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The new generalized odd Median Based Unit Rayleigh

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Introduction: In this paper, the author presents the generalized form of the Median-Based Unit Rayleigh (MBUR) distribution, a novel statistical distribution that is specifically defined within the interval (0, 1). This generalization adds a new parameter to the MBUR distribution that significantly addresses the unique characteristics of data represented as ratios and proportions. The author considers a distinct technique for appending a new parameter to the unit distribution consuming the general formula for the order statistics.

Methods: The paper offers a thorough and meticulous derivation of the (PDF) for this distribution, illuminating each phase of the process with clarity and precision. It delves deep into the intricacies of the generalized odd MBUR (GOMBUR) distribution's properties, presenting a rigorous examination of the accompanying functions that are vital for robust statistical evaluation. These functions-comprising the (CDF), survival function, hazard rate, reversed hazard rate function, and raw moments.

Results and discussion: The paper discusses real data analysis and how the generalization improves such analysis. The author conducts a comparative analysis of the Generalized Odd Median Base Unit Rayleigh (GOMBUR) and the Median Based Unit Rayleigh (MBUR). The primary objective is to evaluate the additional benefits provided by the new shape parameter in the estimation process, focusing on various validity indices, goodness-of-fit statistics, estimated variances of the parameters, and their corresponding standard errors. Parameter estimation is performed using the Maximum Likelihood Estimator (MLE), with the Nelder-Mead optimization method employed for this purpose. The results obtained from this study can be summarized in the following points. (i) Incorporating new parameters into the MBUR model significantly enhances its flexibility, enabling it to accommodate a variety of data shapes with differing characteristics, such as skewness and kurtosis. (ii) This added parameter enhances the estimation process, resulting in improved validity indices, including (AIC), (CAIC), (BIC), and (HQIC). Additionally, it enhances the goodness of fit by reducing test statistics such as the (AD), (CVM), and (KS) tests, while increasing the Log-Likelihood. (iii) The two forms of the model yield different values for the parameter (n) but provide the same value for the parameter (alpha). The variances of the estimated (alpha) are identical, and the covariance between the parameters is minimal—significantly lower than that observed when fitting other distributions like the Beta and Kumaraswamy. Furthermore, the determinant of the estimated variance-covariance matrix from fitting the GOMBUR-1 model is among the lowest compared to those from the Beta and Kumaraswamy distributions.

KEYWORDS

generalized odd MBUR, Median Based Unit Rayleigh, maximum likelihood estimator, hazard rate function, COVID-19 death rate in Canada

1 Introduction

A multitude of real-world phenomena can be elegantly captured as proportions, ratios, or fractions nestled within the bounded interval (0, 1). These captivating representations are not merely abstract concepts; they reflect the intricate relationships found in various fields such as biology, where the delicate balance of ecosystems is analyzed; finance, where the flow of market ratios unfold; and mortality rates, which provide profound insights into human health and longevity. Additionally, recovery rates in medical science showcase the resilience of life, while economics delves into the nuanced distributions of wealth and resources. Engineering and hydrology further enrich this tapestry, modeling everything from structural integrity to the flow of water in our environments. The measurement sciences, too, rely on these continuous distributions, breathing life into data that inform our understanding of the world.

Some of these distributions include: the Johnson SB distribution [1], Beta distribution [2], Unit Johnson distribution [3], Topp-Leone distribution [4], Unit Gamma distribution [5–8], Unit Logistic distribution [9], Kumaraswamy distribution [10], Unit Burr-III distribution [11], Unit Modified Burr-III distribution [12], Unit Burr-XII distribution [13], Unit-Gompertz distribution [14], Unit-Lindley distribution [15], Unit-Weibull distribution [16], and Unit Muth distribution [17].

