategy flexibility

Introduction

Research in mathematics education has found strategy flexibility an important component of mathematics expertise (Verschaffel, 2024). Strategy flexibility is defined as student's ability to select from among different strategies the one that best produces the solution to a given mathematics problem or task (Blöte et al., 2001; Hästö et al., 2019; Heinze et al., 2009; Huntley et al., 2007; Maciejewski and Star, 2016; Maciejewski and Star, 2019; Rittle-Johnson and Star, 2007; Rittle-Johnson and Star, 2009; Star and Rittle-Johnson, 2008; Shaw et al., 2020; Star et al., 2009; Star et al., 2015; Star et al., 2022; Torbeyns et al., 2009; Xu et al., 2017). It is understood not as an innate ability of students (Parikka et al., 2022), which suggest that it is a dynamic skill that can improve intentionally with mindful teaching and purposeful practice (Maciejewski and Star, 2016; Rittle-Johnson et al., 2012). Previous studies have found that students become more flexible in using strategies when they engage with non-routine problems or compare multiple solution methods for routine tasks (e.g., Arslan and Yazgan, 2015; Rittle-Johnson et al., 2012; Star and Rittle-Johnson, 2008; Maciejewski and Star, 2016). Research further indicates that flexibility can be cultivated through exposure to diverse strategies (Rittle-Johnson et al., 2012), structured comparisons of worked examples (Newton et al., 2010; Rittle-Johnson and Star, 2007), explicit prompts and direct instruction (Star and Rittle-Johnson, 2008), and collaborative learning opportunities (Mercier and Higgins, 2013). This body of research highlights that teachers can cultivate students' strategy flexibility in mathematics through providing stimulating learning experiences.

Using the history of mathematics is found as one of the ways of motivating students mathematics learning. It has received remarkably little attention in flexibility research, even though it has been discussed as powerful context for problem-solving teaching and learning in mathematics (e.g., McGinn and Boote, 2003; Meavilla and Flores, 2007; Rizos and Gkrekas, 2023; Swetz, 1986; Zimmermann, 2004). Studies suggest that history of mathematics is a resource for a long-term study of problem-solving processes and mathematics heuristics (e.g., Zimmermann, 2004). According to Swetz (1986), teaching students problem-solving through original sources in the history of mathematics "…helps to demonstrate problem-solving strategies and sharpen mathematics skills." For this reason, we can argue that the context of the history of mathematics can provide fertile grounds for nurturing strategy flexibility.

Meanwhile, none of the existing papers had reported exact findings on improving flexibility by teaching original sources in the history of mathematics. Against this odd, the present study investigates how teaching students by exposing them to earlier methods for solving equations might foster strategy flexibility. This direction is particularly novel, as most empirical studies in mathematical flexibility has mostly leaned towards linear equation solving even without historical integration (e.g., Schneider et al., 2011; Xu et al., 2017). Much remains unknown about how students improve on strategy flexibility across more advanced or higher degree polynomial equations when history of mathematics is used as teaching context. Hence, this study seeks to answer the research question "Can integrating the history of mathematics into teaching enhance students" strategy flexibility in solving higher-degree non-linear equations? To answer this question, we used the historical context of the history of cubic equations. This helped address our gap within a traditionally understudied instructional concept of cubic equations where intervention studies are hard to come by (e.g., Baki and Guven, 2009), despite it been rich content area of algebra.

Literature review

Strategy flexibility

The position of the National Council of Teachers of Mathematics (NCTM) on strategy flexibility is firm. As cited in Star et al. (2022), the NCTM (2014) has emphasised that "All students need to have a deep and flexible knowledge of a variety of procedures, along with an ability to make critical judgments about which procedures or strategies are appropriate for use in particular situations." In addition to this, research has seen more papers published around strategy flexibility in different content areas of mathematics such as algebra and calculus (e.g., Maciejewski and Star, 2016; Maciejewski and Star, 2019; Rittle-Johnson and Star, 2007; Shaw et al., 2020; Star et al., 2009; Star et al., 2022; Torbeyns et al., 2009; Xu et al., 2017). Within the research area of algebra, flexibility in performing arithmetic tasks and linear equation solving has been dominating (e.g., Hickendorff, 2022; Jóelsdóttir and Andrews, 2025; Star et al., 2022; Xu et al., 2017). Among different demographic groups and geographical locations of students, the concept has also been studied (Verschaffel, 2024). The literature review by Verschaffel (2024) showed that though more has been done about the construct of strategy flexibility, there are "many aspects of it that are still not well-understood" and that there is the need for more studies.

Theoretical explanation of strategy flexibility has been underpinned on the theory of adaptive expertise (Hatano and Inagaki 1986). It is not only in this study that you would see this but in the broader literature on the concept of strategy flexibility in mathematics (e.g., Baroody and Dowker, 2003; Baroody, 2013; Hatano and Oura, 2003; Heinze et al., 2009). Adaptive expertise theory originally distinguishes between routine expertise and adaptive expertise. According to Hatano and Inagaki (1986), routine expertise is characterised by efficient implementation of well-practiced procedures and adaptive expertise on the other hand involves the ability to modify or generate strategies in response to new problem situations (Baroody, 2013; Baroody and Dowker, 2003). Based on the theory, adaptive experts are not just fast or more accurate when they execute procedures but they also understand when and why to use particular strategies and can adapt their approach when confronted with unfamiliar or complex tasks (Schwartz et al., 2005; Crawford et al., 2005).

Within this theoretical lens, strategy flexibility in mathematics aligns closely with the adaptive dimension of expertise. Baroody and Dowker (2003) identifies flexibility as the underlying principle of adaptive expertise because it requires coordinating conceptual understanding with procedural knowledge to make informed strategic decisions. Star (2007) also characterises strategic flexibility as a form of deep procedural knowledge, in which learners know multiple strategies and can evaluate their relative appropriateness. This perspective is supported empirically by studies showing that flexible strategy use correlates with richer conceptual understanding and more sophisticated procedural knowledge (McMullen et al., 2017; Rittle-Johnson et al., 2012; Schneider et al., 2011). Thus, strategy flexibility is not merely about knowing and choosing multiple strategies but the ability to discern which one best fits a given situation (Star et al., 2022).

