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Dynamical behavior of a stochastic SEIQRV infectious model with an Ornstein-Uhlenbeck process and general incidence

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Considering the influence of quarantine and vaccination factors, this study examines an SEIQRV infectious disease model that incorporates an Ornstein-Uhlenbeck process and a general incidence function. By accounting for disease-induced mortality rates among infected individuals, the article establishes the existence and uniqueness of a global solution for any arbitrary positive initial value. An adequate condition for disease extinction is also provided. Simultaneously, by reconstructing a sequence of random Lyapunov functions, we demonstrate the existence of a unique stationary distribution indicating that the disease persists over a period of time in a biological sense. Based on these findings, the precise expression for the probability density function of the stochastic model near the quasi-equilibrium state is derived. Finally, the theoretical results are verified through a series of numerical simulations.

KEYWORDS

SEIQRV model, Ornstein-Uhlenbeck process, extinction, stationary distribution, density function

1 Introduction

SARS-CoV-2, a coronavirus that emerged in late 2019, causes a severe respiratory illness that can progress to fatal pneumonia [1]. The WHO has characterized the epidemic as COVID-19 [2]. SARS-CoV-2, is primarily transmitted through three routes: respiratory aerosol inhalation, immediate human exposure, and droplet transmission [3]. It is worth noting that the implementation of quarantine and vaccination measures significantly influences epidemic dynamics. To understand the transmission dynamics of infectious diseases, many scholars have proposed numerous mathematical models to predict and control the development of diseases. For instance, Tang et al. [4] developed a generalized SEIR model of the SEIR-type considering isolation and treatment, which elucidated the transmission dynamics of COVID-19 and evaluated the effect of public health interventions on disease infections. Poonia et al. [5] developed a SEIRV model incorporating vaccination rate, highlighting the critical role of social distancing and vaccination in epidemic control.

In 2020, Fosu et al. [6] proposed a SEIQRV general epidemiological model for COVID-19 as follows:

$$\begin{cases} dS = [\Lambda - \bar{\beta}SI - (\mu + \delta)S + \nu E]dt, \\ dE = [\bar{\beta}SI - (\sigma_2 + \gamma + \nu + \mu)E]dt, \\ dI = [\gamma E - (\sigma_1 + \omega_1 + \mu)I]dt, \\ dQ = [\sigma_1 I + \sigma_2 E - (\omega_2 + \mu)Q]dt, \\ dR = [\omega_1 I + \omega_2 Q - \mu R]dt, \\ dV = [\delta S - \mu V]dt. \end{cases} \quad (1)$$

where $S(t)$, $E(t)$, $I(t)$, $Q(t)$, $R(t)$, and $V(t)$ represent the number of susceptible, exposed, infected, quarantined, recovered, and vaccinated classes at time t , respectively, and their initial values are satisfied:

$$S(0) > 0, E(0) > 0, I(0) > 0, Q(0) \geq 0, R(0) \geq 0, V(0) \geq 0.$$

The parameters of Equation 1 are defined as follows: $\bar{\beta}$ denotes the transmission rate from susceptible to exposed individuals; Λ represents the recruitment rate of susceptible populations; μ is the natural death rate; δ denotes the vaccination rate among susceptibles; ν is the transfer rate from exposed back to susceptible; γ is the progression rate from exposed to infected; σ_1 and σ_2 represent the quarantine rates for infected and exposed individuals, respectively; ω_1 and ω_2 are the recovery rates of infected and quarantined individuals, respectively.

On this basis, considering that the mechanism of disease transmission may be influenced by several factors, we introduce the disease-caused mortality rate h of infected individuals and assume that this rate exceeds the natural mortality rate. The resulting deterministic Equation 2 takes the following form:

$$\begin{cases} dS = [\Lambda - \bar{\beta}SI - (\mu + \delta)S + \nu E]dt, \\ dE = [\bar{\beta}SI - (\sigma_2 + \gamma + \nu + \mu)E]dt, \\ dI = [\gamma E - (\sigma_1 + \omega_1 + \mu + h)I]dt, \\ dQ = [\sigma_1 I + \sigma_2 E - (\omega_2 + \mu)Q]dt, \\ dR = [\omega_1 I + \omega_2 Q - \mu R]dt, \\ dV = [\delta S - \mu V]dt. \end{cases} \quad (2)$$

Similar to Fosu et al. [6], here are certain conclusions about Equation 2:

(1) The basic reproduction number of Equation 2 is $R_0 = \frac{\Lambda \bar{\beta} \gamma}{(\mu + \delta)(\omega_1 + \sigma_1 + \mu + h)(\gamma + \sigma_2 + \nu + \mu)}$.

(2) The disease-free equilibrium point is $E_0 = (S_0, E_0, I_0, Q_0, R_0, V_0) = (\frac{\Lambda}{\mu + \delta}, 0, 0, 0, 0, \frac{\delta \Lambda}{\mu(\mu + \delta)})$, and the disease-free equilibrium point E_0 is locally asymptotically stable for $R_0 < 1$.

(3) The endemic equilibrium point is $P^* = (S^*, E^*, I^*, Q^*, R^*, V^*)$, and the endemic equilibrium point P^* is globally asymptotically stable for $R_0 > 1$, where $S^* = \frac{(\omega_1 + \sigma_1 + \mu + h)(\gamma + \sigma_2 + \nu + \mu)}{\bar{\beta} \gamma}$, $E^* = \frac{\Lambda - (\mu + \delta)S^*}{\gamma + \sigma_2 + \mu}$, $I^* = \frac{\gamma E^*}{\omega_1 + \sigma_1 + \mu + h}$, $Q^* = \frac{\sigma_1 I^* + \sigma_2 E^*}{\omega_2 + \mu}$, $R^* = \frac{\omega_1 I^* + \omega_2 Q^*}{\mu}$, $V^* = \frac{\delta(\omega_1 + \sigma_1 + \mu + h)(\gamma + \sigma_2 + \nu + \mu)}{\bar{\beta} \gamma \mu}$.

However, due to the inherent variability in real-world populations, viral transmission is subject to stochastic disturbances. Consequently, a growing body of research in infection dynamics focuses on the stochastic properties of dynamical systems.

Stochastic epidemic models that incorporate environmental noise provide a more realistic representation of disease spread than traditional deterministic models [7]. For instance, Shi et al. [8] proposed a randomized COVID-19 SEIR model with an Ornstein-Uhlenbeck process, which contributes to our comprehension of the dynamic mechanisms. Su et al. [9] applied a randomized HBV infectious model having normal morbidity, cell-to-cell spread, and immunological reactions. The consequences of environmental perturbations on the mathematical model can be realized by revising the model's parameters. Two principal ways of changing the parameters are mentioned [10]. One approach is to pretend that the parameters are linear functions of Gaussian white noise. Liu et al. [11] developed an SEIR-based COVID-19 stochastic epidemic model incorporating independent standard Brownian motion to analyze transmission dynamics within infected populations. Shi et al. [12] developed a stochastic SEIRS rabies model incorporating population diffusion. The other option is to utilize the mean-reversion process to interfere with the parameters. Wei et al. [13] established the dynamic behavior of an HIV infection model featuring two transmission modes. Liu et al. [14] investigated a viral infectious disease model incorporating latent individuals and a log-OU process. Ornstein-Uhlenbeck process is notionally and biologically more relevant. It better characterizes environmental heterogeneity in biological systems than a linear function of white noise [15]. Thus, this study adopts the latter assumption. Since $\bar{\beta}$ is biologically important as the rate of transmission between susceptible and exposed individuals, then we suppose that $\bar{\beta}$ of Equation 2 varies arbitrarily due to environmental perturbations and it is amenable to the Ornstein-Uhlenbeck process. In this case, $\bar{\beta}$ could be changed to the form below:

$$d\beta(s) = \alpha(\bar{\beta} - \beta(s))ds + \theta dB(s), \quad (3)$$

where $\alpha > 0$ is for the speed of reversal. $\theta > 0$ stands for the intensity of fluctuation. $B(s)$ is the standard Brownian motion. The solution of Equation 3 can be expressed as follows: $\beta(s) = e^{-\alpha s} \beta(0) + (1 - e^{-\alpha s})\bar{\beta} + \int_0^s e^{-\alpha(s-x)} \theta dB(x)$, and $\beta(s) \sim N(\bar{\beta}, \frac{\theta^2}{2\alpha})$ as $s \rightarrow \infty$. By calculation, it is clear that when $s \rightarrow \infty$, we have

$$\mathbb{E}(\beta(s)) = e^{-\alpha s} \mathbb{E}(\beta(0)) + (1 - e^{-\alpha s})\bar{\beta} \rightarrow \bar{\beta}, \text{Var}(\beta(s)) = e^{-2\alpha s} \text{Var}(\beta(0)) + (1 - e^{-\alpha s}) \frac{\theta^2}{2\alpha} \rightarrow \frac{\theta^2}{2\alpha}.$$

Based on Allen [15], $\beta(s)$ is ergodic. Its asymptotic probability density function is $\pi(\chi) = \frac{\sqrt{\alpha}}{\theta\sqrt{\pi}} e^{-\frac{\alpha(\chi - \bar{\beta})^2}{\theta^2}}$. Assuming $\beta(0) = \bar{\beta}$. Defining $\bar{\beta}(s)$ to be the time-averaged, we have

$$\begin{aligned} \bar{\beta}(s) &= \frac{1}{s} \int_0^s \beta(x) dx + \frac{1}{s} \int_0^s \frac{\theta}{\alpha} (1 - e^{\alpha(x-s)}) dB(x), \mathbb{E}(\bar{\beta}(s)) = \bar{\beta} \\ &= \bar{\beta}, \text{Var}(\bar{\beta}(s)) = \frac{\theta^2 s}{3} + o(s^2). \end{aligned}$$

As $s \rightarrow 0$, the variance $\text{Var}(\bar{\beta}(s))$ tends to zero rather than infinity. This behavior indicates that the Ornstein-Uhlenbeck process is more suitable for characterizing the effects of stochastic perturbations. Because $\bar{\beta}$ is a normal number based on its biological significance, we know that $\beta(s)$ has a significant probability of being positive by the property of the normal distribution when θ becomes small. Letting $\beta^+(t) = \max\{\beta(s), 0\}$. Adding Ornstein-Uhlenbeck

process to Equation 2, we have the stochastic model as below:

$$\begin{cases} dS = [\Lambda - \beta^+(t)SI - (\mu + \delta)S + vE]dt, \\ dE = [\beta^+(t)SI - (\sigma_2 + \gamma + v + \mu)E]dt, \\ dI = [\gamma E - (\sigma_1 + \omega_1 + \mu + h)I]dt, \\ dQ = [\sigma_1 I + \sigma_2 E - (\omega_2 + \mu)Q]dt, \\ dR = [\omega_1 I + \omega_2 Q - \mu R]dt, \\ dV = [\delta S - \mu V]dt, \\ d\beta(t) = \alpha(\bar{\beta} - \beta(t))dt + \theta dB(t). \end{cases} \quad (4)$$

Since the sixth expression in Equation 4 does not affect the dynamical analysis, we only have to investigate the stochastic model as follows:

$$\begin{cases} dS = [\Lambda - \beta^+(t)SI - (\mu + \delta)S + vE]dt, \\ dE = [\beta^+(t)SI - (\sigma_2 + \gamma + v + \mu)E]dt, \\ dI = [\gamma E - (\sigma_1 + \omega_1 + \mu + h)I]dt, \\ dQ = [\sigma_1 I + \sigma_2 E - (\omega_2 + \mu)Q]dt, \\ dR = [\omega_1 I + \omega_2 Q - \mu R]dt, \\ d\beta(t) = \alpha(\bar{\beta} - \beta(t))dt + \theta dB(t). \end{cases} \quad (5)$$

The remaining parts of the article are structured according to the following description. Section 2 investigates whether a unique global solution exists for Equation 5. Section 3 gives an adequate condition for the extinction of the epidemic. Section 4 derives a stationary distribution for the Equation 5, which describes the continuation of the disorder. Section 5 derives the exact expression of the probability density function through solving the relevant six-dimensional Fokker-Planck equation. Section 6 illustrates theoretical outcomes with a few numerical simulations. Section 7 ends with several conclusions.

2 Existence of unique global solution

To analyze the dynamic behavior of Equation 5, it is necessary first to establish the existence of a global solution. This is a fundamental requirement for all subsequent proofs. Let $\mathcal{M}(t) = (S(t), E(t), I(t), Q(t), R(t), \beta(t))$.

Theorem 2.1. For any initial value $\mathcal{M}(0) \in \mathbb{R}_+^5 \times \mathbb{R}$, there will exist a unique global solution $\mathcal{M}(t)$ for Equation 5, and it will stay in $\mathbb{R}_+^5 \times \mathbb{R}$ with probability one.