Generalizing a distribution by adding new parameters can undeniably enrich and amend the estimation process in numerous manners. First, adding parameters empowers the distribution to apprehend different data conducts such as skewness, kurtosis and different tail behaviors. For example, this is glorified when generalized gamma [18] distribution extends the gamma distribution to accommodate modeling wide range of data with different characteristics. The extra parameter in the generalized Pareto [19] distribution controlling the tail behavior facilitates it to better analyze the extreme values. Second, the newly added parameters amplify the goodness of fit to empirical data and reduce the systematic bias. An example for such an effect, when the data exhibit extreme values; the four parameter Beta distribution [20] extending the standard beta distribution can better model these heavy tail features data. Third, extending the distribution with new parameters augments it flexibility to align more properly with the basic data hence diminishing the bias in parameter estimation, obtaining more efficient estimators with minor variance and improve the implementation of MLE. The Generalized Weibull Distribution [21] enhances the estimates of failures rates in reliability studies. Fourth, the newly introduced parameters behave as regularizers that take control against over-fitting thereby improving stability in estimation. For example, the additional shape parameter in the Exponentiated Weibull Distribution [21] enables the modeling of both decreasing and increasing failure rates. Likewise, the skewness parameter in the Skew-Normal Distribution [22] promotes the distribution to better model the asymmetric data. Fifth, in real-world analysis like Generalized Logistic Distribution [23], implanting a shape parameter can regulate the rate of decay in growth models. The incorporated extra parameter in the Generalized Gamma [24] Distribution aids modeling diverse hazard rates.

The generalization of the unit distribution can be achieved through different mechanisms like power transformation to obtain the power Johnson B [25], power Generalized Johnson SB [26], and power unit inverse Lindley distribution [27]. The generalization can also be conducted using the T-X family method (transformed-transformer mechanism) like the transmuted power unit inverse Lindley distribution [28], Kumaraswamy generalized family of distribution [29], generalized distribution based on T-Topp - Leone family of distributions [30], and the generalized unit half logistic geometric distributions [31]. The author discusses in this paper a different method for adding a new parameter to the unit distribution using the general formula for the order statistics.

It's important to recognize the limitations of this generalization. While adding more parameters can enhance a model's complexity and capability, it also brings challenges that must be carefully considered. Increased parameters may complicate computations and require sophisticated optimization techniques to manage effectively. Additionally, a larger set of parameters can introduce identifiability issues, where the effects of certain parameters become unclear. Many parameters may even lack practical significance, questioning their value in real-world applications. Therefore, it's crucial to weigh the advantages of incorporating new parameters against the potential difficulties they could introduce, ensuring that we make informed decisions in our modeling approaches.

This paper is structured into several sections for clarity and coherence. Section 2 provides a comprehensive discussion of the methodology employed to derive the new distribution. Section 3 delves into its fundamental characteristics, including the probability density function (PDF), cumulative distribution function (CDF), survival function (S), hazard function (HF), reversed hazard function (RHF), and quantile function. Section 4 demonstrates the maximum likelihood estimation method. Section 5 offers an in-depth discussion that encompasses an analysis of real data as well as a detailed examination of the findings. In conclusion, Section 6 provides a comprehensive overview of our findings and offers valuable recommendations for future research, inviting further exploration and innovation in the field.

2 Methodology

2.1. Derivation of the PDF

The general formula of the median order statistics for an odd sample size can be written as in Equation 1.

$$f_{i:m} = \frac{m!}{\left(\frac{m-1}{2}\right)!\left(\frac{m-1}{2}\right)!}\left[F_X(x)\right]^{\frac{m-1}{2}}\left[1 - F_X(x)\right]^{\frac{m-1}{2}} f_X(x) \quad (1)$$

Where (i) is the i^{th} odd order statistics and m is the sample size. Replacing (m) sample size (which is an odd number) with m = 2n + 1 as shown in Equations 2.A and 2.B.

$$f_{i:m} = \frac{(2n+1)!}{\left(\frac{2n+1-1}{2}\right)!\left(\frac{2n+1-1}{2}\right)!}\left[F_X(x)\right]^{\frac{2n+1-1}{2}}\left[1-F_X(x)\right]^{\frac{2n+1-1}{2}} f_X(x) \quad (2.A)$$

$$f_{i:m} = \frac{(2n+1)!}{(n)!\,(n)!}\left[F_X(x)\right]^n\left[1-F_X(x)\right]^n f_X(x) \quad (2.B)$$

Substituting Equation 3 which is the CDF and the PDF of Rayleigh distribution in Equation 2.B yields Equation 4.