Central to the above perspective is the idea of situational appropriateness which concerns the ability to choose, from one's repertoire of strategies, the one that aligns best with the structure and demands of the problem situation (Dover and Shore, 1991; Star and Newton, 2009). Yet determining what strategy counts as "most appropriate" is often subtle and context-dependent. In the perspective of Verschaffel et al. (2009), appropriateness may reflect efficiency (fewest steps or quickest execution), reliability (the method least prone to error), or, in more advanced mathematical thinking, "elegance," which is a valued but difficult-to-define aesthetic criterion. Thus, while the importance of flexible, informed strategy choice is well emphasised, identifying and enacting situationally appropriate strategy is very difficult part of mathematics education. Hence the need for more intervention research into this area as Verschaffel (2024) has recommended.

Strategy flexibility in equation solving

Algebra has proven to be a fruitful domain for research on strategy flexibility and the trend continues recently (e.g., Star et al., 2015; Star et al., 2022; Xu et al., 2017). The focus on algebra make sense, since many view it as the first point at which students experience the abstract thinking that underly the power of mathematics (Susac et al., 2014). Much of what we know of strategy flexibility here is based on a surprisingly set of simple algebraic problem-solving tasks particularly involving arithmetic and linear equation solving (e.g., Hopkins et al., 2025; Lemaire and Lecacheur, 2010; Newton et al., 2010; Star and Seifert, 2006; Star et al., 2022; Torbeyns et al., 2009; Xu et al., 2017; Threlfall, 2009; Hickendorff, 2022). In many mathematics curricula around the world, students are introduced to linear equation solving the most at all times and levels as the path to transition to complex algebraic reasoning and problem-solving (Otten et al., 2019). It has been reported that the task of solving linear equations is core to algebraic understanding and rather remains difficult aspect of algebraic thinking for pretertiary students (Wati and Fitriana, 2018), hence the need for strategic flexibility has prompted substantial research attention so far. Researchers have thus gravitated toward linear equation-solving, a domain where a standard algorithm is universally taught and derivations from it are easy to identify, which also make this concept so ripe for flexibility research.

From the above point, it follows that given linear equations, standard algorithm(s) exist but most at times there are other alternative strategies that are arguably more optimal (Buchbinder et al., 2015; Star and Seifert, 2006). Decades of research have shown that students often default to familiar procedures even when more efficient or elegant alternatives exist (e.g., Newton et al., 2010). Studies comparing students' use of inverse-operations, balancing, and intuitive structural shortcuts (e.g., Star and Newton, 2009; Star et al., 2015) have generated important insights precisely because linear equations are simple, well-structured, and cognitively manageable. Research in this area (e.g., Lemaire and Siegler, 1995; Star et al., 2015) has illuminated important principles about flexibility, but tasks involving linear equation solving remain far removed from the demands of complex algebraic reasoning. While this evidence forms the backbone of what is currently known about strategy flexibility in mathematics, it reflects mathematical situations that are among the least complex that students encounter.

The focus of strategy flexibility on narrow algebra tasks such as linear equation solving becomes even more evident when we consider what is missing in research on solving higher-order polynomial equations. Even quadratic equations (with degree of 2) which are possibly the first form of non-linear equations that offer meaningful strategic variation, appear only occasionally in the literature on strategy flexibility (e.g., Ernvall-Hytönen et al., 2024; Küttel et al., 2025), despite the wealth of different strategies they support when solving them (e.g., factoring, completing the square, or applying the quadratic formula). Beyond quadratic equations, the literature remains almost silent. Equations of the form of cubic or quartic, systems of nonlinear equations, or more structurally intricate polynomial equations rarely appear in strategy flexibility literature. Solving these equations demand deeper structural thinking and offer multiple, truly variable strategies that could serve to be precisely the kind of problem-solving environments where strategy flexibility should matter most. Taken together, the existing literature paints a picture of a research field that has generated valuable conclusions, but largely within highly simplified algebraic contexts. Linear equations and arithmetic tasks have provided a productive starting point, but they represent only a narrow slice of the algebraic terrain students encounter at the pretertiary level. The relative absence of studies involving higher-order polynomial equations suggests an important opportunity to extend the research on strategy flexibility into the richer, more complex algebraic spaces where mathematics structure becomes more nuanced, strategies proliferate, and choices matter in genuinely consequential ways.

History of mathematics and problem-solving

Problems and problem-solving lie at the heart of the history of mathematics (Ernest, 1998; Torres-Peña et al., 2024). Thus, the intersection between history and problem-solving flexibility can be argued for. As echoed in the experiences of McGinn and Boote (2003) in their self-study with solving questions with ancient mathematics sources such as Plimpton 322. At first glance they perceived such problem seemed computationally trivial but upon deeper engagement with it, it revealed an underlying pedagogical purpose. The value in the problem lies "less in obtaining an answer" and more in experiencing a range of methods and understanding the process of solving it. This self experience of McGinn and Boote also suggest that when learners solve a problem using ancient methods alongside modern ones, they gain "greater appreciation for each of the methods" and begin to see how different strategies can coexist, complement one another, or illuminate the problem from distinct angles. Importantly, these comparisons allowed students to recognise precursors of modern methods in earlier methods, therefore helping them to "learn about solving problems in mathematics" (De Vittori, 2022) and to use strategies flexibly across related problems.

The study by Rizos and Gkrekas (2023) also share much insights about how problem-solving in the context of historical sources could lead to flexible skills. In their study, when students encountered historically grounded, open-ended problems such as the classical "hundred fowls" problem," they were compelled to recognise that mathematics problems do not always possess a single prescribed method or a uniquely "appropriate" solution strategy. Students exposed to these historical problems "understood, at least to some extent, that a problem can have different solutions" and began to lose their certainty that mathematics is strictly about fixed procedures that leads to predetermined results (Fauvel, 1991). Instead, the historical context acted as "an environment that can activate the students strategic thinking and in parallel give them the ability to come out with their different strategies." In the study by Zimmermann (2004), when students analyse these historical methods, they gain insight into heuristics that are broader and more adaptable than the narrow procedural methods typically emphasised in standard textbooks.

Swetz (1986) further emphasises that solving problems in historical context foster problem-solving skills precisely because they reveal both the continuity and evolution of mathematics methods. Students got the chance to investigate how mathematicians across cultures and centuries approached similar problems in different ways. As a result, they get a richer sense of how strategies emerged, adapted, and sometimes became obsolete. As Meavilla and Flores (2007) notes, comparing modern solution processes with original ones "illustrates the evolution of solution processes," which prompts students to question their default methods and consider alternative approaches. Swetz (1986) example of Liu Hiu's *Sea Island Mathematical Classic* demonstrates this vividly that problems that today fall under trigonometry were historically solved using proportions. Such contrasts invite students to explore why a strategy works, when it is efficient, and how different representations can lead to viable, even elegant solutions. This could become that precise comparison that activates flexible strategy thinking as students come to see that they "could be innovative" in problem solving, not bound to the teacher taught methods alone.