Proof. Since the coefficients of system (1.5) are locally Lipschitz continuous on $\mathbb{R}_+^5 \times \mathbb{R}$ for any initial value $\mathcal{M}(0) \in \mathbb{R}_+^5 \times \mathbb{R}$, it follows that there exists a unique local solution $\mathcal{M}(t)$ on $t \in [0, \tau_e)$, where τ_e denotes the explosion time. To justify that $\mathcal{M}(t)$ is global, we simply prove $\tau_e = \infty$ a.s..

Let $k_1 > 0$ be large enough to ensure $\mathcal{M}(0) \in \mathbb{R}_+^5 \times \mathbb{R}$ remains inside the interval $[\frac{1}{k_1}, k_1]$. Then, for every integer $m \geq k_1$, define a stopping time as follows:

$$\tau_m = \inf\{t \in [0, \tau_e] : \min\{S(t), E(t), I(t), Q(t), R(t), e^{\beta(t)}\} \leq \frac{1}{m} \text{ or } \max\{S(t), E(t), I(t), Q(t), R(t), e^{\beta(t)}\} \geq m\}. \quad (6)$$

Set $\inf \phi = \infty$ (where ϕ represents the empty set). Obviously, τ_m increases monotonically with $m \rightarrow \infty$. Define $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, that is, $\tau_\infty \leq \tau_e$. In case we demonstrate that $\tau_\infty = \infty$, then $\tau_e = \infty$, indicating that $\mathcal{M}(t)$ is global and $\mathcal{M}(t) \in \mathbb{R}_+^5 \times \mathbb{R}$.

Now, assume for contradiction that there exist constants $\bar{T} > 0$, $\varepsilon_m \in (0, 1)$ obeys $\mathbb{P}\{\tau_\infty \leq \bar{T}\} > \varepsilon_m$. Accordingly, there is an integer $k_2 \geq k_1$ such that $\mathbb{P}\{\tau_\infty \leq \bar{T}\} \geq \varepsilon_m, \forall m \geq k_2$. What's more, $\forall m \geq k_1, t \leq \tau_m$, there has

$$\begin{aligned} \frac{d(S + E + I + Q + R)}{dt} &\leq \Lambda - \mu(S + E + I + Q + R) - \delta S \leq \Lambda \\ &\quad - \mu(S + E + I + Q + R) \end{aligned} \quad (7)$$

then

$$S(t) + E(t) + I(t) + Q(t) + R(t) \leq \frac{\Lambda}{\mu}, \quad (8)$$

where $S(0) + E(0) + I(0) + Q(0) + R(0) \leq \frac{\Lambda}{\mu}$. Then define a non-negative C^2 -function $\mathcal{U}(S, E, I, Q, R, \beta) : \mathbb{R}_+^5 \times \mathbb{R} \rightarrow \mathbb{R}_+$, as follows:

$$\begin{aligned} \mathcal{U}(S, E, I, Q, R, \beta) = & S - 1 - \ln S + E - 1 - \ln E + I - 1 \\ & - \ln I + Q - 1 - \ln Q \\ & + R - 1 - \ln R + \frac{\beta^2(t)}{2} \end{aligned}$$

Set $A = \sup_{\beta \in \mathbb{R}} \{(\alpha \bar{\beta} + \frac{\Lambda}{\mu}) | \beta(t) | - \alpha | \beta(t) |^2\}$. Using Itô's formula, we obtain

$$\begin{aligned} \mathcal{U} &= (1 - \frac{1}{S})dS + (1 - \frac{1}{E})dE + (1 - \frac{1}{I})dI + (1 - \frac{1}{Q})dQ \\ &\quad + (1 - \frac{1}{R})dR + \beta(t)d\beta(t) + \frac{1}{2}\theta^2 \\ &= \Lambda - \mu(S + E + I + Q + R) - \delta S - \frac{\Lambda}{S} + \beta^+(t)I + (\delta + \mu) \\ &\quad - \frac{vE}{S} - \frac{\beta^+(t)SI}{E} \\ &\quad + (\sigma_2 + \gamma + v + \mu) - \frac{\gamma E}{I} + (\sigma_1 + \omega_1 + \mu + h) - \frac{\sigma_1 I}{Q} \\ &\quad - \frac{\sigma_2 E}{Q} + (\omega_2 + \mu) - \frac{\omega_1 I}{R} \\ &\quad - \frac{\omega_2 Q}{R} + \mu + \alpha\beta(t)(\bar{\beta} - \beta(t)) + \frac{1}{2}\theta^2 \\ &\leq \Lambda + \beta^+(t)\frac{\Lambda}{\mu} + (\delta + \mu) + (\sigma_2 + \gamma + v + \mu) \\ &\quad + (\sigma_1 + \omega_1 + \mu + h) \\ &\quad + (\omega_2 + \mu) + \mu + \alpha\beta(t)(\bar{\beta} - \beta(t)) + \frac{1}{2}\theta^2 \\ &\leq (\alpha \bar{\beta} + \frac{\Lambda}{\mu}) | \beta(t) | - \alpha | \beta(t) |^2 + \Lambda + 5\mu + \delta + \gamma \\ &\quad + v + h + \omega_1 + \omega_2 + \sigma_1 + \sigma_2 + \frac{1}{2}\theta^2 \\ &\leq \Lambda + \Lambda + 5\mu + \delta + \gamma + v + h + \omega_1 + \omega_2 + \sigma_1 + \sigma_2 + \frac{1}{2}\theta^2 \\ &=: \tilde{k}. \end{aligned} \quad (9)$$

Here, $\tilde{k}$ is a positive constant. Integrating inequality from 0 to t and taking the expectation on both sides

$$\begin{aligned} 0 &\leq \mathbb{E}[\mathcal{U}(S(\tau_m \wedge \bar{T}), E(\tau_m \wedge \bar{T}), I(\tau_m \wedge \bar{T}), Q(\tau_m \wedge \bar{T}), R(\tau_m \wedge \bar{T}), \\ &\quad \beta(\tau_m \wedge \bar{T}))] \\ &= \mathbb{E}[\mathcal{U}(\mathcal{M}(0))] + \mathbb{E}\left[\int_0^{\tau_m \wedge \bar{T}} \mathcal{U}(\mathcal{M}(\tau)) d\tau\right] \\ &\leq \mathbb{E}[\mathcal{U}(\mathcal{M}(0))] + \tilde{k}\bar{T}. \end{aligned}$$

For $m \geq k_2$, let $\Omega_m = \tau_m \leq \bar{T}$, and there exists $\mathbb{P}(\Omega_m) \geq \varepsilon_m$. Thus for any $\xi \in \Omega_m$, there exists at minimum one in $(S(\tau_m, \xi), E(\tau_m, \xi), I(\tau_m, \xi), Q(\tau_m, \xi), R(\tau_m, \xi), e^{\beta(\tau_m, \xi)})$ reaches either m or $\frac{1}{m}$. So

$$\begin{aligned} &\mathbb{E}[\mathcal{U}(\mathcal{M}(0))] + \tilde{k}\bar{T} \\ &\geq \mathbb{E}[\mathcal{U}(S(\tau_m \wedge \bar{T}), E(\tau_m \wedge \bar{T}), I(\tau_m \wedge \bar{T}), Q(\tau_m \wedge \bar{T}), R(\tau_m \wedge \bar{T}), \\ &\quad \beta(\tau_m \wedge \bar{T}))] \\ &\geq \mathbb{E}[1_{\Omega_m(\xi)} \mathcal{U}(S(\tau_m \wedge \bar{T}), E(\tau_m \wedge \bar{T}), I(\tau_m \wedge \bar{T}), Q(\tau_m \wedge \bar{T}), \\ &\quad R(\tau_m \wedge \bar{T}), \beta(\tau_m \wedge \bar{T}))] \\ &\geq \varepsilon[(m-1) - \ln m] \wedge \left(\frac{1}{m} - 1 + \ln m\right) \wedge \frac{1}{4}(\ln m)^4. \end{aligned}$$

When $m \rightarrow \infty$, we can obtain $+\infty < \mathbb{E}[\mathcal{U}(\mathcal{M}(0))] + \tilde{k}\bar{T} < +\infty$. Thus, $\tau_\infty = +\infty$ a.s.; that is, $\tau_e = +\infty$. Hence, the Equation 5 admits a unique global solution $\mathcal{M}(t)$.

Remark 2.1. By Theorem 2.1, we know that Equation 5 has a unique global solution $\mathcal{M}(t) \in \mathbb{R}_+^5 \times \mathbb{R}$. And if $S(0) + E(0) + I(0) + Q(0) + R(0) \leq \frac{\Lambda}{\mu}$, there has $S(t) + E(t) + I(t) + Q(t) + R(t) \leq \frac{\Lambda}{\mu}$. Thus, we assume that $T^* = \{(S, E, I, Q, R, \beta) \in \mathbb{R}_+^5 \times \mathbb{R} : 0 < S + E + I + Q + R \leq \frac{\Lambda}{\mu}\}$ is the invariant set.

3 Extinction

Having established the existence and uniqueness of the system's solution, this section presents the sufficient conditions for eradicating COVID-19, providing a theoretical basis for targeted disease control efforts. Define

$$R_0^E = \sqrt{R_0} + \frac{\gamma\theta\Lambda}{\mu(\sigma_2 + \gamma + v + \mu)\sqrt{R_0\pi\alpha} \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\}}.$$

Theorem 3.1. If $R_0^E < 1$, it has

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{\ln\left(\frac{\bar{\omega}_1}{\sigma_2 + \gamma + v + \mu}E + \frac{\bar{\omega}_2}{\sigma_1 + \omega_1 + \mu + h}I\right)}{t} \\ &\leq \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\}(R_0^E - 1) < 0, \end{aligned}$$

, where $\bar{\omega}_1 = \frac{\gamma}{(\sigma_1 + \omega_1 + \mu + h)\sqrt{R_0}}$, $\bar{\omega}_2 = 1$.

Proof. Define

$$Y(E, I) = \frac{\bar{\omega}_1}{\sigma_2 + \gamma + v + \mu}E + \frac{\bar{\omega}_2}{\sigma_1 + \omega_1 + \mu + h}I.$$

Then $\frac{\bar{\omega}_1}{\sigma_2 + \gamma + v + \mu} \frac{E}{Y} + \frac{\bar{\omega}_2}{\sigma_1 + \omega_1 + \mu + h} \frac{I}{Y} = 1$, which means $\frac{I}{Y} < \frac{\sigma_1 + \omega_1 + \mu + h}{\bar{\omega}_2}$. Next, define the matrix M_0 as follows:

$$\begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu(\sigma_2 + \gamma + v + \mu)} \\ \frac{\gamma}{\sigma_1 + \omega_1 + \mu + h} & 0 \end{pmatrix}.$$

By direct calculation, one can obtain Equation 10:

$$(\bar{\omega}_1, \bar{\omega}_2)(M_0 - \mathbb{I}_2) \begin{pmatrix} E \\ I \end{pmatrix} = (\sqrt{R_0} - 1)(\bar{\omega}_1 E + \bar{\omega}_2 I). \quad (10)$$

Using Itô's formula to $\ln Y$, we have

$$\begin{aligned} L(\ln Y) &= \frac{1}{Y} \left(\frac{\bar{\omega}_1 \beta^+(t) SI}{(\sigma_2 + \gamma + v + \mu)} - \bar{\omega}_1 E \right. \\ &\quad \left. + \frac{\bar{\omega}_2 \gamma E}{\sigma_1 + \omega_1 + \mu + h} - \bar{\omega}_2 I \right) \\ &\leq \frac{1}{Y} \left(\frac{\bar{\omega}_1 \bar{\beta} \Lambda I}{\mu(\sigma_2 + \gamma + v + \mu)} - \bar{\omega}_1 E + \frac{\bar{\omega}_2 \gamma E}{\sigma_1 + \omega_1 + \mu + h} \right. \\ &\quad \left. - \bar{\omega}_2 I \right) + \frac{1}{Y} \frac{\bar{\omega}_1 (\beta^+(t) - \bar{\beta}) \Lambda I}{\mu(\sigma_2 + \gamma + v + \mu)} \\ &= \frac{1}{Y} (\bar{\omega}_1, \bar{\omega}_2)(M_0 - \mathbb{I}_2) \begin{pmatrix} E \\ I \end{pmatrix} + \frac{1}{Y} \frac{\bar{\omega}_1 (\beta^+(t) - \bar{\beta}) \Lambda I}{\mu(\sigma_2 + \gamma + v + \mu)} \\ &\leq \frac{1}{Y} (\sqrt{R_0} - 1)(\bar{\omega}_1 E + \bar{\omega}_2 I) + \frac{I}{Y} \frac{\bar{\omega}_1 |\beta(t) - \bar{\beta}| \Lambda}{\mu(\sigma_2 + \gamma + v + \mu)} \\ &\leq \frac{\bar{\omega}_1 E + \bar{\omega}_2 I}{\frac{\bar{\omega}_1}{\sigma_2 + \gamma + v + \mu} E + \frac{\bar{\omega}_2}{\sigma_1 + \omega_1 + \mu + h} I} (\sqrt{R_0} - 1) \\ &\quad + \frac{\sigma_1 + \omega_1 + \mu + h}{\bar{\omega}_2} \frac{\bar{\omega}_1 |\beta(t) - \bar{\beta}| \Lambda}{\mu(\sigma_2 + \gamma + v + \mu)} \\ &\leq \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\} (\sqrt{R_0} - 1) \\ &\quad + \frac{\sigma_1 + \omega_1 + \mu + h}{\bar{\omega}_2} \frac{\bar{\omega}_1 |\beta(t) - \bar{\beta}| \Lambda}{\mu(\sigma_2 + \gamma + v + \mu)} \\ &\leq \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\} (\sqrt{R_0} - 1) \\ &\quad + \frac{\gamma \Lambda}{(\mu(\sigma_2 + \gamma + v + \mu)\sqrt{R_0})} |\beta(t) - \bar{\beta}|. \end{aligned} \quad (11)$$