$$F_X(x) = 1 - exp\left(\frac{-x^2}{\alpha^2}\right) \quad and$$
$$f_X(x) = \frac{2x}{\alpha^2} exp\left(\frac{-x^2}{\alpha^2}\right) \quad (3)$$

$$f_{i:m} = \frac{(2n+1)!}{(n)!\,(n)!}\left[1-\exp\left(\frac{-x^2}{\alpha^2}\right)\right]^n\left[\exp\left(\frac{-x^2}{\alpha^2}\right)\right]^n \frac{2x}{\alpha^2} exp\left(\frac{-x^2}{\alpha^2}\right) \quad (4)$$

Using the transformation in Equation 5 and the Jacobian in Equation 6 and then substituting both in Equation 4 yields the new PDF in Equation 7. This is the first version of generalization of the MBUR.

$$let\ y = e^{-x^2} \quad (5.A)$$
$$-\ln y = x^2 \quad , \quad \left[-\ln y\right]^{0.5} = x \quad (5.B)$$

$$\frac{dx}{dy} = \frac{1}{2}\left[-\ln y\right]^{-0.5}\frac{-1}{y} \quad (6)$$

$$f_Y(y) = \frac{\Gamma(2n+2)}{\Gamma(n+1)\,\Gamma(n+1)}\frac{1}{\alpha^2}\left[1-y^{\alpha^{-2}}\right]^n\left[y\right]^{\frac{n+1}{\alpha^2}-1}, \quad n \geq 0\,,\ \alpha > 0,\ 0 < y < 1 \quad (7)$$

The second version of generalization is obtained by substituting the CDF and the PDF of Rayleigh in Equation 8 which yields Equation 9

$$f_{i:n} = \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}\left[F_X(x)\right]^{\frac{n-1}{2}}\left[1-F_X(x)\right]^{\frac{n-1}{2}} f_X(x) \quad (8)$$

$$f_{i:n} = \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}\left[1-\exp\left(\frac{-x^2}{\alpha^2}\right)\right]^{\frac{n-1}{2}}\left[\exp\left(\frac{-x^2}{\alpha^2}\right)\right]^{\frac{n-1}{2}}\frac{2x}{\alpha^2} exp\left(\frac{-x^2}{\alpha^2}\right) \quad (9)$$

Substituting the same transformation of Equation 5 and the same Jacobian of Equation 6 in Equation 9 yields the new PDF in Equation 10. This is the second version of generalization of the MBUR distribution.

$$f_Y(y) = \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\frac{1}{\alpha^2}\left[1-y^{\alpha^{-2}}\right]^{\frac{n-1}{2}}\left[y\right]^{\frac{n+1}{2\alpha^2}-1}, n \geq 1\,,\ \alpha > 0,\ 0 < y < 1 \quad (10)$$

Theorem 1: Both versions in Equations 7 and 10 are valid PDF.

Proof: PDF version in Equation 7:

To prove that the PDF in Equation 7 is a valid PDF, the integral in Equation 11 should equal 1. Applying the transformation in Equation 12 and substitute in Equation 11:

$$\frac{\Gamma(2n+2)}{\Gamma(n+1)\,\Gamma(n+1)}\int_0^1 \frac{1}{\alpha^2}\left[1-y^{\alpha^{-2}}\right]^n\left[y\right]^{\frac{n+1}{\alpha^2}-1} dy = 1 \quad (11)$$

$$let: \quad y^{\frac{1}{\alpha^2}} = w\,, \quad so \quad y = w^{\alpha^2} \quad this\ gives \quad dy = \alpha^2 w^{\alpha^2-1} dw \quad (12)$$

$$\frac{\Gamma(2n+2)}{\Gamma(n+1)\,\Gamma(n+1)}\int_0^1 \frac{1}{\alpha^2}\left[1-w\right]^n\left[w^{\alpha^2}\right]^{\frac{n+1}{\alpha^2}-1}\alpha^2 w^{\alpha^2-1} dw = 1$$

For the PDF version in Equation 10, applying the same transformation will integrate the PDF in Equation 10 to 1.

3 Some properties of the generalized odd MBUR distribution

Theorem 2: the cumulative distribution function (CDF) of the generalized odd MBUR is:

$P\left(Y < y\right) = I_w\left(n+1, n+1\right)$ for version 1 and $P\left(Y < y\right) = I_w\left(\frac{n+1}{2}, \frac{n+1}{2}\right)$ for version 2.