Research on strategy flexibility in algebra has demonstrated that students often rely on familiar procedures even when alternative strategies are available (Star and Seifert, 2006; Newton et al., 2010; Star et al., 2015). Exposing students to historical methods may offer a distinctive pathway for addressing this issue. When students engage with solution methods developed in different historical contexts, they are exposed to strategies grounded in alternative representations, assumptions, and problem structures (Swetz, 1986; Meavilla and Flores, 2007). This exposure changes the idea of "just" solving equations towards identifying the structure inherent in equations as learners must interpret the mathematical logic underlying each method rather than merely apply a memorised algorithm. Studies drawing on historical problem-solving contexts show that such engagement encourages learners to compare methods, identify structural similarities and differences, and evaluate the efficiency and suitability of strategies across problems (McGinn and Boote, 2003; Rizos and Gkrekas, 2023; Zimmermann, 2004). These processes align closely with the theory of adaptive expertise, which emphasises flexible transfer, strategic choice, and responsiveness to problem structure rather than routine performance. By situating algebraic problem solving within historically grounded methods, students are provided with opportunities to develop adaptive expertise through sustained structural reasoning and meaningful strategic comparison. However, despite the theoretical alignment between historical approaches and adaptive expertise, empirical studies examining this connection in complex algebraic contexts remain scarce, motivating the present study.

Present study

The history of mathematics present diverse original strategies for solving algebraic equations. It also exposes complete accounts of how each strategy evolved to become what we know of today. Looking at the history of cubic equations, we can see that different methods were invented to solve different forms of the same equation. This historical context is appropriate for helping students improve on their strategy flexibility. Thus, in the present study our focus is to exploit the history of cubic equations as a context for developing students strategy flexibility in solving cubic equations in senior high school algebra. The premise of this study follows that, in the history, diverse methods for solving cubic equations can be found, but these methods are largely absent from standard school mathematics curricula. Focus has only been on standard methods of factorisation, synthetic division, factor theorem and remainder theorem, which limits students strategy repertoire and choice. While most of the earlier strategies such as those of del Ferro, Tartaglia, and Cardano are very robust, when students are exposed to them it can make them get hold of more choices of innovative methods of solving cubic equations. In addition, when students are taught how these methods emerged; their limitations and strengthens, it will strengthen their ability to make situational strategy choices at the right time. Moving students through the history of cubic equations, thus will help them understand *why* and *how* each method matters. In this sense, the history of cubic equations is a lens through which students can appreciate the value of innovative methods, that they are not taught. The purpose of this study is to determine whether students who are taught cubic equations through a historical context will demonstrate improved strategy flexibility. Accordingly, we propose the following hypotheses:

> H_0: Students taught cubic equations in history-integrated lessons will demonstrate the same level of strategy flexibility as their counterparts taught in no-history integrated lessons.
>
> H_1: Students taught cubic equations in history-integrated lessons will demonstrate higher level of strategy flexibility as their counterparts taught in no-history integrated lessons.

Methods

Participants

Prior to the commencement of the study, sample size was determined using power analysis computed in G*Power (Faul et al., 2009; Kang, 2021). The results based on significance level (α) of 0.05, a statistical power ($1–\beta$) of 0.80, and an estimated medium effect size (Cohen's d = 0.50) indicated that sample size of 128 student (control [n = 64; males = 50, females = 14] and experimental group [n = 64; males = 46, females = 18] combined) is deemed appropriate (males = 94, females = 30). The medium effect size estimate was based on conventional benchmarks set by Cohen (1998), which were adopted as widely used reference guidelines in educational research and interpreted as heuristic indicators rather than rigid cut-off values, acknowledging that practical significance should be considered in relation to the study context. Participants ages range between 11 and 17 years. Participants were enrolled in two senior high schools located in the Ashanti region of Ghana. The number of students were conveniently selected from 4 science classes (two classes from each school) to form the experimental and control groups, because the number of students in one intact class (as recommended by quasi-experiments) cannot make up exactly the number of students in either of the groups. Form 2 science students were selected purposely because, by this level, they have typically covered standard methods for solving cubic equations (factor theorem, synthetic division, long division, and factorisation). Conversations with the class teacher indicated that standard methods were first introduced in Form 1. At this stage, students are cognitively matured enough to engage with more advanced and innovative methods inspired by the history of mathematics, which made them suitable candidates for the study. In addition to the students been at the same academic level and taking the same course, results of independent sample t-test of the pretest scores showed that the groups did not differ in terms of initial strategy flexibility $\left(t = -323, p = 0.747 > 0.05\right)$. This confirmed that the groups are homogenous and thus could start the study at comparable levels.

Instrument

In this study, our operationalisation of strategy flexibility aligns more to the practical flexibility dimension of procedural flexibility in Xu et al. (2017). This dimension treats strategy flexibility as the spontaneous use of innovative method on the first attempt. Previous studies had also measured strategy flexibility this way (e.g., Star et al. 2015). For example, Star et al. (2015) instructed students to identify the innovative first step for the solution of an equation and their responses were taken as knowledge of innovative strategies. While in the literature, scholars commonly define "innovative" methods as those that can be used to solve problems following the fewest steps and the greatest degree of algorithmic simplicity (e.g., Heinze et al., 2009; Star and Rittle-Johnson, 2008; Star and Newton, 2009), for the sake of this study, an "innovative" method is one that shortens the solution process or uses structural insight that is history-inspired. So, a method for solving cubic equations is deemed innovative, if it is especially: (i) history-inspired, (ii) not well emphasised in mathematics curricula (iii) enables more structural exploitation, and (iv) uses cleaner transformations. Any method apart from these is deemed standard. Typical curriculum order of methods for working cubic equations by looking for rational roots using factor theorem, and then using long division or synthetic division to reduce the cubic into quadratic×linear factors is defined as standard methods taught in school algebra.