Integrating from 0 to t and dividing by t on both sides, we obtain

$$\begin{aligned} &\frac{\ln Y(t) - \ln Y(0)}{t} \leq \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\} \\ &\quad (\sqrt{R_0} - 1) \\ &\quad + \frac{\gamma \Lambda}{\mu(\sigma_2 + \gamma + v + \mu)\sqrt{R_0}} \frac{1}{t} \int_0^t |\beta(\tau) - \bar{\beta}| d\tau. \end{aligned} \quad (12)$$

Taking the superior limit,

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{\ln Y(t)}{t} \\ &\leq \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\} (\sqrt{R_0} - 1) \\ &\quad + \frac{\gamma \Lambda}{\mu(\sigma_2 + \gamma + v + \mu)\sqrt{R_0}} \frac{\theta}{\sqrt{\pi\alpha}} \\ &= \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\} (\sqrt{R_0} \\ &\quad + \frac{\gamma \Lambda \theta}{\mu(\sigma_2 + \gamma + v + \mu)\sqrt{R_0\pi\alpha} \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\}} - 1) \\ &\quad \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\} (\sqrt{R_0^E} - 1), \end{aligned} \quad (13)$$

where $R_0^E = \sqrt{R_0} + \frac{\gamma\theta\Lambda}{\mu(\sigma_2+\gamma+v+\mu)\sqrt{R_0\pi\alpha}\min\{\sigma_2+\gamma+v+\mu,\sigma_1+\omega_1+\mu+h\}}$. Thus if $R_0^E < 1$, we can obtain $\limsup_{t \rightarrow +\infty} \frac{\ln Y(t)}{t} \leq \min\{\sigma_2 + \gamma + v + \mu, \sigma_1 + \omega_1 + \mu + h\}(\sqrt{R_0^E} - 1) < 0$ a.s.. That means $\limsup_{t \rightarrow +\infty} \frac{\ln E(t)}{t} < 0, \limsup_{t \rightarrow +\infty} \frac{\ln I(t)}{t} < 0$ a.s., which is to say that $\lim_{t \rightarrow +\infty} E(t) = 0, \lim_{t \rightarrow +\infty} I(t) = 0$. This indicates that the disease ultimately becomes extinct with probability one. For COVID-19, achieving eradication requires maintaining $R_0^E < 1$ over the long term. From a mathematical theory perspective, the eradication of COVID-19 is feasible. However, in the real world, achieving this goal is extremely challenging due to complex environmental factors, including population mobility, viral mutations, and animal reservoirs. It is crucial to note that $R_0^E < 1$ describes the epidemic's transmission trend, not the instantaneous infection status. In the short term, residual infected individuals may persist within the population. However, due to the significantly reduced transmission efficiency of the virus, this will not trigger new large-scale outbreaks. Over time, the number of infected individuals will gradually decrease until the final infected person recovers or is eliminated, at which point the epidemic will be officially declared over.

Remark 3.1. There is $R_0^E \rightarrow \sqrt{R_0}$ as $\theta \rightarrow 0$, and $R_0^E < 1$ can derive $R_0 < 1$. This indicates that $R_0^E < 1$ can be regarded as a unifying criterion for both deterministic and stochastic models of disease extinction.

4 Stationary distribution

Deterministic models have been effective in capturing the continuity of COVID-19 infection dynamics. However, such models do not possess a fixed equilibrium state in the stochastic sense, due to the inherent randomness in disease transmission and progression. To more accurately characterize the long-term behavior of the epidemic under uncertainty, we turn to the concept of a stationary distribution within the stochastic framework. In this section, we derive sufficient conditions for the existence and stability of a stationary distribution for the stochastic system described by Equation 5. These conditions provide theoretical insights into the circumstances under which the infection is likely to persist endemically over time.

Lemma 4.1. ([16–18]) For any initial value $X(0) = X_0 \in T^*$, assuming there is a bounded closed domain $\Gamma \in T^*$ with a regular boundary, if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}(x, X_0, \Gamma) dx > 0, \text{ a.s.,}$$

where $\mathbb{P}(x, X_0, \Gamma)$ is the transition probability of $X(t)$. That means Equation 5 has at least one ergodic stationary distribution.

Define

$$R_0^S = \frac{\tilde{\beta}\Lambda\gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)(\sigma_2 + \gamma + v + \mu + c_1 \frac{\Lambda\theta}{\mu\sqrt{\pi\alpha}})}, \quad (14)$$

, where $\tilde{\beta} = (\int_0^{+\infty} x^{\frac{1}{4}} \pi(x) dx)^4, c_1 = \frac{\tilde{\beta}\Lambda\gamma}{(\delta + \mu)^2(\sigma_1 + \omega_1 + \mu + h)}$.

Theorem 4.1. Assume $R_0^S > 1$, the solution $\mathcal{M}(t)$ of Equation 5 has at least one stationary distribution $\pi(\cdot)$ on T^* .

Proof. Using Itô's formula to Equation 5,

$$\begin{aligned} L(-\ln S) &= -\frac{\Lambda}{S} - \frac{vE}{S} + \beta^+(t)I + \delta + \mu \leq \\ &\quad -\frac{\Lambda}{S} + \beta^+(t)I + \delta + \mu, \\ L(-\ln E) &= -\frac{\beta^+(t)SI}{E} + \sigma_2 + \gamma + v + \mu, \\ L(-\ln I) &= -\frac{\gamma E}{I} + \sigma_1 + \omega_1 + \mu + h, \\ L(-\ln Q) &= -\frac{\sigma_1 I}{Q} - \frac{\sigma_2 E}{Q} + \omega_2 + \mu \leq -\frac{\sigma_2 E}{Q} + \omega_2 + \mu, \\ L(-\ln R) &= -\frac{\omega_1 I}{R} - \frac{\omega_2 Q}{R} + \mu, \end{aligned}$$

$$\begin{aligned} &L(\ln(\frac{\Lambda}{\mu} - S - E - I - Q - R)) \\ &= \frac{-\Lambda + \mu(S + E + I + Q + R) + \delta S + hI}{\frac{\Lambda}{\mu} - S - E - I - Q - R} \\ &= -\mu + \frac{\delta S + hI}{\frac{\Lambda}{\mu} - S - E - I - Q - R}. \end{aligned}$$

Define $V_1 = -\ln E - c_1 \ln S - c_2 \ln I$, where c_1 will be confirmed in Equation 16.

Using Itô's formula to V_1 , we have

$$\begin{aligned} LV_1 &\leq -\frac{\beta^+(t)SI}{E} + \sigma_2 + \gamma + v + \mu - \frac{c_1\Lambda}{S} + c_1\beta^+(t)I \\ &\quad + c_1(\delta + \mu) - \frac{c_2\gamma E}{I} + c_2(\sigma_1 + \omega_1 + \mu + h), \\ &\leq -3\sqrt[3]{\beta^+(t)c_1c_2\Lambda\gamma} + c_1\beta^+(t)I + c_1(\delta + \mu) \\ &\quad + c_2(\sigma_1 + \omega_1 + \mu + h) + \sigma_2 + \gamma + v + \mu, \\ &= -3\sqrt[3]{\tilde{\beta}c_1c_2\Lambda\gamma} + c_1\beta^+(t)I + c_1(\delta + \mu) \\ &\quad + c_2(\sigma_1 + \omega_1 + \mu + h) + \sigma_2 + \gamma + v + \mu \\ &\quad + 3(\sqrt[3]{\tilde{\beta}c_1c_2\Lambda\gamma} - \sqrt[3]{\beta^+(t)c_1c_2\Lambda\gamma}), \\ &= -3\sqrt[3]{\tilde{\beta}c_1c_2\Lambda\gamma} + c_1(\delta + \mu) + c_2(\sigma_1 + \omega_1 + \mu + h) \\ &\quad + \sigma_2 + \gamma + v + \mu + c_1\tilde{\beta}I + c_1I|\beta^+(t) - \tilde{\beta}| \\ &\quad + 3(\sqrt[3]{\tilde{\beta}c_1c_2\Lambda\gamma} - \sqrt[3]{\beta^+(t)c_1c_2\Lambda\gamma}), \\ &\leq -3\sqrt[3]{\tilde{\beta}c_1c_2\Lambda\gamma} + c_1(\delta + \mu) + c_2(\sigma_1 + \omega_1 + \mu + h) \\ &\quad + \sigma_2 + \gamma + v + \mu + c_1\tilde{\beta}I + c_1\frac{\Lambda}{\mu}|\beta^+(t) - \tilde{\beta}| \\ &\quad + c_1\frac{\Lambda\theta}{\mu\sqrt{\pi\alpha}} - c_1\frac{\Lambda\theta}{\mu\sqrt{\pi\alpha}} + 3(\sqrt[3]{\tilde{\beta}c_1c_2\Lambda\gamma} \\ &\quad - \sqrt[3]{\beta^+(t)c_1c_2\Lambda\gamma}). \end{aligned} \quad (15)$$

Let c_1 and c_2 satisfy the following equalities

$$c_1(\delta + \mu) = c_2(\sigma_1 + \omega_1 + \mu + h) = \frac{\tilde{\beta}\Lambda\gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)},$$

and note

$$H(\beta(t)) = c_1 \frac{\Lambda}{\mu} |\beta^+(t) - \bar{\beta}| - c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}} + 3(\sqrt[3]{\bar{\beta} c_1 c_2 \Lambda \gamma} - \sqrt[3]{\beta^+(t) c_1 c_2 \Lambda \gamma}),$$

then

$$LV_1 \leq -\frac{\bar{\beta} \Lambda \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)} + \sigma_2 + \gamma + \nu + \mu + c_1 \bar{\beta} I + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}} + H(\beta(t)). \quad (16)$$

Define $V_2 = V_1 + \frac{c_1 \bar{\beta}}{\sigma_1 + \omega_1 + \mu + h} I$. Applying Itô's formula to V_2 ,

$$\begin{aligned} LV_2 &\leq -\frac{\bar{\beta} \Lambda \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)} + \sigma_2 + \gamma + \nu + \mu \\ &+ c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}} + \frac{c_1 \bar{\beta} \gamma}{\sigma_1 + \omega_1 + \mu + h} E + H(\beta(t)), \\ &= -(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}})(R_0^S - 1) \\ &+ \frac{c_1 \bar{\beta} \gamma}{\sigma_1 + \omega_1 + \mu + h} E + H(\beta(t)), \end{aligned} \quad (17)$$

$$\text{where } R_0^S = \frac{\bar{\beta} \Lambda \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}})}.$$