Proof: for version 1:

$$P\left(Y < y\right) = \frac{\Gamma(2n+2)}{\Gamma(n+1)\,\Gamma(n+1)}\int_0^y \frac{1}{\alpha^2}\left[1-y^{\alpha^{-2}}\right]^n\left[y\right]^{\frac{n+1}{\alpha^2}-1} dy \quad (13)$$

Apply the transformation of Equation 12 and substitute in Equation 13 yields Equation 14

$$P\left(Y < y\right) = \frac{\Gamma(2n+2)}{\Gamma(n+1)\,\Gamma(n+1)}\int_0^w \left[1-w\right]^n\left[w\right]^n dw = \frac{B_w(n+1, n+1)}{B(n+1, n+1)} \quad (14)$$

For version 2:

$$P\left(Y < y\right) = \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\int_0^y \frac{1}{\alpha^2}\left[1-y^{\alpha^{-2}}\right]^{\frac{n-1}{2}}\left[y\right]^{\frac{n+1}{2\alpha^2}-1} dy \quad (15)$$

Apply the transformation of Equation 12 and substitute in equation 15 yields Equation 16

$$P\left(Y < y\right) = \frac{\Gamma(n+1)}{\Gamma\frac{(n+1)}{2}\Gamma\left(\frac{n+1}{2}\right)}\int_0^w \left[1-w\right]^{\frac{n-1}{2}}\left[w\right]^{\frac{n-1}{2}} dw = \frac{B_w\left(\frac{n+1}{2}, \frac{n+1}{2}\right)}{B\left(\frac{n+1}{2}, \frac{n+1}{2}\right)} \quad (16)$$

Lemma 1: the survival function (S) for version 1 is shown in Equation 17

$$S\left(y\right) = 1 - P\left(Y < y\right) = 1 - I_w\left(n+1, n+1\right) \quad (17)$$

Lemma 2: the survival function (S) for version 2 is shown in Equation 18

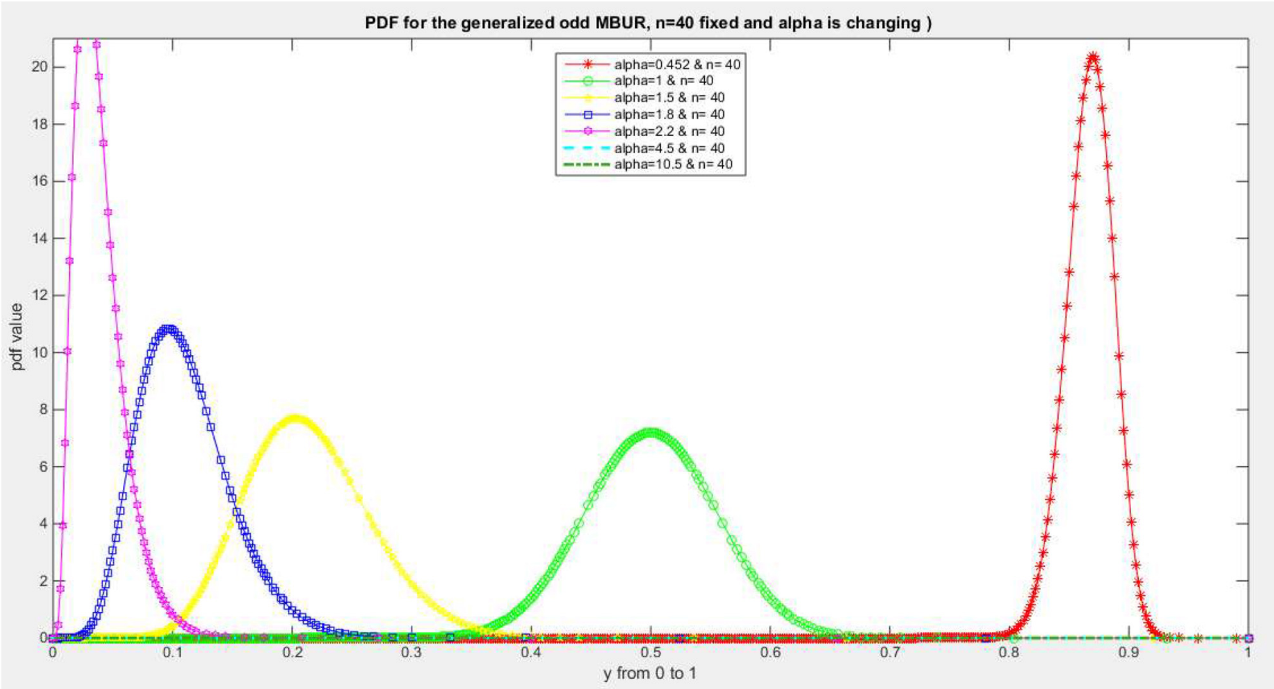

FIGURE 1
Shows PDF of the first version for different levels of alpha and $n = 40$.