Accordingly, an essay-type test is designed with 9 cubic equations as the measurement instrument for strategy flexibility. In the test, different categories (Type A to Type C) of cubic equation based on their structural forms were given such that each could be solved using a standard method but where a more innovative method could also be used. We followed the approach by Xu et al. (2017) where they measured students use of innovative strategy in the Phase One of their Tri-phase Flexibility Assessment. Students were instructed to solve each cubic equation as quickly and accurately as they could, which encouraged them to choose and apply an innovative method (Star et al., 2022). If a student used an innovative method for a given cubic equation, that student is said to have demonstrated strategy flexibility. Type A equations were of the form of $x^3 + bx + c = 0$. The innovative method here involved students to easily use the substitution $x = u + v$. Type B were of the form of $x^3 + ax^2 + c = 0$. The innovative method involved setting $x = y - b/3$ to remove the x^2 term before applying the cubic formula. Type C were of the form of the general cubic $x^3 + bx^2 + cx + d = 0$. The innovative method involves using the substitution $x = y - b/3$ to eliminate the x^2 term or simplify coefficients before applying the cubic formula. All these first steps that are found innovative as in this study were those developed by ancient mathematicians: del Ferro, Tartaglia, and Cardano for cubic equations of the form $x^3 + bx + c = 0$, $x^3 + ax^2 + c = 0$, and $x^3 + bx^2 + cx + d = 0$ respectively. Emphasis has been placed here on these since they are the methods that have historical significance when teaching cubic equations. The equations were set that none of them when solving would end up into the irreducible case "casus irreducibillis" where students have to find real roots using complex numbers. Note that the order in which the equations have been given in the test reflects the order of increasing task difficulty as it was done in Xu et al. (2017) and was seen to be a good practice. Type C equations were the most challenging. The 9 cubic equations are listed in the Table 1.

To enrich the data collected using the test instrument, an interview guide was developed to collect qualitative data. The interview guide focuses on students who participated in the history-integrated lessons from the experimental group. The interview was conducted from 10 students to provide further explanations to why there was improved strategy flexible among the experimental group. According to Guest et al. (2006), this sample size was seen to be suitable because theoretical data saturation occurred after the 10th interviewee. On the interview guide, two questions were asked. These questions are: (i) How did learning to solve cubic equations through its history help you solve cubic equations? (ii) Did the history-integrated lessons change the way you think about solving problems? How? Responses taken from students on these questions were recorded on smart phone and analysed thematically to add more depth to the findings of the study.

Procedures

Ethical clearance

Before commencing the study, ethical clearance was obtained from the ethics committee of the School of Graduate Studies of the Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, followed by formal approval from school administrators of the selected senior high schools.

Design

The present study followed the procedures of quasi-experimental study design. To ensure the rigour (internal validity) of this study, there was the need to pay particular attention to controlling for contamination that could result from cross-group effect (Anisi et al., 2025). Hence, the study implemented the history-integrated lessons on the experimental group in one school while designating another as the control group in a different school so that the risk of participants in the control group being inadvertently exposed to the history-integrated lessons will be substantially reduced. This approach minimises the potential for sharing of lesson information among participants or informal dissemination of lesson materials, which

TABLE 1 Examples of cubic equations classified by type, with standard and innovative first-step solution strategies.

Type	Cubic form	Cubic equations	Standard first step	Innovative first step
A	$x^3+bx+c=0$	1. $x^3+3x-4=0$	split into two recognisable groups; rational root test	Substitute $x=u+v$
		2. $x^3+6x-8=0$	split into two parts that share a common factor; rational root test	Substitute $x=u+v$
		3. $x^3-8=0$	Identify difference of cubes; rewrite as $x^3=8$ and take cube root directly	Substitute $x=u+v$
B	$x^3+ax^2+c=0$	4. $x^3+6x^2-20=0$	Rational root test	Set $x=y-2$
		5. $x^3+3x^2-4=0$	Rewrite by splitting −4 and regrouping for possible factorisation; rational root test	Set $x=y-1$
		6. $x^3+x^2-12=0$	Rational root test	Set $x=y-\frac{1}{3}$
C	$x^3+bx^2+cx+d=0$	7. $x^3-3x^2+3x-1=0$	Recognise the binomial expansion of $(x-1)^3$; rational root test	Set $x=y+1$
		8. $x^3+3x^2+4x+12=0$	Group in pairs to look for common factors; rational root test	Set $x=y-1$
		9. $x^3+6x^2+12x+8=0$	Check for perfect cube pattern; rational root test	Set $x=y-2$

The table lists three types of cubic equations (A: Missing x^2 term, B: missing x term, C: general cubic) along with specific example equations. For each equation, the standard first step shows commonly taught approaches, such as rational root tests, factoring, or recognising special patterns, while the innovative first step demonstrates substitutions or transformations designed to simplify the equation efficiently. All substitutions are indicated explicitly (e.g., $x=u+v$ or $x=y-k$).

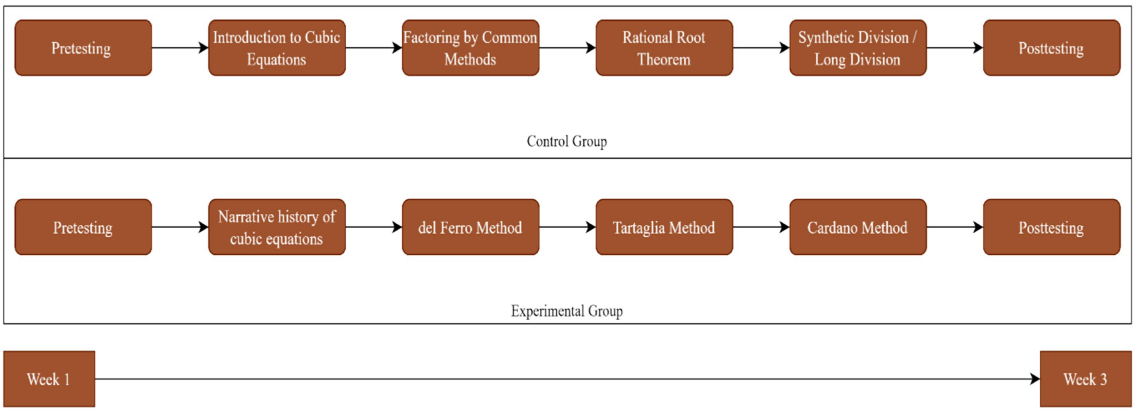

FIGURE 1
Study procedures and lesson breakdown.

could otherwise compromise the integrity of the comparison between groups (Keogh-Brown et al., 2007).

According to Figure 1, pre-tests were administered simultaneously to both the experimental and control groups to establish baseline equivalence and determine if the participants are relatively homogeneous. After this, the lessons for both groups were delivered by the first author himself and spanned two (2) weeks. Each lesson took 1 h for 2 meetings a week with each group (a total of 4 h for each group overall). The lesson plan was designed such that it did not interfere with the normal classroom timetable by making arrangement with two teachers in the two different schools. These teachers volunteered their classes for 2 h each week for the 2 weeks just for the lessons. At the end of the lessons for each group, post-test on strategy flexibility was administered to both groups to measure students' ability to employ innovative methods to solve cubic equations at their first attempt. In all, the study took 4 weeks to completion: 1 week for pretesting, 2 weeks for the implementation of the lessons, and 1 week for the post-testing.