Denote

$$\begin{aligned} \bar{W} &= MV_2 - \ln S - \ln I - \ln Q - \ln R - \\ &\ln\left(\frac{\Lambda}{\mu} - S - I - E - Q - R\right) + \frac{\beta^2(t)}{2}, \end{aligned}$$

and

$$\begin{aligned} B &= \sup_{\beta(t) \in R} \{5\mu + \delta + h + \sigma_1 + \omega_1 + \omega_2 \\ &+ (\alpha \bar{\beta} + \frac{\Lambda}{\mu}) |\beta(t)| - \frac{1}{2} \alpha |\beta(t)|^2 + \frac{1}{2} \theta^2\}, \end{aligned}$$

where M is a sufficiently large positive constant satisfying the inequality $-M(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}})(R_0^S - 1) + B \leq -2$. It has been observed that $\bar{W} \rightarrow +\infty$ as (S, E, I, Q, R, β) toward the boundary of T^* . Thus, it is essential that it has a minimum value $\bar{W}_{min}$. Finally, we obtain a C^2 -function W : $W(S, E, I, Q, R, \beta) = \bar{W}(S, E, I, Q, R, \beta) - \bar{W}_{min}$. Using Itô's formula to $W(S, E, I, Q, R, \beta)$,

we obtain

$$\begin{aligned} LW &\leq -M(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}})(R_0^S - 1) \\ &+ M \frac{c_1 \bar{\beta} \gamma}{\sigma_1 + \omega_1 + \mu + h} E + MH(\beta(t)) - \frac{\Lambda}{S} \\ &+ \beta^+(t) I + \delta + 5\mu - \frac{\gamma E}{I} + \sigma_1 + \omega_1 + h - \frac{\sigma_2 E}{Q} \\ &+ \omega_2 + -\frac{\omega_1 I}{R} - \frac{\omega_2 Q}{R} \\ &- \frac{\delta S + h I}{\frac{\Lambda}{\mu} - S - E - I - Q - R} + \alpha \beta(t)(\bar{\beta} - \beta(t)) + \frac{1}{2} \theta^2 \\ &\leq -M(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}})(R_0^S - 1) \\ &+ 5\mu + \delta + h + \sigma_1 + \omega_1 + \omega_2 + (\alpha \bar{\beta} + \frac{\Lambda}{\mu}) |\beta(t)| \\ &- \frac{1}{2} \alpha |\beta(t)|^2 + \frac{1}{2} \theta^2 + M \frac{c_1 \bar{\beta} \gamma}{\sigma_1 + \omega_1 + \mu + h} E \\ &- \frac{\Lambda}{S} - \frac{\gamma E}{I} - \frac{\sigma_2 E}{Q} - \frac{\omega_1 I}{R} - \frac{\omega_2 Q}{R} \\ &- \frac{\delta S}{\frac{\Lambda}{\mu} - S - E - I - Q - R} - \frac{1}{2} \alpha |\beta(t)|^2 + MH(\beta(t)) \\ &:= G(S, E, I, Q, R, \beta(t)) + MH(\beta(t)). \end{aligned} \quad (18)$$

Then we define a closed subset of D_ε as follows: $D_\varepsilon = \{(S, E, I, Q, R, \beta(t) \in T^*) | S \geq \varepsilon, E \geq \varepsilon, I \geq \varepsilon^2, Q \geq \varepsilon^2, R \geq \varepsilon^3, S + E + I + Q + R \leq \frac{\Lambda}{\mu} - \varepsilon^2, \varepsilon \leq \beta \leq \frac{1}{\varepsilon}\}$. Suppose that ε is a minimal positive constant sufficient to satisfy the inequalities below.

$$\begin{aligned} -2 + M \frac{c_1 \bar{\beta} \gamma}{\sigma_1 + \omega_1 + \mu + h} \varepsilon &\leq -1, -2 \\ + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\Lambda}{\varepsilon} &\leq -1, \\ -2 \\ + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\gamma}{\varepsilon} &\leq -1, -2 \\ + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\omega_1}{\varepsilon} &\leq -1, \\ -2 \\ + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\omega_2}{\varepsilon} &\leq -1, -2 \\ + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\sigma_2}{\varepsilon} &\leq -1, \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\delta}{\varepsilon} &\leq -1, -2 \\ + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h) \mu} - \frac{\alpha}{2\varepsilon^2} &\leq -1. \end{aligned} \quad (19)$$

The complement of D_ε can be divided into eight domains below:

$$\begin{aligned} D_{1,\varepsilon}^c &= \{(S, E, I, Q, R, \beta(t)) \in T^* | E < \varepsilon\}, D_{2,\varepsilon}^c \\ &= \{(S, E, I, Q, R, \beta(t)) \in T^* | S < \varepsilon\}, \\ D_{3,\varepsilon}^c &= \{(S, E, I, Q, R, \beta(t)) \in T^* | E \geq \varepsilon, I < \varepsilon^2\}, D_{4,\varepsilon}^c \\ &= \{(S, E, I, Q, R, \beta(t)) \in T^* | I \geq \varepsilon^2, R < \varepsilon^3\}, \\ D_{5,\varepsilon}^c &= \{(S, E, I, Q, R, \beta(t)) \in T^* | Q \geq \varepsilon^2, R < \varepsilon^3\}, D_{6,\varepsilon}^c \\ &= \{(S, E, I, Q, R, \beta(t)) \in T^* | E \geq \varepsilon, Q < \varepsilon^2\}, \\ D_{7,\varepsilon}^c &= \{(S, E, I, Q, R, \beta(t)) \in T^* | S \geq \varepsilon, S + E + I + Q + R \\ &> \frac{\Lambda}{\mu} - \varepsilon^2\}, \\ D_{8,\varepsilon}^c &= \{(S, E, I, Q, R, \beta(t)) \in T^* | |\beta(t)| > \frac{1}{\varepsilon}\}. \end{aligned}$$

Then, we can obtain the following result from the inequalities

$$\begin{aligned} \text{Case 1: } (S, E, I, Q, R, \beta(t)) \in D_{1,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma}{\sigma_1 + \omega_1 + \mu + h} \varepsilon &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 2: } (S, E, I, Q, R, \beta(t)) \in D_{2,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\Lambda}{\varepsilon} &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 3: } (S, E, I, Q, R, \beta(t)) \in D_{3,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\gamma}{\varepsilon} &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 4: } (S, E, I, Q, R, \beta(t)) \in D_{4,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\omega_1}{\varepsilon} &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 5: } (S, E, I, Q, R, \beta(t)) \in D_{5,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\omega_2}{\varepsilon} &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 6: } (S, E, I, Q, R, \beta(t)) \in D_{6,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\sigma_2}{\varepsilon} &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 7: } (S, E, I, Q, R, \beta(t)) \in D_{7,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\delta}{\varepsilon} &\leq -1. \end{aligned}$$

$$\begin{aligned} \text{Case 8: } (S, E, I, Q, R, \beta(t)) \in D_{8,\varepsilon}^c, G(S, E, I, Q, R, \beta(t)) &\leq \\ -2 + M \frac{c_1 \bar{\beta} \gamma \Lambda}{(\sigma_1 + \omega_1 + \mu + h)\mu} - \frac{\alpha}{2\varepsilon^2} &\leq -1. \end{aligned}$$

Given the above derivations, it is clear that it has a sufficiently small constant ε and a closed set D_ε such that $G(S, E, I, Q, R, \beta(t)) \leq -1, \forall (S, E, I, Q, R, \beta(t)) \in T^* \setminus D_\varepsilon$.

For any $(S, E, I, Q, R, \beta(t)) \in T^*$, denote $X = \sup\{G(S, E, I, Q, R, \beta(t)) + MH(\beta(t))\}$. Then $G(S, E, I, Q, R, \beta(t)) \leq X < +\infty, \forall (S, E, I, Q, R, \beta(t)) \in T^*$.

For any initial value $\mathcal{M}(0) \in T^*$, integrating and applying

mathematical expectation, we obtain

$$\begin{aligned} 0 &\leq \frac{\mathbb{E}[W(\mathcal{M}(t))]}{t} = \frac{\mathbb{E}[LW(\mathcal{M}(0))]}{t} + \frac{1}{t} \int_0^t \mathbb{E}[W(\mathcal{M}(\tau))] d\tau \\ &\leq \frac{\mathbb{E}[LW(\mathcal{M}(0))]}{t} + \frac{1}{t} \int_0^t \mathbb{E}[G(\mathcal{M}(\tau))] d\tau \\ &\quad + 3M\sqrt[3]{c_1 c_2 \Lambda \gamma} \frac{1}{t} \int_0^t \mathbb{E}[(\sqrt[3]{\tilde{\beta}} - \sqrt[3]{\beta^+}(\tau))] d\tau \\ &\quad + Mc_1 \frac{\Lambda}{\mu t} \int_0^t \mathbb{E}[|\beta(\tau) - \bar{\beta}| - \frac{\theta}{\sqrt{\pi\alpha}}] d\tau. \end{aligned} \quad (20)$$

According to Cai et al. [19] and Zhang and Yuan [20], $\beta(t)$ has ergodic property, so

$$\begin{aligned} &\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t |\beta(\zeta) - \bar{\beta}| d\zeta \\ &= \int_{-\infty}^{+\infty} |x - \bar{\beta}| \pi(x) dx = \frac{\theta}{\sqrt{\pi\alpha}}, \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \sqrt{\beta^+}(\zeta) d\zeta \\ &= \left(\int_0^{+\infty} x^{\frac{1}{4}} \pi(x) dx \right)^2. \end{aligned} \quad (21)$$

We can derive from that

$$\begin{aligned} 0 &\leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{E}[G(\mathcal{M}(\tau))] d\tau \\ &= \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{E}[G(\mathcal{M}(\tau)) I_{\mathcal{M}(\tau) \in D_\varepsilon}] d\tau \\ &\quad + \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{E}[G(\mathcal{M}(\tau)) I_{\mathcal{M}(\tau) \in \Omega^* \setminus D_\varepsilon}] d\tau \\ &\leq X \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{P}(\mathcal{M}(\tau)) \in D_\varepsilon d\tau \\ &\quad - \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{P}\{(\mathcal{M}(\tau)) \in \Omega^* \setminus D_\varepsilon\} d\tau \\ &\leq -1 + (X+1) \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{P}\{(\mathcal{M}(\tau)) \in D_\varepsilon\} d\tau. \end{aligned} \quad (22)$$

Hence, we obtain $\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{P}\{(\mathcal{M}(\tau)) \in D_\varepsilon\} d\tau \geq \frac{1}{X+1} > 0$ a.s.. Therefore $\forall (\mathcal{M}(0)) \in T^*, \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{P}\{\tau, (\mathcal{M}(\tau)), D_\varepsilon\} d\tau \geq \frac{1}{X+1} > 0$.

According to Lemma 4.1, Equation 5 has a stationary distribution $\pi(\cdot)$ on T^* . Thus, the proof of Theorem 4.1 is completed.

Furthermore, in order to observe the influence of the stochastic perturbation on R_0^S for numerical simulation, we need to evaluate the value of $\tilde{\beta}$. Let $\tilde{\beta}_0 \leq \tilde{\beta} \leq \tilde{\beta}_1$, the calculations for $\tilde{\beta}_0$ and $\tilde{\beta}_1$ are provided below.

For easier computation, define $\rho = \frac{\theta^2}{2\alpha}$, then $\pi(\chi) = \frac{1}{\sqrt{2\pi\rho}} e^{-\frac{(\chi-\bar{\beta})^2}{2\rho^2}}$. Let $F(\chi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\chi^2}{2}}$ be the probability density function of the standard normal distribution and $\Phi(\chi) = \int_{-\infty}^{\chi} F(s) ds$, then $\Phi(+\infty) = 1, \Phi(-\infty) = 0$. Take $\Gamma(\chi) = \int_0^{+\infty} s^{\chi-1} e^{-s} ds (\chi > 0)$, then $\Gamma(\chi+1) = \chi \Gamma(\chi), \Gamma(\frac{1}{2}) = \sqrt{\pi}$. Let $K = \int_0^{+\infty} \chi^{\frac{1}{4}} \pi(\chi) d\chi$. Combining the above definitions and letting

$x = \frac{\chi - \bar{\beta}}{\rho}$, we have

$$\begin{aligned} K &= \int_0^{+\infty} \chi^{\frac{1}{4}} \pi(\chi) d\chi = \int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} (\bar{\beta} + \rho x)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \bar{\beta}^{\frac{1}{4}} \int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} (1 + \frac{\rho x}{\bar{\beta}})^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \end{aligned} \quad (23)$$

Since $(1 + a)^x \geq 1 + \frac{x(x-1)}{2} a^2$ ($0 < x < 1$), the Equation 23 can be expressed as

$$\begin{aligned} K &= \bar{\beta}^{\frac{1}{4}} \int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} (1 + \frac{\rho x}{\bar{\beta}})^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \geq \bar{\beta}^{\frac{1}{4}} \\ &\int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} (1 - \frac{3\rho^2 x^2}{32\bar{\beta}^2}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\geq \bar{\beta}^{\frac{1}{4}} [\int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \frac{3\rho^2}{32\bar{\beta}^2 \sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2}} dx] \\ &= \bar{\beta}^{\frac{1}{4}} [\Phi(+\infty) - \Phi(-\frac{\bar{\beta}}{\rho})] - \frac{3\rho^2}{16\bar{\beta}^{-\frac{7}{4}} \sqrt{2\pi}} \int_0^{+\infty} x^2 e^{-\frac{x^2}{2}} dx \\ &= \bar{\beta}^{\frac{1}{4}} \Phi(\frac{\bar{\beta}}{\rho}) - \frac{3\rho^2}{16\bar{\beta}^{-\frac{7}{4}} \sqrt{2\pi}} \int_0^{+\infty} \sqrt{2} s^{\frac{1}{2}} e^{-s} ds \\ &= \bar{\beta}^{\frac{1}{4}} \Phi(\frac{\bar{\beta}}{\rho}) - \frac{3\rho^2}{16\bar{\beta}^{-\frac{7}{4}} \sqrt{2\pi}} \Gamma(\frac{3}{2}) = \bar{\beta}^{\frac{1}{4}} [\Phi(\frac{\bar{\beta}}{\rho}) - \frac{3\rho^2}{32\bar{\beta}^2}] \\ &:= (\tilde{\beta}_0)^{\frac{1}{4}}. \end{aligned} \quad (24)$$