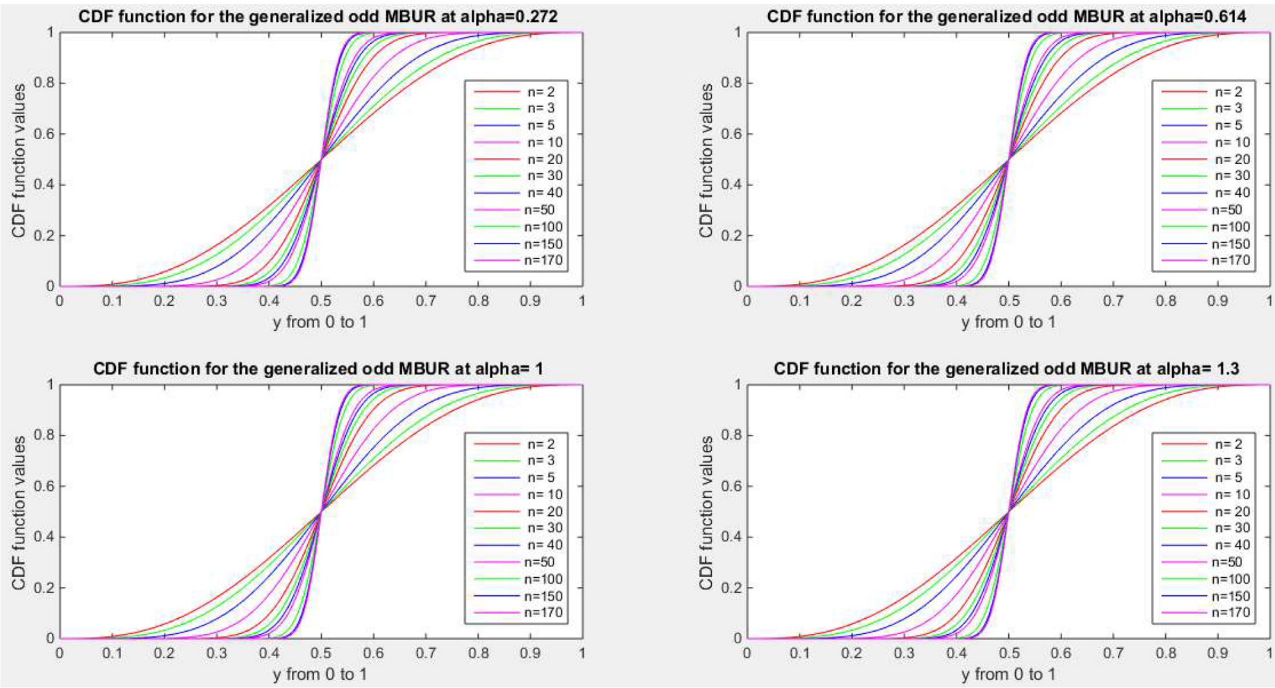

FIGURE 2
Shows CDF of the first version for different levels of alpha and n.