According to Figure 1, the experimental group received lessons on cubic equations solving integrated with the history of cubic equations. The lesson started with brief narrative of the history behind cubic equations. Students were also introduced to the historical methods developed to solve cubic equations and how each method links to specific structural forms of cubics (e.g., $x^3+mx=n$). More emphasis was placed on the structural classifications offered by del Ferro, Tartaglia and Cardano and how knowledge on different forms of cubics played roles in solving cubic equations by using these methods.

The control group on the other hand were taught standard methods for working cubic equations following sequence of steps as emphasised in the additional mathematics curriculum. After students were introduced to cubic equations and their graphs, they were taken through how to factorise by common methods approach where they look for factorable patterns in cubic expressions. Then they were introduced to the rational root theorem rational root theorem (if $x-k$ is a factor of $f(x)=ax^3+bx^2+cx+d$, then $f(a)=0$, which implies that $x=a$ is a root of $ax^3+bx^2+cx+d=0$), followed by the synthetic division or long division to break the cubic expression into the form of $(ax^2+bx+c)(x-k)=0$. Students were taught that at this point the quadratic part could be solved using factorisation, completing squares or the quadratic formula. Figure 1 shows a complete breakdown of the lessons for each meeting.

Rating strategy

The scripts of the 128 students were carefully checked and rated by two independent raters. All of the raters were masters students trained by the first author to do this job for the sake of this study. The rating procedures was divided into two rounds. In the first round, the first rater rated the scripts of the control group and the second rater rated that of the experimental group independently. When they were done, they interchanged the scripts for a second round of independent rating. To determine scores for strategy flexibility, a marking scheme was developed. Raters looked at students' solutions of each equation and identified if standard method or innovative method is used, despite algorithmic errors. The rating scheme is such that, a student is said to have used standard method when they use the factor theorem followed by synthetic or long division collectively, the rational-root heuristics as it is taught in the curriculum. Other than this, a student is said to have used innovative method especially if he/she made use of an innovative first step. Understand that by the raters analysing the first one or two steps in the solution, they could determine if an innovative or standard method is used (Xu et al., 2017). If a student was able to use an innovative method, the he/she was said to have shown flexibility and was awarded 1 mark. A student who was only able to use a standard method in the first attempt was deemed to have shown no flexibility and was thus awarded 0 mark. Given that a student received a flexibility score for every cubic equation solved, the maximum score was 9 (for the nine questions). The percent agreement between the scores of the two raters was calculated as the proportion of items on which raters give exactly the same rating and the value of 95% indicated that the flexibility assessment was reliable (McHugh, 2012).

Data analysis

Because the purpose was to investigate if participating in the history-integrated lessons improved students strategy flexibility, we compare the strategy flexibility scores between the two groups. The independent sample t-test was used for this. We analysed the pretest data to establish baseline equivalence between the experimental and control groups. Baseline equivalence ensures that any post-test differences in flexibility cannot be attributed to pre-existing disparities in strategy flexibility. Levene's test for equality of variances was conducted to assess whether the variability in pretest flexibility scores was statistically similar for the two groups. Based on this outcome, the independent samples t-test for equal variances was used. After this, the post-test data was analysed to examine whether there exists difference in the mean strategy flexibility scores between the experimental group and the control group. To complement the comparative tests, descriptive statistics (means, standard deviations) were also computed to provide a clear picture of group differences. Additionally, effect sizes (Cohen's d) were computed to quantify the magnitude of the effect of history-integrated lesson on strategy flexibility. The effect size helped us to understand the practical significance beyond the *p*-value interpretation.

To further explain and enrich the results obtained from the quantitative data, interview data were collected and analysed using thematic analysis (Ahmed et al., 2025). First the audio recorded interviews were transcribed carefully one after the other (participant by participant). After this, we begun the analysis by following the six steps offered by Braun and Clarke (2006). The analysis begun with repeated reading of the transcripts to achieve familiarity with the data. Initial codes were then generated inductively to capture meaningful features of the responses. These codes were subsequently organised into broader themes that reflected recurring ideas or insights. The themes were reviewed and refined to ensure coherence, internal consistency, and accurate representation of the dataset. Finally, each theme was clearly defined and supported with illustrative quotations (Braun and Clarke, 2021, 2006; Braun et al., 2023). This was done to allow the qualitative findings to deepen and contextualise the quantitative results.

Results

Differences in strategy flexibility

Pretest scores

A preliminary analysis was conducted to determine whether the experimental and control groups differed in strategy flexibility prior to the lesson. Descriptive statistics and group comparisons for the pretest scores are presented in Table 2.

TABLE 2 Descriptive statistics for pretest flexibility scores by group, along with results of Levene's test for equality of variances and independent-samples *t*-test.

Groups	Mean	Std. Dev.	Levene's test		*t*-test			
Experimental	2.31	1.08	*F*	*p*	*t*	*df*	*p*	Cohen's *d*
Control	2.38	1.11	0.09	0.76	−0.32	126	0.75	−0.057

Results indicate no significant difference between the experimental and control groups ($p > 0.05$), and effect size was negligible.

In Table 2, Levene's test indicated that the assumption of homogeneity of variances was met for the pretest scores $[F(1{,}126) = 0.091, p > 0.05]$. This supported the use of the *t*-test with equal variances. Results from the independent samples *t*-test showed no statistically significant difference between the two groups in terms of strategy flexibility at pretest, $\left[t(126) = -0.323, p > 0.05\right]$, with a negligible effect size (Cohen's $d = -0.057$). These results demonstrate that both groups entered the study with comparable levels of strategy flexibility.

Post-test scores

We determined whether the history-integrated lesson led to improved strategy flexibility compared to the no-history-integrated lesson. Descriptive statistics for the post-test scores and group comparisons are displayed in Table 3.

According to Table 3, the independent samples *t*-test revealed a statistically significant difference between the experimental and control groups on post-test flexibility scores $[t(126) = 4.25, p < 0.05]$. Students who received history-integrated lessons demonstrated higher strategy flexibility than their counterparts in the no-history-integrated lessons group. The effect size (Cohen's $d = 0.75$) indicated a moderate effect size. It indicated that the mean of the experimental group is 0.75 standard deviations higher than the mean of the control group. This effect implies practical significance not just statistical significance. Many researchers in education consider effects above 0.40 to be educationally worthwhile (e.g., Hattie, 2023); thus 0.75 indicates a substantially effective lesson.