Similarly, since $(1 + a)^x \leq 1 + xa$ ($0 < x < 1$), the Equation 23 can be also expressed as follows:

$$\begin{aligned} K &= \bar{\beta}^{\frac{1}{4}} \int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} (1 + \frac{\rho x}{\bar{\beta}})^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \leq \bar{\beta}^{\frac{1}{4}} \\ &\int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} (1 + \frac{\rho x}{4\bar{\beta}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &\leq \bar{\beta}^{\frac{1}{4}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} \frac{\rho x}{4\bar{\beta}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \bar{\beta}^{\frac{1}{4}} [1 + \frac{\rho}{4\sqrt{2\pi}\bar{\beta}} \int_{-\frac{\bar{\beta}}{\rho}}^{+\infty} x e^{-\frac{x^2}{2}} dx] \\ &= \bar{\beta}^{\frac{1}{4}} [1 + \frac{\rho}{4\sqrt{2\pi}\bar{\beta} e^{\frac{\bar{\beta}^2}{2\rho^2}}}] := (\tilde{\beta}_1)^{\frac{1}{4}}. \end{aligned}$$

It follows that $\tilde{\beta}_0 \rightarrow \tilde{\beta} \rightarrow \tilde{\beta}_1$ when $\theta \rightarrow 0$ from the expressions for $\tilde{\beta}_0$ and $\tilde{\beta}_1$, $\tilde{\beta}_0 = \bar{\beta} [\Phi(\frac{\bar{\beta}}{\rho}) - \frac{3\rho^2}{32\bar{\beta}^2}]^4$, $\tilde{\beta}_1 = \bar{\beta} [1 + \frac{\rho}{4\sqrt{2\pi}\bar{\beta} e^{\frac{\bar{\beta}^2}{2\rho^2}}}]^4$.

Define

$$R_{00}^S = \frac{\tilde{\beta}_0 \Lambda \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{S_0 \theta}{\sqrt{\pi \alpha}})}, \quad (25)$$

It is clear that $R_{00}^S \leq R_0^S$ and there is $R_{00}^S \rightarrow R_0^S$ when $\theta \rightarrow 0$. Therefore, if $R_{00}^S > 1$, then $R_0^S > 1$. Moreover, we can observe

$$\begin{aligned} R_{00}^S &= \frac{\tilde{\beta}_0 \Lambda \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{S_0 \theta}{\sqrt{\pi \alpha}})} \\ &\leq \frac{\tilde{\beta} \Lambda \gamma}{(\mu + \delta)(\omega_1 + \sigma_1 + \mu + h)(\gamma + \sigma_2 + \nu + \mu)} = R_0, \end{aligned}$$

which indicates that if $R_{00}^S > 1$, then $R_0 > 1$. Consequently, when $R_{00}^S > 1$, the endemic equilibrium point P^* of Equation 2 is stable and Equation 5 has a stationary distribution $\pi(\cdot)$. This suggests that $R_{00}^S > 1$ can be viewed as a harmonized threshold for COVID-19 prevalence for both deterministic and stochastic systems. Similar to R_0^E , the essence of R_0^S lies in the average number of healthy individuals that a single infected person can transmit the disease to within a susceptible population. It does not directly represent the actual number of infections occurring during an outbreak but serves as a core metric for measuring the strength of an infectious disease's transmission potential. For COVID-19, sustained epidemic circulation requires the basic reproduction number R_0^S to remain stably above 1 over the long term. Thus, when the virus's baseline transmissibility $\bar{\beta}$ is sufficiently high (the numerator is sufficiently large) and can offset the effects of control measures (represented by interventions such as δ , σ_1 , σ_2 , and h) through mutation or waning immunity (reflected in the dynamics of parameters ν and S^*), R_0^S can stabilize above 1 over the long term, transforming the outbreak from a transient event into sustained endemicity.

5 Probability density function

Section 4 demonstrates the existence of a stationary distribution, indicating that the disease will persist. To better elucidate the dynamics and statistical properties of the stochastic system, we will provide an exact expression for the probability density function of the steady-state distribution in this section.

When $R_0 > 1$, there will exist a quasi-equilibrium $\tilde{P}^* = (S^*, E^*, I^*, Q^*, R^*, \beta^*)$, which satisfies the equations below:

$$\begin{cases} \Lambda - \beta^* S^* I^* - (\mu + \delta) S^* + \nu E^* = 0, \\ \beta^* S^* I^* - (\sigma_2 + \gamma + \nu + \mu) E^* = 0, \\ \gamma E^* - (\sigma_1 + \omega_1 + \mu + h) I^* = 0, \\ \sigma_1 I^* + \sigma_2 E^* - (\omega_2 + \mu) Q^* = 0, \\ \omega_1 I^* + \omega_2 Q^* - \mu R^* = 0, \\ \alpha(\bar{\beta} - \beta^*) = 0. \end{cases} \quad (26)$$

Taking $C_1 = S - S^*$, $C_2 = E - E^*$, $C_3 = I - I^*$, $C_4 = Q - Q^*$, $C_5 = R - R^*$, $C_6 = \beta - \beta^*$, the corresponding linearized system for Equation 5 is

$$\begin{cases} dC_1 = (-m_{11} C_1 + m_{12} C_2 - m_{13} C_3 - m_{16} C_6) dt, \\ dC_2 = (m_{21} C_1 - m_{22} C_2 + m_{23} C_3 + m_{26} C_6) dt, \\ dC_3 = (m_{32} C_2 - m_{33} C_3) dt, \\ dC_4 = (-m_{42} C_2 + m_{43} C_3 - m_{44} C_4) dt, \\ dC_5 = (m_{53} C_3 - m_{54} C_4 - m_{55} C_5) dt, \\ dC_6 = -m_{66} C_6 dt + \theta dB(t). \end{cases} \quad (27)$$

where

$$\begin{aligned} m_{11} &= \bar{\beta}I^* + (\delta + \mu), m_{12} = \nu, m_{13} = \bar{\beta}S^* = m_{23}, m_{16} = S^*I^* \\ &= m_{26}, m_{21} = \bar{\beta}I^*, \\ m_{22} &= \sigma_2 + \gamma + \nu + \mu, m_{32} = \gamma, m_{33} = \sigma_1 + \omega_1 + \mu + h, m_{42} \\ &= \sigma_2, m_{43} = \sigma_1, \\ m_{44} &= \omega_2 + \mu, m_{53} = \omega_1, m_{54} = \omega_2, m_{55} = \mu, m_{66} = \alpha. \end{aligned}$$

Next, we denote

$$G = \begin{pmatrix} -m_{11} & m_{12} & -m_{13} & 0 & 0 & -m_{16} \\ m_{21} & -m_{22} & m_{13} & 0 & 0 & m_{16} \\ 0 & m_{32} & -m_{33} & 0 & 0 & 0 \\ 0 & -m_{42} & m_{43} & -m_{44} & 0 & 0 \\ 0 & 0 & m_{53} & -m_{54} & -m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_{66} \end{pmatrix},$$

$$\Psi = \text{diag}(0, 0, 0, 0, 0, \theta),$$

$$C(t) = (C_1, C_2, C_3, C_4, C_5, C_6)^T, \mathbf{B}(t) = (0, 0, 0, 0, 0, B(t))^T.$$

Then, the Equation 5 can be expressed by a matrix as follows:

$$dC(t) = GC(t)dt + \Psi d\mathbf{B}(t).$$

Theorem 5.1. If $R_0^S > 1$, $a_2 \neq 0$, $a_5 \neq 0$ and $a_7 \neq 0$, the solution $\mathcal{M}(t)$ of Equation 5 around the quasi-stationary equilibrium $\tilde{P}^*$ follows a normal probability density function $\Phi(S, E, I, Q, R, \beta)$:

$$\begin{aligned} \Phi(S, E, I, Q, R, \beta) &= (2\pi)^{-3} |\Sigma|^{-\frac{1}{2}} \\ &\exp\left[-\frac{1}{2}(C_1, C_2, C_3, C_4, C_5, C_6)\Sigma^{-1}(C_1, C_2, C_3, C_4, C_5, C_6)^T\right], \end{aligned}$$

where

$$\begin{aligned} \Sigma &= b_1^2 \theta^2 (J_6 J_5 J_4 J_3 J_2 J_1)^{-1} \Sigma^* [(J_6 J_5 J_4 J_3 J_2 J_1)^T]^{-1}, \\ \Sigma^* &= \begin{pmatrix} c_{11} & 0 & c_{13} & 0 & c_{15} & 0 \\ 0 & -c_{13} & 0 & -c_{15} & 0 & c_{26} \\ c_{13} & 0 & c_{15} & 0 & -c_{26} & 0 \\ 0 & -c_{15} & 0 & c_{26} & 0 & c_{46} \\ c_{15} & 0 & -c_{26} & 0 & -c_{46} & 0 \\ 0 & c_{26} & 0 & c_{46} & 0 & c_{66} \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ J_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_{42}}{m_{32}} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 1 \end{pmatrix}, \\ J_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{a_3}{a_2} & 1 & 0 \\ 0 & 0 & 0 & -\frac{a_4}{a_2} & 0 & 1 \end{pmatrix}, J_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{a_6}{a_5} & 1 \end{pmatrix}, \\ J_6 &= \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ 0 & a_2 a_5 a_7 m_{32} & b_7 & b_8 & b_9 & (m_{11} - m_{21})^4 \\ 0 & 0 & a_2 a_5 a_7 & b_{10} & b_{11} & -(m_{11} - m_{21})^3 \\ 0 & 0 & 0 & a_5 a_7 & a_7 (-m_{11} + m_{21} - m_{55}) & (m_{11} - m_{21})^2 \\ 0 & 0 & 0 & 0 & a_7 & -m_{11} + m_{21} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Proof. For the proof of Theorem 5.1, we proceed in two steps. Step 1: We prove that the eigenvalues of matrix G all contain negative real parts. Step 2: We demonstrate that Σ is a positive definite matrix.

Step 1. Defining the characteristic polynomial of a matrix G to be $\varphi_G(\lambda)$, we have the following:

$$\varphi_G(\lambda) = (\lambda + m_{66})(\lambda + m_{55})(\lambda^4 + q_1 \lambda^3 + q_2 \lambda^2 + q_3 \lambda + q_4),$$

where

$$\begin{aligned} q_1 &= m_{11} + m_{22} + m_{33} + m_{44} > 0, \\ q_2 &= (m_{11}m_{22} - m_{12}m_{21}) + m_{11}m_{33} + m_{11}m_{44} \\ &\quad + (m_{22}m_{33} - m_{23}m_{32}) + m_{22}m_{44} + m_{33}m_{44} > 0, \\ q_3 &= m_{11}m_{22}m_{33} - m_{11}m_{23}m_{32} - m_{12}m_{21}m_{33} + m_{13}m_{21}m_{32} \\ &\quad + (m_{11}m_{22}m_{44} - m_{12}m_{21}m_{44}) + m_{11}m_{33}m_{44} \\ &\quad + (m_{22}m_{33}m_{44} - m_{23}m_{32}m_{44}) \\ &= (m_{11} - m_{21})(m_{22}m_{33} - m_{32}m_{23}) \\ &\quad + (m_{21}m_{22}m_{33} - m_{12}m_{21}m_{33}) \\ &\quad + (m_{11}m_{22}m_{44} - m_{12}m_{21}m_{44}) + m_{11}m_{33}m_{44} \\ &\quad + (m_{22}m_{33}m_{44} - m_{23}m_{32}m_{44}) > 0, \\ q_4 &= m_{11}m_{22}m_{33}m_{44} - m_{11}m_{23}m_{32}m_{44} - m_{12}m_{21}m_{33}m_{44} \\ &\quad + m_{13}m_{21}m_{32}m_{44} \\ &= m_{44}[(m_{11} - m_{21})(m_{22}m_{33} - m_{32}m_{23}) \\ &\quad + m_{21}m_{33}(m_{22} - m_{12})] > 0. \end{aligned}$$

By calculation, we can obtain $q_1 q_2 - q_3 > 0$, $q_1 q_2 q_3 - q_3^2 - q_1^2 q_4 > 0$. Therefore, according to the Hurwitz criterion, the eigenvalues of the matrix G all contain negative real parts.