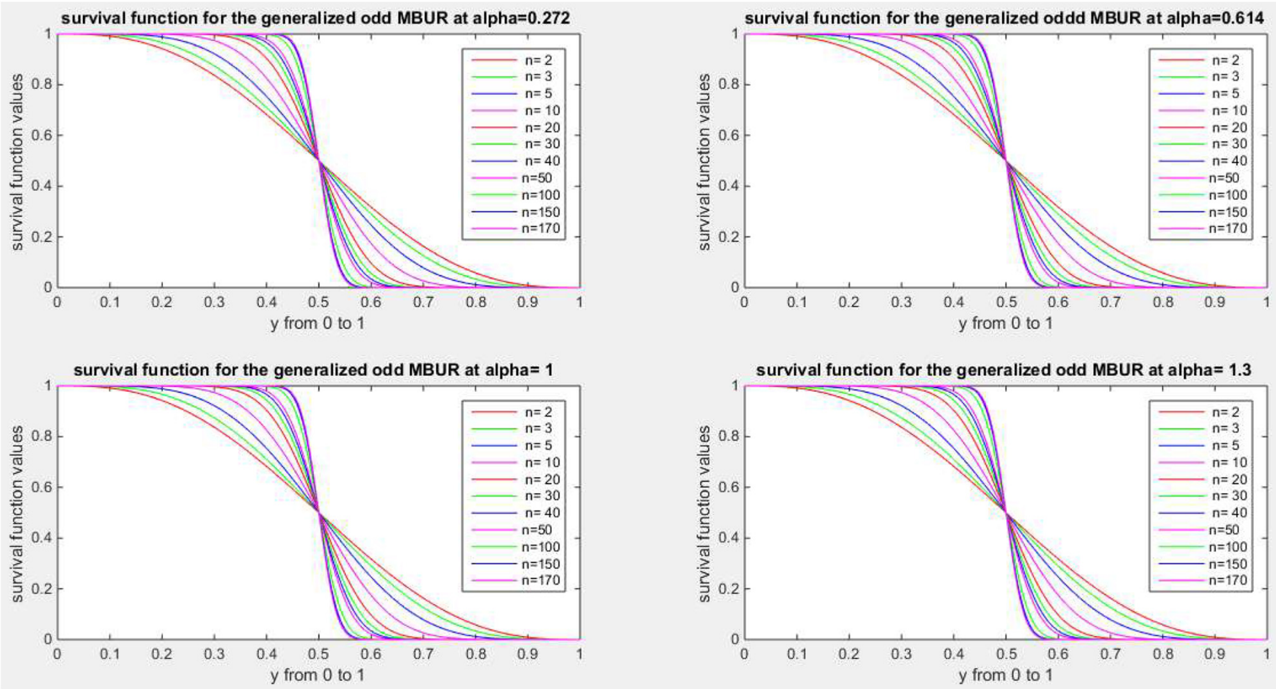

FIGURE 3
Shows survival function of the first version for different levels of alpha and *n*.