Pretest–post-test gain scores

To further evaluate improvement in strategy flexibility after the lessons, gain scores (post-test minus pretest) were computed for each participant. This provided a direct measure of improvement in strategy flexibility attributable to the lessons. Group means and standard deviations and comparisons for gain scores are shown in Table 4.

The independent samples *t*-test conducted on the gain scores showed that the experimental group experienced significantly greater improvement than the control group $[t(126) = 3.82, p < 0.05]$. This result reinforces the finding that the history-integrated lesson improved students' strategy flexibility than the no history-integrated lesson confirming the results on Table 3. To have a visual view of these gains, a graphical representation of the pretest and post-test mean scores is provided in Figure 2. The figure illustrates the comparable starting points of both groups and the clear divergence in strategy flexibility scores following the history-integrated lessons.

TABLE 3 Summary of post-test flexibility scores for experimental and control groups, along with independent-samples *t*-test results.

Groups	Mean	Std. Dev.	*t*-test			
Experimental	4.92	2.60	*t*	*p*	df	Cohen's *d*
Control	3.36	1.38	4.25	< 0.001	126	0.75

The experimental group showed a statistically significant improvement compared with the control group ($p < 0.001$), with a large effect size (Cohen's $d = 0.75$).

TABLE 4 Pretest-to-post-test gain scores for experimental and control groups, with results of the independent-samples *t*-test.

Groups	Mean	Std. Dev.	t-test			
Experimental	2.61	2.83	*t*	*p*	*df*	Cohen's *d*
Control	0.98	1.90	3.82	< 0.001	126	0.68

The experimental group showed significantly greater improvement than the control group [$t(126) = 3.82$, $p < 0.001$], with a large effect size (Cohen's $d = 0.68$), indicating the intervention effectively increased strategy flexibility.

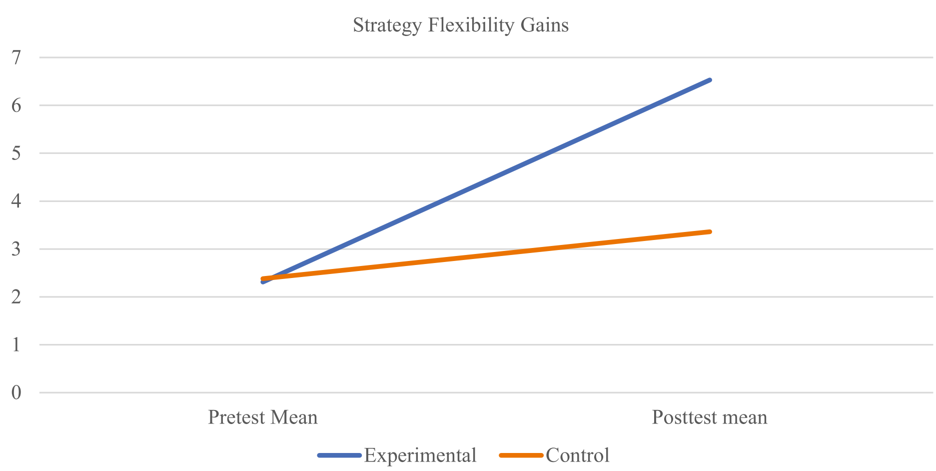

FIGURE 2
Comparison of pretest and post-test mean flexibility scores for experimental and control groups.

Students' experiences with the history-integrated lessons

Expansion of strategy repertoire

Participants consistently mentioned that learning different ancient algebraic methods for solving cubic equations broadened their knowledge of approaches for solving this kind of polynomial equations. Many noted that they now realised there is more than one way to solve the same equations. For example, S5 reported, *"When I learn the history, I see that mathematicians themselves tried different ways before finding the right one. I feel okay to try new methods and check if they work."* S1 confirmed that, *"It made me realise that there is not just one way to solve a problem and that sometimes, different methods can give the same answer."* This showed that students got the opportunity to expand their strategy repertoire after taking part in the history-integrated lessons which could contribute to the improvement in strategy flexibility (Table 5).

Situational strategy selection

Participants described how the history-integrated lessons encouraged them to reflect on which method to use and why that method should be used. They reported that they became more thoughtful in selecting strategies for solving cubic equations rather than blindly applying them just because they are good at using them. S2 said that "*I felt like I was solving the equation like one of the mathematicians in the past. I did not just memorise steps. I could choose the right method for each equation I was given to solve in the question.*" S8 also confirmed that "*I now try to understand why a method works before using it since I got to know that not every method can be used for any equation. I will not memorise formula anymore but I will try to understand the given equation.* S5 said "*Before, I always stuck to one method because I was afraid to try something else, but now I can pick the best method for each equation.*" Students also reported that after passing through the history-integrated lessons, it helped them become more confident in selecting from among different methods, the one that best solves an equation based on cubic forms. For instance, S3 replied "*Once I understood Del Ferro's method, I compared it with that of what I was taught in class and see between them which one is better. Now, when I see any cubic equation, I can decide which method is easiest or fastest.*" In the view of S9, "*Comparing the old methods with new ones helped me understand which method is faster or easier in different situations. I asked myself which method will work best for this equation. Instead of doing the first thing that comes to my mind, I was able to think and choose a strategy.*" This show that students could now compare methods and choose the best one depending on the given problem situation.

Motivational dispositions

Most of the participants reported that when they heard about the stories of how each of the methods for solving cubic equations came about it made them interested in not just applying the method but also keeping them glued to the lesson. In fact, it made the lessons memorable, and relatable, increased engagement and attention throughout the lesson. For instance, S4 reported that "*The story of how del Ferro method discovered his method and how it evolved to become the cubic formula published by Cardano made the lesson interesting.*" S10 also recounted that "*Learning the history was like listening to a story. It helped me understand the idea behind the formulas instead of just memorising them.*" This shows that integrating historical context enhances motivation and cognitive engagement of students which likely contributed to the improvement in flexibility scores observed above.

TABLE 5 Themes, codes, and illustrative participant responses regarding the effects of learning historical methods for solving cubic equations.