Step 2. Based on Markov theory [21], the probability density function $\Phi(S, E, I, Q, R, \beta)$ of the Equation 5 can be expressed by the following Fokker-Planck equation,

$$\begin{aligned} &-\frac{\theta^2}{2} \frac{\partial^2}{\partial C_6^2} \Phi + \frac{\partial}{\partial \beta} [(-m_{66} C_6) \Phi] + \frac{\partial}{\partial C_1} [(-m_{11} C_1 \\ &\quad + m_{12} C_2 - m_{13} C_3 - m_{16} C_6) \Phi] \\ &\quad + \frac{\partial}{\partial C_2} [(m_{21} C_1 - m_{22} C_2 + m_{23} C_3 + m_{26} C_6) \Phi] \\ &\quad + \frac{\partial}{\partial C_3} [(m_{32} C_2 - m_{33} C_3) \Phi] \\ &\quad + \frac{\partial}{\partial C_4} [(-m_{42} C_2 + m_{43} C_3 - m_{44} C_4) \Phi] \\ &\quad + \frac{\partial}{\partial C_5} [(m_{53} C_3 - m_{54} C_4 - m_{55} C_5) \Phi] = 0. \end{aligned} \quad (28)$$

The equation can be expressed as a Gaussian distribution $\Phi(C) = c \exp\{-\frac{1}{2} C \Upsilon C^T\}$, where c is a constant and Υ satisfies $\Upsilon \Psi^2 \Upsilon + G^T \Upsilon + \Upsilon G = 0$. Then if Υ is a positive definite matrix, denote $\Upsilon^{-1} = \Sigma$, we obtain $\Psi^2 + G \Sigma + \Sigma G^T = 0$.

Through the calculations in the Appendix 1, we obtain the

transition matrix G_6 as follows:

$$G_6 = \begin{pmatrix} -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & -d_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where $d_1 = m_{66} + m_{55} + q_1$, $d_2 = (m_{55} + q_1)m_{66} + m_{55}q_1 + q_2$, $d_3 = (m_{55}q_1 + q_2)m_{66} + m_{55}q_2 + q_3$, $d_4 = (m_{55}q_2 + q_3)m_{66} + m_{55}q_3 + q_4$, $d_5 = (m_{55}q_3 + q_4)m_{66} + m_{55}q_4$, $d_6 = m_{66}m_{55}q_4$.

Next, we can transform the equation into the following form:

$$J_6 J_5 J_4 J_3 J_2 J_1 \Psi^2(J_6 J_5 J_4 J_3 J_2 J_1)^T + G_6 [J_6 J_5 J_4 J_3 J_2 J_1 \Sigma^2(J_6 J_5 J_4 J_3 J_2 J_1)^T] + 6 J_5 J_4 J_3 J_2 J_1 \Sigma^2(J_6 J_5 J_4 J_3 J_2 J_1)^T] G_6^T = 0. \quad (29)$$

Assuming $\Sigma^* = \frac{1}{b_1^2 \theta^2} J_6 J_5 J_4 J_3 J_2 J_1 \Sigma^2(J_6 J_5 J_4 J_3 J_2 J_1)^T$, then $\Theta^2 + G_6 \Sigma^* + \Sigma^* G_6^T = 0$,

$$\text{where } \Theta = \text{diag}(1, 0, 0, 0, 0, 0), \quad \Sigma^* = \begin{pmatrix} c_{11} & 0 & c_{13} & 0 & c_{15} & 0 \\ 0 & -c_{13} & 0 & -c_{15} & 0 & c_{26} \\ c_{13} & 0 & c_{15} & 0 & -c_{26} & 0 \\ 0 & -c_{15} & 0 & c_{26} & 0 & c_{46} \\ c_{15} & 0 & -c_{26} & 0 & -c_{46} & 0 \\ 0 & c_{26} & 0 & c_{46} & 0 & c_{66} \end{pmatrix},$$

$$\begin{aligned} c_{11} &= \frac{(d_1 d_6 - d_2 d_5)^2 + (d_4 d_5 - d_3 d_6)(d_2 d_3 + d_5 - d_1 d_4)}{c_{00}}, \\ c_{13} &= -\frac{d_2 d_5^2 + d_3^2 d_6 - d_1 d_5 d_6 - d_3 d_4 d_5}{c_{00}}, \\ c_{15} &= \frac{d_1 d_3 d_6 - d_1 d_4 d_5 + d_5^2}{c_{00}}, c_{26} = \frac{d_1^2 d_6 - d_1 d_2 d_5 + d_3 d_5}{c_{00}}, \\ c_{46} &= \frac{d_1^2 d_4 + d_3^2 - d_1 d_2 d_3 - d_1 d_5}{c_{00}}, \\ c_{66} &= \frac{(d_1 d_4 - d_5)^2 + (d_1 d_2 - d_3)(d_2 d_5 - d_3 d_4 - d_1 d_6)}{c_{00}}, \\ c_{00} &= 2d_6[d_5((d_1 d_4 - d_5)^2 - (d_1 d_2 - d_3)(d_3 d_4 - d_2 d_5)) \\ &\quad + d_6(d_1(d_1^2 d_6 - d_1 d_2 d_5 + d_3 d_5) - d_3(d_1^2 d_4 + d_3^2 \\ &\quad - d_1 d_2 d_3 - d_1 d_5) \\ &\quad - d_1 d_5(d_1 d_2 - d_3))]. \end{aligned}$$

Σ^* is a positive definite matrix, which means that $\Sigma = b_1^2 \theta^2 (J_6 J_5 J_4 J_3 J_2 J_1)^{-1} \Sigma^* [(J_6 J_5 J_4 J_3 J_2 J_1)^T]^{-1}$ is a positive definite matrix. Therefore, the normal probability density function around the quasi-stationary equilibrium $\tilde{P}^*$ can be expressed by

$$\begin{aligned} \Phi(S, E, I, Q, R, \beta) &= (2\pi)^{-3} |\Sigma|^{-\frac{1}{2}} \exp[-\frac{1}{2} (S - S^*, E - E^*, I - I^*, \\ &\quad Q - Q^*, R - R^*, \beta - \beta^*) \\ &\quad \Sigma^{-1} (S - S^*, E - E^*, I - I^*, Q - Q^*, R - R^*, \beta - \beta^*)^T]. \end{aligned}$$

Remark 5.1. By Theorem 5.1 it follows that the solution $\mathcal{M}(t)$ of Equation 5 satisfies the normal density function $\mathbb{N}((S^*, E^*, I^*, Q^*, R^*, \beta^*), \Sigma)$ around $\tilde{P}^*$. If

$R_0^S = \frac{\bar{\beta} \Lambda \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Lambda \theta}{\mu \sqrt{\pi \alpha}})} > 1$, then $(S(t), E(t), I(t), Q(t), R(t))$ has a unique normal density function

$$\begin{aligned} \Phi(S, E, I, Q, R) &= (2\pi)^{-\frac{5}{2}} |\Sigma^{(5)}|^{-\frac{1}{2}} \\ &\quad \exp[-\frac{1}{2} (C_1, C_2, C_3, C_4, C_5) \Sigma^{(5)-1} (C_1, C_2, C_3, C_4, C_5)^T], \end{aligned}$$

where we define $\Sigma^{(5)}$ as the fifth-order principal subform of Σ . Therefore $S(t), E(t), I(t), Q(t)$ and $R(t)$ will converge to the marginal density function $\Phi_S, \Phi_E, \Phi_I, \Phi_Q$ and Φ_R , respectively:

$$\begin{aligned} \Phi_S(S) &= \frac{1}{\sqrt{2\pi} \varphi_1} e^{-\frac{(S-S^*)^2}{2\varphi_1^2}}, \quad \Phi_E(E) = \frac{1}{\sqrt{2\pi} \varphi_2} e^{-\frac{(E-E^*)^2}{2\varphi_2^2}}, \\ \Phi_I(I) &= \frac{1}{\sqrt{2\pi} \varphi_3} e^{-\frac{(I-I^*)^2}{2\varphi_3^2}}, \quad \Phi_Q(Q) = \frac{1}{\sqrt{2\pi} \varphi_4} e^{-\frac{(Q-Q^*)^2}{2\varphi_4^2}}, \\ \Phi_R(R) &= \frac{1}{\sqrt{2\pi} \varphi_5} e^{-\frac{(R-R^*)^2}{2\varphi_5^2}}, \end{aligned}$$

where φ_i^2 is the i -th element on the main diagonal of Σ , respectively.

6 Numerical simulations

We will use Milstein's higher-order method for numerical simulations. Then, transform the Equation 5 into the following discrete system:

$$\begin{cases} S_{i+1} = S_i + [\Lambda - x_i S_i I_i - \bar{\beta} S_i I_i - (\mu + \delta) S_i + \nu E_i] \Delta t, \\ E_{i+1} = E_i + [x_i S_i I_i + \bar{\beta} S_i I_i - (\sigma_2 + \gamma + \nu + \mu) E_i] \Delta t, \\ I_{i+1} = I_i + [\gamma E_i - (\sigma_1 + \omega_1 + \mu + h) I_i] \Delta t, \\ Q_{i+1} = Q_i + [\sigma_1 I_i + \sigma_2 E_i - (\omega_2 + \mu) Q_i] \Delta t, \\ R_{i+1} = R_i + [\omega_1 I_i + \omega_2 Q_i - \mu R_i] \Delta t, \\ x_{i+1} = x_i - \alpha x_i \Delta t + \theta \sqrt{\Delta t} \xi_i + \frac{\theta^2}{2} (\xi_i^2 - 1) \Delta t. \end{cases} \quad (30)$$

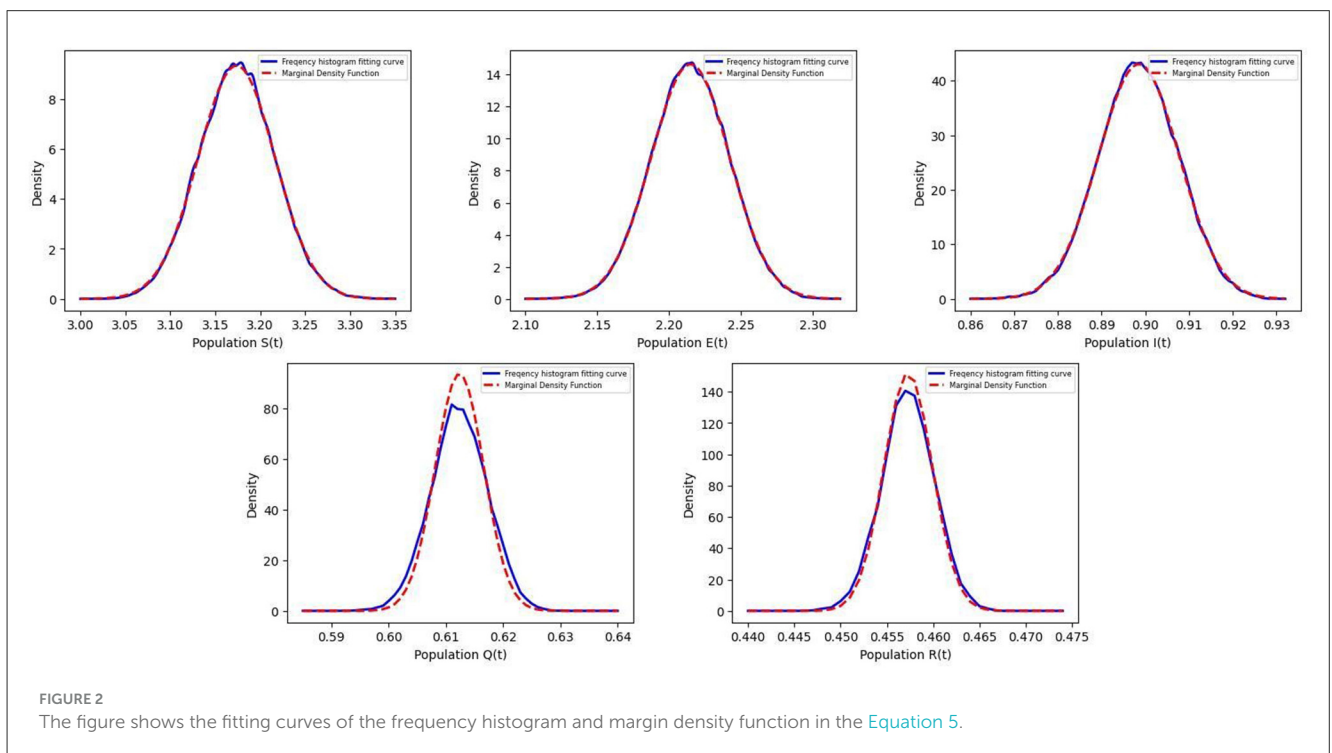
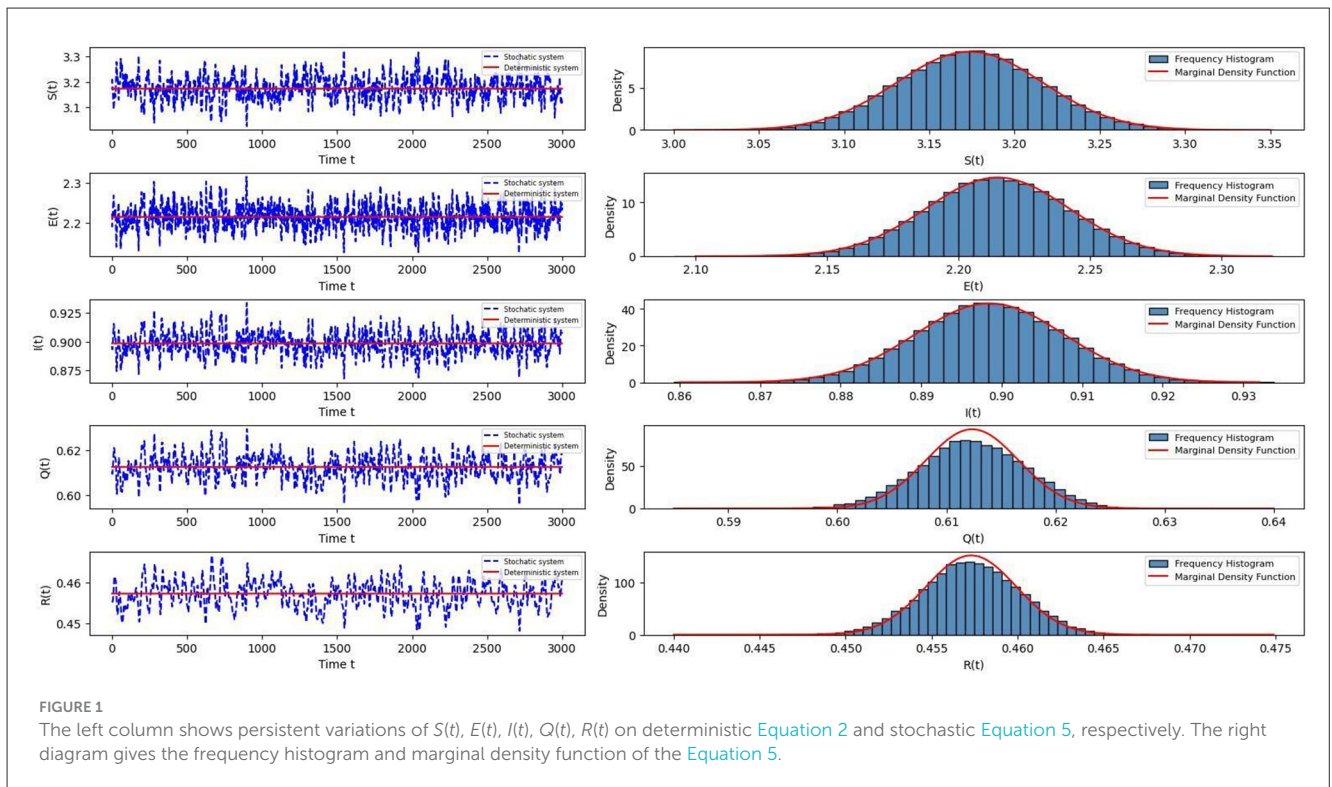
where $\Delta t > 0$ represents the time increment, $(S_i, E_i, I_i, Q_i, R_i, x_i)$ is the corresponding value of the i th iteration of Equation 5. In addition, ξ_i denotes the random variables that satisfy the standard normal distribution.