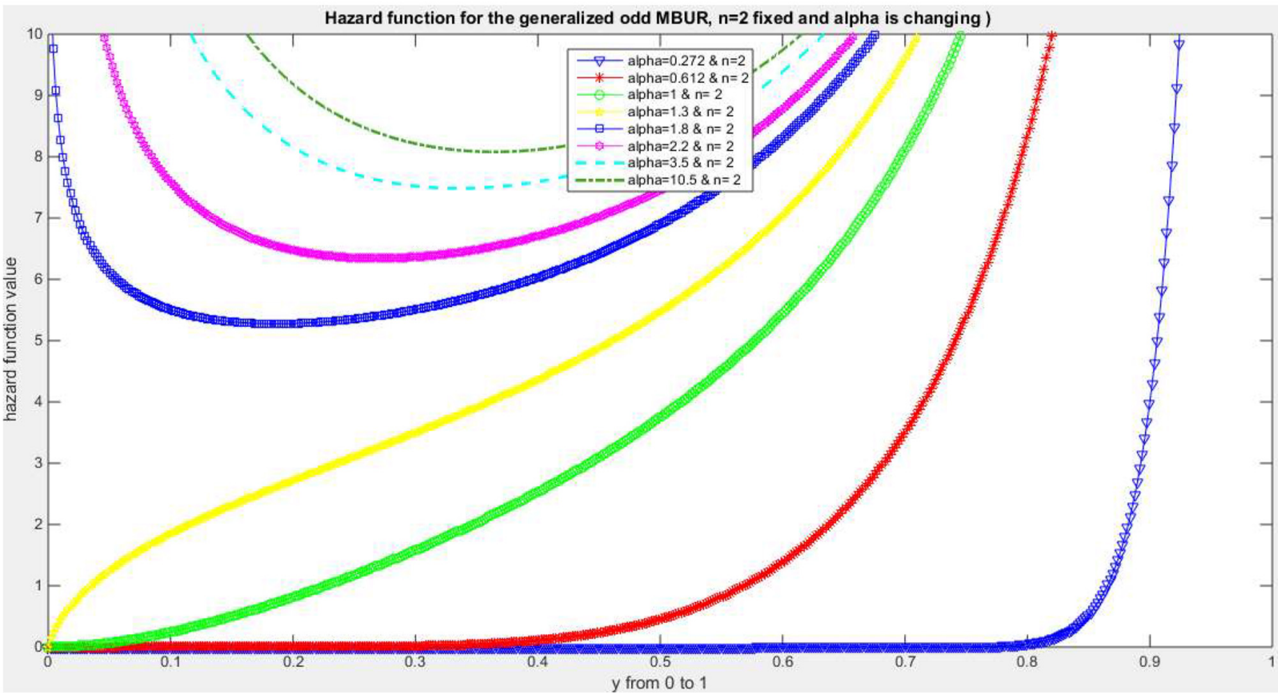

FIGURE 4
Shows hazard rate function of the first version for different levels of alpha and $n = 2$.

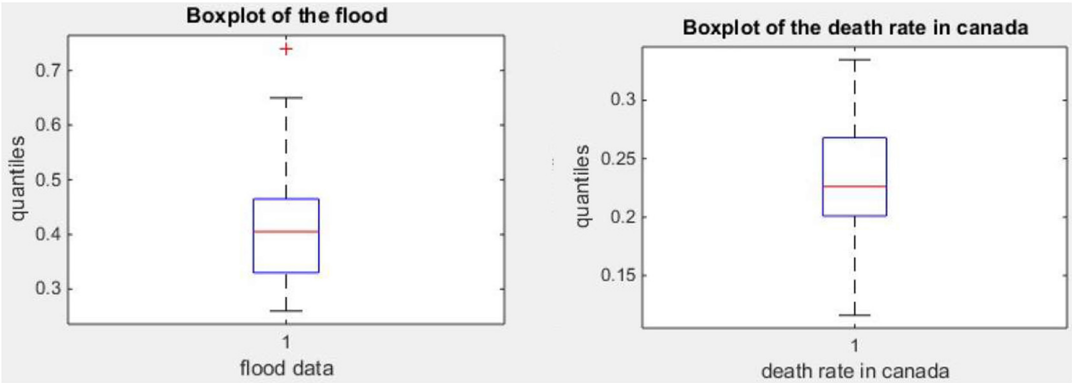

FIGURE 5
Boxplot of the flood data and COVID-19 death rate in Canada.

TABLE 1 Illustrates the descriptive statistics for the above datasets.

Dataset	Min	Mean	Standard deviation	Skewness	Kurtosis	Q(25)	Q(50)	Q(75)	Max
Flood data	0.2600	0.4225	0.1244	1.1625	4.2363	0.3300	0.4050	0.4650	0.7400
COVID-19 death rate	0.1159	0.2305	0.0520	−0.0897	2.7360	0.2011	0.2262	0.2678	0.3347

TABLE 2A Shows the results of analysis of flood data.

Results	Beta		Kumaraswamy		MBUR	Topp-Leone	Unit-Lindley
Theta	$\alpha = 6.8318$		$\alpha = 3.3777$		1.0443	2.2413	1.6268
	$\beta = 9.2376$		$\beta = 12.0057$				
Var	7.22	7.2316	0.3651	2.8825	0.007	0.2512	0.0819
	7.2316	8.0159	2.8825	29.963			
SE(a)	2.687		0.6042		0.0837	0.5012	0.2862
	$P < 0.025$		$P < 0.01$		$P < 0.01$	$P < 0.01$	$P < 0\ .01$
SE(b)	2.8312		5.4738		–	–	–
	$P < 0.01$		$P < 0.025$				
AIC	−24.3671		−21.9465		−10.9233	−12.7627	−12.3454
CAIC	−23.6613		−21.2407		−10.7011	−12.5405	−12.1231
BIC	−22.3757		−19.9551		−9.9276	−11.767	−11.3496
HQIC	−23.9784		−21.5578		−10.7289	−12.584	−12.151
LL	14.1836		12.9733		6.4617	7.3814	7.1727
K-S	0.2063		0.2175		0.3202	0.3409	0.2625
H_0	Fail to reject		Fail to reject		Fail to reject	Reject	Fail to reject
P-value	0.3174		0.2602		0.0253	0.0141	0.0311
AD	0.7302		0.9365		2.7563	2.9131	2.3153
CVM	0.1242		0.1653		0.531	0.5857	0.4428
determinant	5.5784		2.6314		–	-	-

TABLE 2B To be continued: comparison between GOMBUR 1 and GOMBUR 2.