Theme	Code	Relevant sample response
Expansion of Strategy Repertoire	Realising availability of multiple solution methods	"It made me realise that there is not just one way to solve an equation and that sometimes, different methods can give the same answer." (S1)
	Willingness to try alternative strategies	"I feel okay to try new methods and check if they work." (S5)
Situational Strategy Selection	More thoughtful selection of methods	"Before, I just followed the method we learned in class, but learning the history made me think carefully about which strategy to use for each question." (S2)
	Comparing historical and modern methods	"But once I learned it, I could compare it with other methods we had learned like synthetic method and factorisation. Now when I see a cubic equation, I can decide which method is easiest or fastest." (S3)
	Understanding the origins of methods aids recall	"When I know why a method was discovered and what form of equation it was used to solve, I remember the steps better." (S4)
	Selecting methods based on equation type	"I can now tell which method works better depending on the type of cubic equation." (S6)
Motivational dispositions	Interest in ancient methods	"I like Cardano's method the most because it was very different from what we learnt in class." (S1)
	Appreciation of the discovery of methods	"del Ferro's method was interesting because I now understand why mathematicians made new ways to solve equations." (S2)
	Increased enjoyment from narrative learning	"I really enjoyed the stories of how these methods were discovered because it made the lessons more interesting." (S4)
	Increased confidence	"I feel more confident solving cubic equations now." (S5)
	Learning as an enjoyable story experience	"Learning through the history was like listening to a story, and it made the lesson more fun." (S10)
	Reduced fear of mistakes	"This made me less scared of making mistakes by myself." (S7)

Discussion

The research field of strategy flexibility has recently only begun to investigate flexibility in more advanced content areas in algebra. The present study builds on previous studies of flexibility in algebraic tasks that identify innovative first step as a measure of strategy flexibility (e.g., Arslan and Yazgan, 2015; Maciejewski and Star, 2016; Mercier and Higgins, 2013; Newton et al., 2010; Rittle-Johnson and Star, 2007; Star and Rittle-Johnson, 2008; Rittle-Johnson et al., 2012; Star and Rittle-Johnson, 2008; Star et al., 2015; Star et al., 2022; Xu et al., 2017). Meanwhile these studies had focused on arithmetic or linear equations solving, where opportunities for strategic variation are relatively limited. Few studies have examined flexibility in higher-degree equations such as quadratics (e.g., Ernvall-Hytönen et al., 2024; Küttel et al., 2025), and even fewer have explored the pedagogical value of history of mathematics.

According to the studies on quadratic equations solving, students exhibit very limited spontaneous flexibility when working with non-linear equations and that flexibility becomes more difficult as equation structure becomes more complex. This study extends these works to the domain of cubic equations solving that seem complex and non-linear. It shows that flexible first-step innovation is teachable even when tasks require reasoning beyond the linear procedures commonly explored in previous research. This study suggest that strategy flexibility is not confined to elementary algebraic equations; it can also be cultivated within higher-degree polynomial equation contexts when lessons are designed to be sufficiently rich and well planned especially using the history of mathematics as a context.

The purpose of this study then was to investigate whether integrating history into lessons on cubic equations could enhance senior high school students' strategy flexibility. The findings indicate that there exists significant statistical difference in the mean post-test strategy flexibility scores between the experimental and control group. The students in the experimental group, who received history-integrated lessons, achieved significantly higher mean flexibility score than those in the control group. Because both groups were comparable at pretest, the post-test differences can be attributed to the history-integrated lessons. These results support our alternate hypothesis and provide empirical evidence that historical context can meaningfully enhance students' ability to apply innovative methods or adapt to new methods for solving cubic equations (Star et al., 2015, 2022; Xu et al., 2017).

In addition, findings from the interview data provide further information that could be used to explain how strategy flexibility improved among the experimental group. Taken together, the present study suggests that integrating the history of mathematics into pedagogy can support the growth of flexible strategy use. The results are discussed below by linking to the results identified from the analysis of the interview data.

The higher mean reported which shows improvement in strategy flexibility as it was observed in the experimental group can be interpreted through the lens of adaptive expertise theory (Hatano and Inagaki, 1986; Baroody and Dowker, 2003). The theory first of all distinguishes between routine and adaptive expertise. Routine expertise reflects students' proficiency to employ familiar, well-practiced, or standard procedures to solve problems, whereas adaptive expertise involves the ability of students to flexibly select, modify, or generate new or non-standard strategies in response to a novel problem situation that demands urgent attention (Baroody, 2013). The present results are in line with this theoretical perspective.

In line with the theory, students in the control group who were taught standard textbook strategies (e.g., factor theorem) for solving cubic equations primarily developed routine expertise and relied solely on these methods that they are taught. At the end, they struggled to adapt when faced with cubic equations requiring non-standard or structurally sensitive strategies. Students in the experimental group who got exposed to the history-integrated lessons shared they became more willing to learn unfamiliar historical methods and at the end showed they could turn to the most appropriate methods depending on the form of cubic equations that they were given, since they now know of new methods in addition to the standard ones they were taught by their teacher (Star, 2007). The ability to make such quick decisions is central to adaptive expertise (Crawford et al., 2005).

Moreover, the history-integrated lessons appeared to have supported the improvement in strategy flexibility by exposing students to ancient solution methods and how methods were evolved. This corresponds with the idea that when students are exposed to different methods for solving problems based on the context of each problem and which method fits for a particular context, it fosters deep procedural understanding, which forms the backbone of flexibility (McMullen et al., 2017; Rittle-Johnson et al., 2012; Rittle-Johnson and Star, 2007). This assertion was identified from the students' interview reports.

Students who were interviewed consistently reported that exposure to historical methods expanded their knowledge and awareness about alternative methods that exist and as such they could use such methods based on the context of a given cubic equation which could link to why they performed higher in the flexibility assessment (Star, 2007; Rittle-Johnson and Star, 2007). This corroborates with prior research which has shown that prompting students to compare multiple strategies increases their awareness of efficiency and situational appropriateness even in routine problem contexts (e.g., Arslan and Yazgan, 2015; Rittle-Johnson et al., 2012; Star and Rittle-Johnson, 2008; Maciejewski and Star, 2016) which is also an important principle of adaptive expertise (Verschaffel et al., 2009).

The present findings support the practice of historical juxtaposition. Students can compare and draw the similarities and contrasts among ancient and modern methods of solving cubic equations. This practice naturally prompts students' strategy comparison even though there was nothing pertaining to explicit prompts and direct instruction (Star and Rittle-Johnson, 2008) that students were engaged in. Students could organically compare del Ferro-Tartaglia-Cardano's methods with modern methods of factorisation, synthetic division and so on, which appeared to enhance their strategic reasoning.