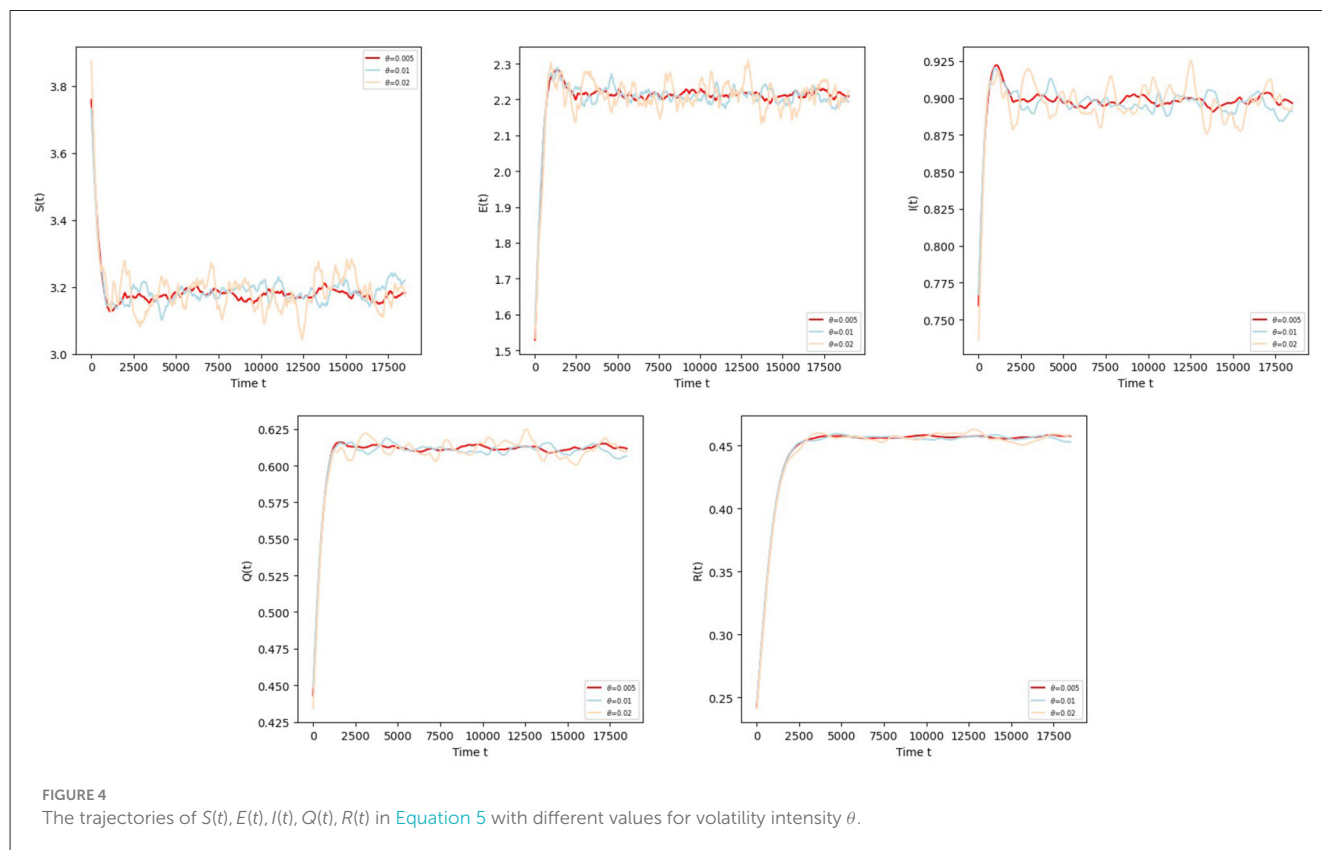
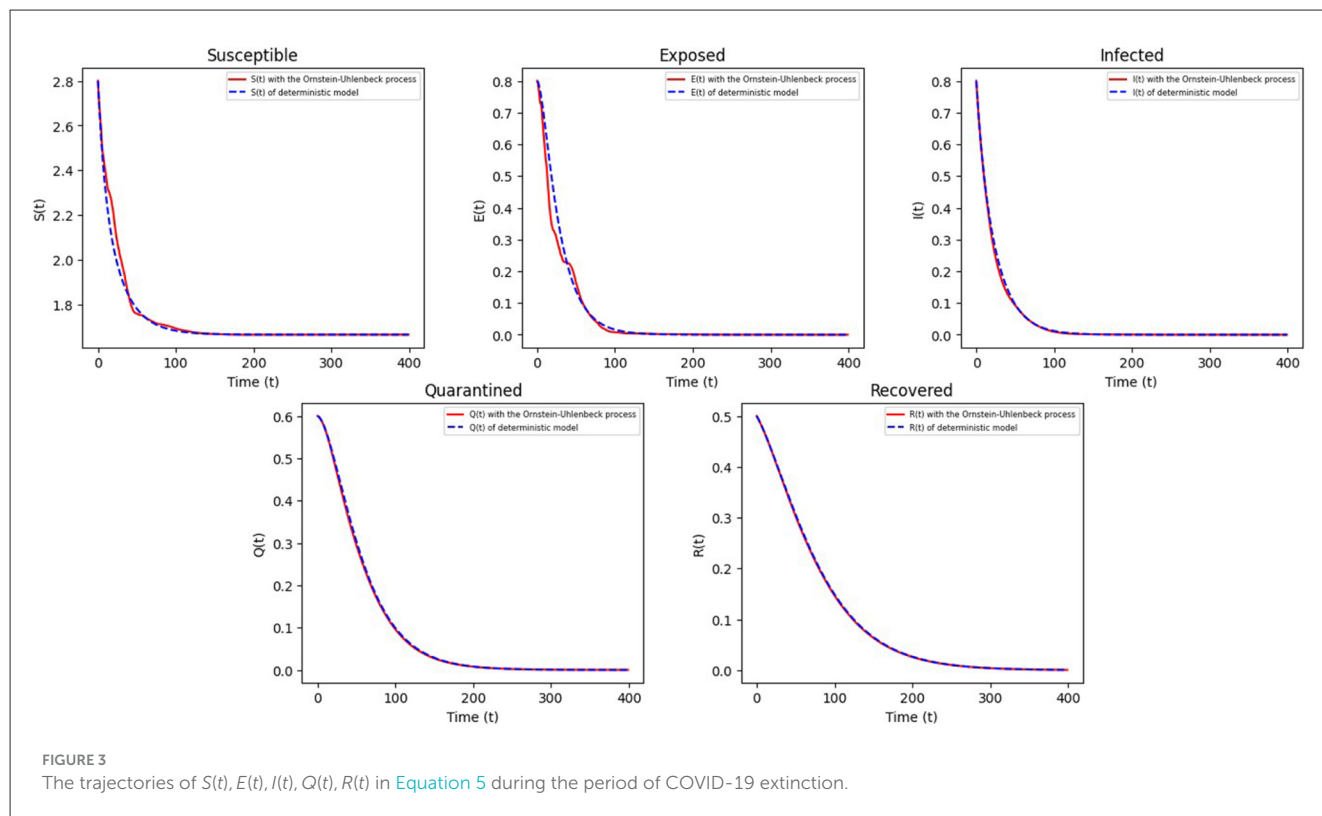
Example 6.1 Let $\bar{\beta} = 0.4$, $\alpha = 0.6$, $\theta = 0.01$, $\Lambda = 1.5$, $\mu = 0.15$, $\delta = 0.04$, $\nu = 0.11$, $\gamma = 0.2$, $\sigma_1 = 0.013$, $\sigma_2 = 0.055$, $\omega_1 = 0.03$, $\omega_2 = 0.068$ and $h = 0.3$, then we can calculate $\bar{\beta} = (0.7952)^4$, $R_0 = 2.4876 > 1$, $R_0^S = 1.2732 > 1$ and $(S^*, E^*, I^*, Q^*, R^*, \beta^*) = (3.1737, 2.2148, 0.8985, 0.6124, 0.4573, 0.4)$. Moreover, the solution $\mathcal{M}(t)$ obeys the following normal density function:

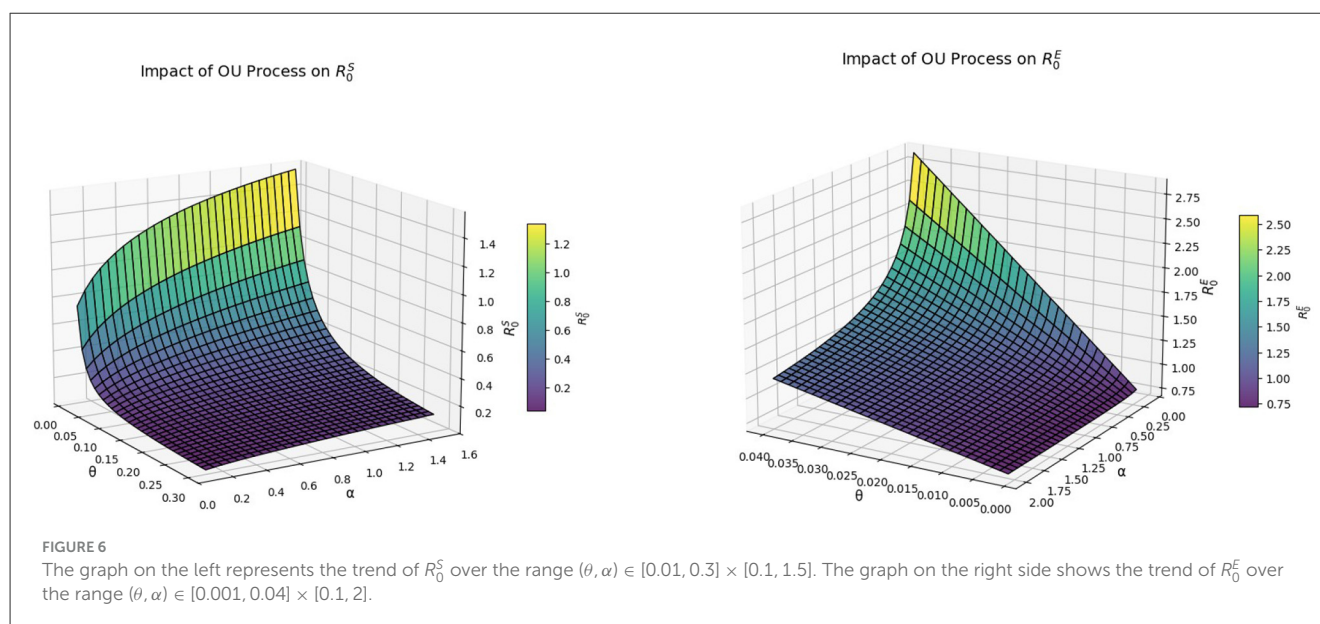
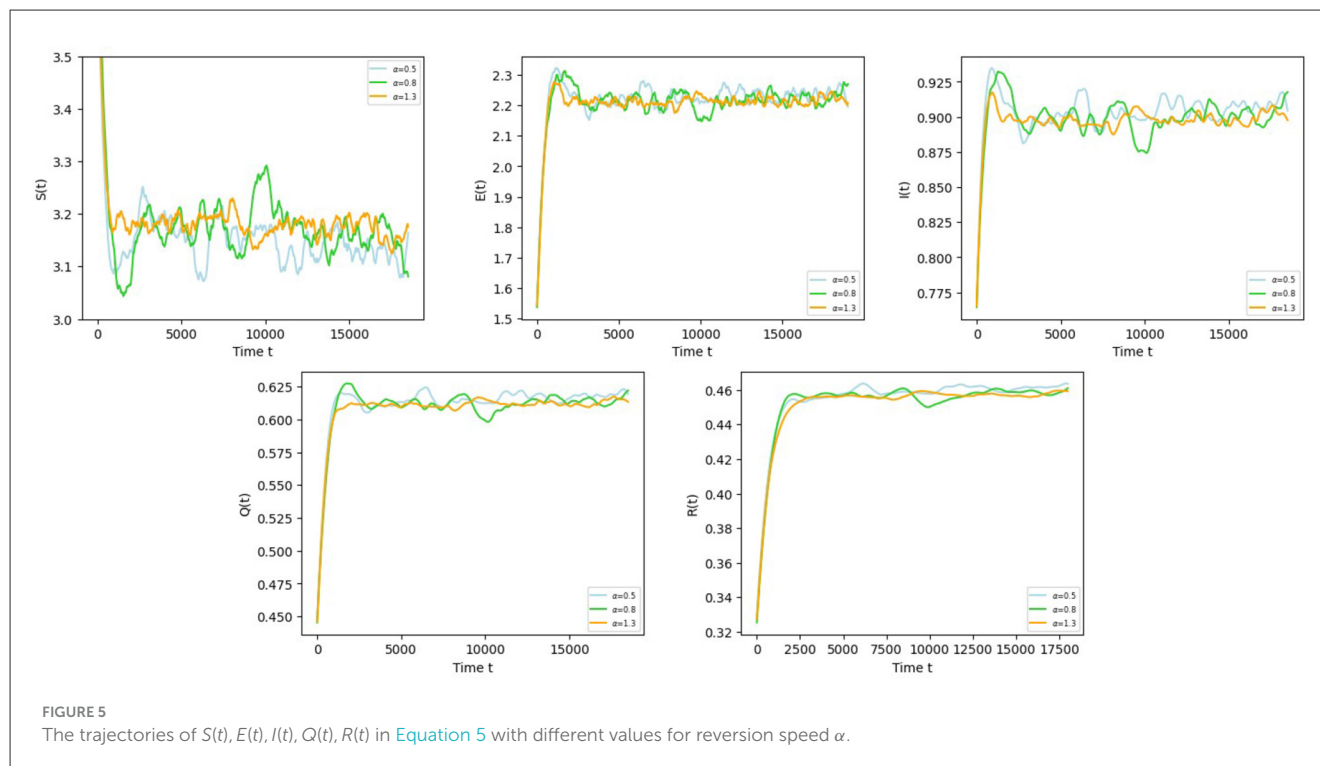
$$\begin{aligned} \Phi(S, E, I, Q, R, \beta) \\ \sim \mathbb{N}((3.1737, 2.2148, 0.8985, 0.6124, 0.4573, 0.4)^T, \Sigma), \end{aligned}$$

where

$$\Sigma = \begin{bmatrix} 1.8315 & -1.0907 & -0.3804 & 0.1646 & -0.0645 & -0.2256 \\ -1.0907 & 0.7431 & 0.2119 & -0.0806 & 0.0201 & 0.1773 \\ -0.3804 & 0.2119 & 0.0860 & -0.0375 & 0.0142 & 0.0325 \\ 0.1646 & -0.0806 & -0.0375 & 0.0181 & -0.0089 & -0.0114 \\ -0.0645 & 0.0201 & 0.0142 & -0.0089 & 0.0069 & 0.0023 \\ -0.2256 & 0.1773 & 0.0325 & -0.0114 & 0.0023 & 0.0833 \end{bmatrix} \times 10^{-3}.$$







We obtain the marginal functions of $S(t)$, $E(t)$, $I(t)$, $Q(t)$ and $R(t)$:

$$\begin{aligned}\Phi_S &= 9.3219e^{-273.0002(S-3.1737)^2}, \Phi_E = 14.6345e^{-672.8268(E-2.2148)^2}, \\ \Phi_I &= 43.0292e^{-5816.7079(I-0.8985)^2}, \\ \Phi_Q &= 93.7587e^{-27616.7741(Q-0.6124)^2}, \\ \Phi_R &= 152.0486e^{-72629.7551(R-0.4573)^2}.\end{aligned}$$

The dynamical behavior of the solutions is illustrated on the left-hand side of Figure 1, which shows that the disease exhibits

significant persistence over time. The right-hand side of Figure 1 gives the frequency histogram and marginal density function of Equation 5, which indicates that the Equation 5 has a stationary distribution. Figure 2 presents the frequency fitted-density function and marginal density function for $S(t)$, $E(t)$, $I(t)$, $Q(t)$, and $R(t)$. From Figure 2, it can be observed that the density function image is almost identical to the corresponding frequency histogram fitting curves, which demonstrates that the probability density function of the stationary distribution follows a Gaussian distribution.

Example 6.2 Choosing $\bar{\beta} = 0.24$, $\alpha = 0.6$, $\theta = 0.25$, $\Lambda = 0.6$, $\delta = 0.16$, $\nu = 0.28$, $\gamma = 0.2$, $\sigma_1 = 0.15$, $\sigma_2 = 0.032$, $\omega_1 = 0.045$,

$\omega_2 = 0.06$ and other parameters referring to Example 6.1, then we can calculate $R_0 = 0.1617 < 1$, $R_0^E = 0.9512 < 1$ and there exists a disease-free equilibrium $E_0 = (0.216, 0, 0, 0, 0)$ that is locally asymptotically stable. It is clearly shown in Figure 3 that the disease will become extinct over time.

Example 6.3 Taking three different values for the influence of volatility intensity θ :

$$\theta = 0.005, \theta = 0.01, \theta = 0.02.$$

Let reversion speed $\alpha = 1.2$, other parameters refer to Example 6.1. We can see that the smaller θ is, the smaller the fluctuation of the sample's trajectory is, and the more stable the disease is after observing the trajectories of S, E, I, Q, R in Figure 4.

Example 6.4 To investigate the effect of reversion speed α on COVID-19 disease, take $\theta = 0.01$ and other parameters refer to Example 6.1. Then, we choose three different values for reversion speed α :

$$\alpha = 0.5, \alpha = 0.8, \alpha = 1.3.$$

From the sample trajectories of S, E, I, Q, R in Figure 5, the larger α is, the smaller the vibration of the sample motion trajectory is, and the more stable the disease is.

Example 6.5 In the following, we examine the impact of the Ornstein-Uhlenbeck process on the COVID-19 disease by modifying the parameter values of volatility intensity θ and reversion speed α in R_0^S and R_0^E . For R_0^S , we choose $\theta \in [0.01, 0.3]$ and $\alpha \in [0.1, 1.5]$ and other parameters refer to example 6.1. For R_0^E , we choose $\theta \in [0.001, 0.04]$ and $\alpha \in [0.1, 2]$ and other parameters refer to example 6.2. As shown in Figure 6, as θ decreases and α increases, R_0^S exhibits an increasing trend, which suggests that the disease will continue to exist in the future, while R_0^E displays a decreasing trend, which indicates that the disease will tend to become extinct after a long time.

7 Conclusion

This study investigates the dynamics of COVID-19 transmission with Ornstein-Uhlenbeck processes. Compared with the common linear combination of white noise, this study assumes that the transmission rate parameter β of the susceptible and exposed satisfies a mean-reversion process. This approach avoids the unbounded variance of β as the time step decreases and offers greater biological realism. In addition, taking into account the effect of transmission mechanisms, we introduced disease-caused mortality for $I(t)$ and assumed that the disease-caused mortality rate h was greater than the natural mortality rate μ . For Equation 5, we first establish the existence and uniqueness of a global positive solution and prove its boundedness via the construction of a nonnegative Lyapunov-type function. On this basis we analyze the dynamical behavior of Equation 5 and derive the following key results:

(1) we fully utilize the ergodicity of the Ornstein-Uhlenbeck process to derive two critical values R_0^S and R_0^E related to disease

persistence and extinction, where

$$R_0^S = \frac{\tilde{\beta} \Delta \gamma}{(\delta + \mu)(\sigma_1 + \omega_1 + \mu + h)(\sigma_2 + \gamma + \nu + \mu + c_1 \frac{\Delta \theta}{\mu \sqrt{\pi \alpha}})},$$

$$R_0^E = \sqrt{R_0} + \frac{\gamma \theta \Lambda}{\mu(\sigma_2 + \gamma + \nu + \mu) \sqrt{R_0 \pi \alpha} \min\{\sigma_2 + \gamma + \nu + \mu, \sigma_1 + \omega_1 + \mu + h\}}.$$

When $R_0^S > 1$, it indicates that the stochastic system has a stationary distribution, meaning that the disease will persist over time. To further determine the size relationship between R_0^S and R_0 , we define an R_{00}^S such that $R_{00}^S \leq R_0^S$ and $R_{00}^S \leq R_0$. It follows that when $R_{00}^S > 1$ there is $R_0^S > 1$ and $R_0 > 1$. This suggests that $R_{00}^S > 1$ can be considered as a harmonized threshold for disease prevalence for both deterministic and stochastic systems.

When $R_0^E < 1$ indicates that the disease will become extinct over time. And $R_0^E < 1$ can derive $R_0 < 1$. This shows that $R_0^E < 1$ can be regarded as a unifying condition for deterministic and stochastic models of disease extinction.

(2) To further investigate the dynamical behavior of the disease, a display expression for the probability density function in the quasi-equilibrium nearby was derived by solving the sixth-order Fokker-Planck Equation 28. And we draw an image of the probability density function as shown in Figure 1.

(3) We verify our theoretical results with numerical simulations and find that both larger fluctuation intensity and smaller speed of regression may lead to extinction of the disease.

This study proposes a stochastic SEIRQV model incorporating an Ornstein-Uhlenbeck process. By introducing a quarantine compartment, the model comprehensively accounts for the combined effects of vaccination and quarantine measures on the transmission of infectious diseases. To evaluate model performance and the effectiveness of isolation measures, we conducted a comparative analysis of simulation results against an existing stochastic SEIV model [22]. Our findings suggest that integrating natural mortality rates with artificial isolation policies enhances our model's precision in estimating disease extinction probabilities and steady-state distributions. The interaction between isolation and other compartments generates a modified critical threshold condition, more accurately reflecting the impact of combined intervention measures. Compared with existing stochastic SEIR models [23], this model captures richer covariance structures across all compartments. Moreover, the variance and covariance terms explicitly depend on isolation and vaccination rates, which provides a stronger theoretical basis for evaluating the variability and potential outcomes of disease control strategies.

In summary, our study presents a more realistic stochastic model that overcomes the limitations of the white noise model in simulating the dynamics of COVID-19 infection. Although we have made a breakthrough in this aspect, we still need to exploit other valid approaches to compensate for the limitations of the study. For example, further explore harmonized thresholds for disease persistence and extinction, find a more suitable method and theory to determine the uniqueness of the stationary distribution, consider incorporating more compartmental models into COVID-19 infectious disease research, and integrate age structures to study differences in susceptibility and contact patterns across age groups.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

W-HL: Writing – review & editing. K-JW: Writing – original draft, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships

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Supplementary material

The Supplementary Material for this article can be found online at: <https://www.frontiersin.org/articles/10.3389/fams.2025.1687991/full#supplementary-material>

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