The findings also align with research that concerns how students' engagement in non-routine problem-solving contexts foster flexible strategy thinking (e.g., Newton et al., 2010; Star and Newton, 2009). In the present study, solving cubic equations were taught based on their structural classifications offered by ancient mathematicians. This provided equation problem contexts that are new and seem unique to students to come about which they might seem non-routine since they have not had encountered them before. This would challenge students to engage with unfamiliar methods they have not employed before, atypical transformations (such as depressing the cubic) that they have not heard of before, and historically derived heuristics that make a lot of sense and meaning, despite the fact that it deviates from the standard algorithms students have been using before the history-integrated lessons. By this, students become more open, curious and developed the willingness to explore which original methods would actually work for such non-routine case.

In support of the above, students reported that they felt more willing to "try new methods" after they learnt from the history-integrated lesson. This finding concurs with that of previous studies which found that students become more flexible in using strategies when they engage with non-routine problems and try new or alternative approaches (e.g., Arslan and Yazgan, 2015; Rittle-Johnson et al., 2012; Star and Rittle-Johnson, 2008; Maciejewski and Star, 2016). McGinn and Boote (2003) found that working through original sources of mathematics problems fosters the ability to recognise that diverse methods had been invented for dealing with such problems. This encourages reflection on problem structure rather than rote algorithmic computations. This parallels the experience of students in the experimental group, who realised that "there is not just one way to solve an equation," indicating a conceptual shift similar to the one observed in McGinn and Boote (2003) experience in their self-research with the Plimpton 322.

Similarly, Zimmermann (2004), and Rizos and Gkrekas (2023) reported that historical contexts activate strategic thinking by exposing students to the heuristics and innovations of past mathematicians. The ability of students in this study to articulate strengths of solution methods, the limitations, and situational suitability strongly mirrors these earlier findings.

Furthermore, the results align with research which highlight the pedagogical benefits of using original historical sources in teaching problem-solving (e.g., Meavilla and Flores, 2007; Swetz, 1986). In line with Swetz (1986), solving problems in historical context illuminate the reasoning behind modern methods. Also, according to Meavilla and Flores (2007), juxtaposing ancient and modern approaches helps students appreciate the evolution of solution techniques. Per these perspectives, students in the experimental group described that when they engaged with the history-integrated lessons, it makes them see that the original historical methods are more meaningful, memorable, and easier to apply.

Last but not the least, the findings showed that recognising diversity in method use encourages reflection on problem structure. This mirrors exactly what McGinn and Boote (2003) found in their study that engaging with ancient sources supports recognition of method diversity and encourages reflection on problem structure. These are the experiences reported by students in the experimental group when they were interviewed.

In summary, the findings of the present study not only align with prior research and theoretical considerations in algebraic cognion and history and pedagogy of mathematics in general, but significantly expand the boundaries of what is known about mathematical flexibility in particular.

Conclusion

The present study investigated whether integrating the history of mathematics into lessons on solving cubic equations could enhance senior high school students' strategy flexibility. Results revealed that students who participated in the experimental group demonstrated significantly higher mean flexibility scores than those who received standard lessons. These findings provide strong evidence that when students learn algebra in historical context, it can meaningfully enhance their ability to apply innovative methods for solving different structural forms of cubic equations rather than using standard procedural strategies that might not be appropriate for certain cubic forms.

Further qualitative results offer deeper insight into how and why this improvement occurred. Students reported that exposure to historical solution methods such as those developed by del Ferro, Tartaglia, and Cardano expanded their strategic repertoire, strengthened their ability to select appropriate strategies based on the structural characteristics of cubic equations, and it fostered their willingness to experiment with unfamiliar methods. The narrative of the history of cubic equations that was used to introduce the lesson also made students appreciate the reasoning behind each method and understand how mathematics algorithms or methods evolve over time. These experiences promoted both cognitive and epistemic aspects of adaptive expertise, which include situational judgment, curiosity, and resilience. All these thereby encourage students to approach to solving problems with a more exploratory and flexible mindset.

The study makes several contributions to research and practice. Theoretically, it demonstrates that integrating history into pedagogy can serve as a powerful environment where students develop adaptive expertise through exposure to historically rich strategies, where students have to reason about structures inherent in equation forms, and think epistemically within lessons. Empirically, it extends the scope of flexibility research into higher-degree polynomial equations which is more complex than linear equations but has received minimal attention in previous studies. Practically, it offers mathematics educators and teachers a concrete approach for cultivating strategic reasoning and procedural flexibility where they could teach beyond supporting mere procedural fluency to support adaptive problem-solving in secondary school algebra classrooms.

Limitations and suggestions for future study

Despite the contributions made by this study, it has several limitations that must be acknowledged honestly. First, the use of a quasi-experimental design with conveniently selected samples limits the strength of causal inferences; random assignment would provide more robust evidence of intervention effects. Second, the sample was drawn from only two schools within a single geographic region of Ghana, which restricts the generalisability of the findings to other educational contexts or the wider senior high school populations. Third, the history-integrated intervention was relatively short. This could prevent conclusions about the long-term retention of strategy flexibility. Finally, the focus on cubic equations in algebra might limit the transferability of findings to other mathematics domains, such as geometry, calculus or statistics.

In closing, we identify implications for future research that emerge from this study and its limitations. Future studies should address these limitations by employing randomised controlled designs and include larger, more diverse samples to enhance generalisability. Longer and more comprehensive interventions are needed to examine the durability of flexibility gains over time and across different mathematics domains. Future research could also investigate the effects of history-integrated instruction on related outcomes, such as conceptual understanding, creativity, problem-solving efficiency, and the development of epistemic dispositions associated with adaptive expertise. Additionally, exploring how history-integration could interact with other pedagogical strategies such as collaborative learning or technology-enhanced instruction may yield further insights into optimising students' mathematics flexibility.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Ethics statement

The studies involving humans were approved by School of Graduate Studies of the Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development. The studies were conducted in accordance with the local legislation and institutional requirements. Written informed consent for participation in this study was provided by the participants' legal guardians/next of kin. Written informed consent was obtained from the minor(s)' legal guardian/next of kin for the publication of any potentially identifiable images or data included in this article.

Author contributions

AGB: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing – original draft, Writing – review & editing. JG: Conceptualization, Formal analysis, Methodology, Resources, Supervision, Writing – review & editing. YA: Conceptualization, Formal analysis, Methodology, Supervision, Validation, Writing – review & editing.

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The author(s) declared that Generative AI was not used in the creation of this manuscript.

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