Results	GOMBUR-1		GOMBUR-2	
Theta	$n = 8.1044$		$n = 17.2087$	
	$\alpha = 1.1168$		$\alpha = 1.1168$	
Variance	8.0302	0.0177	32.1208	0.0354
	0.0177	0.0018	0.0354	0.0018
SE(n)	2.8338		5.6675	
SE(a)	0.0424		0.0424	
AIC	−24.4562		−24.4562	
CAIC	−23.7503		−23.7503	
BIC	−22.4647		−22.4647	
HQIC	−24.0674		−24.0674	
LL	14.2281		14.2281	
K-S Value	0.204		0.204	
H_0	Fail to reject		Fail to reject	
P-value	0.3297		0.3297	
AD	0.7153		0.7153	
CVM	0.1205		0.1205	
Determinant	0.0143		0.0573	
Significant (n)	$P < 0.025$		$P < 0.025$	
Significant (a)	$P < 0.01$		$P < 0.01$	

$$S\left(y\right) = 1 - P\left(Y < y\right) = 1 - I_w\left(\frac{n+1}{2}, \frac{n+1}{2}\right) \quad (18)$$

Lemma 3: The Hazard function or rate (HF or hr) for version 1 is shown in Equation 19

$$hr\left(y\right) = \frac{f_Y\left(y\right)}{1 - P\left(Y < y\right)} = \frac{\frac{\Gamma(2n+2)}{\Gamma(n+1)\Gamma(n+1)} \frac{1}{\alpha^2} \left[1 - y^{\frac{1}{\alpha^2}}\right]^n \left[y\right]^{\frac{n+1}{\alpha^2} - 1}}{1 - I_w\left(n+1, n+1\right)} \quad (19)$$

Lemma 4: The Hazard function or rate (HF or hr) for version 2 is shown in Equation 20

$$hr\left(y\right) = \frac{f_Y\left(y\right)}{1 - P\left(Y < y\right)} = \frac{\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \frac{1}{\alpha^2} \left[1 - y^{\alpha^{-2}}\right]^{\frac{n-1}{2}} \left[y\right]^{\frac{n+1}{2\alpha^2} - 1}}{1 - I_w\left(\frac{n+1}{2}, \frac{n+1}{2}\right)} \quad (20)$$

Lemma 5: the reversed hazard function (RHF) or reversed hazard rate (rhr) for version 1 is shown in Equation 21

$$rhr\left(y\right) = \frac{f_Y\left(y\right)}{P\left(Y < y\right)} = \frac{\frac{\Gamma(2n+2)}{\Gamma(n+1)\Gamma(n+1)} \frac{1}{\alpha^2} \left[1 - y^{\alpha^{-2}}\right]^n \left[y\right]^{\frac{n+1}{\alpha^2} - 1}}{I_w\left(n+1, n+1\right)} \quad (21)$$

Lemma 6: the reversed hazard function (RHF) or reversed hazard rate (rhr) for version 2 is shown in Equation 22

$$rhr\left(y\right) = \frac{f_Y\left(y\right)}{P\left(Y < y\right)} = \frac{\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \frac{1}{\alpha^2} \left[1 - y^{\alpha^{-2}}\right]^{\frac{n-1}{2}} \left[y\right]^{\frac{n+1}{2\alpha^2} - 1}}{I_w\left(\frac{n+1}{2}, \frac{n+1}{2}\right)} \quad (22)$$

The quantile function of the distribution has no closed explicit form.

Theorem 3: the rth raw moment of the first version of the distribution is given by

$$E\left(y^r\right) = \frac{\Gamma\left(2n+2\right)}{\Gamma\left(n+1\right)} \frac{\Gamma\left(n+1+r\alpha^2\right)}{\Gamma\left(2n+2+r\alpha^2\right)}$$

Proof: the expectation of the rth moment in Equation 23 is obtained with the help of the transformation mentioned in Equation 12

$$\begin{aligned} E\left(y^r\right) &= \int_0^1 y^r f_Y\left(y\right) dy = \frac{\Gamma\left(2n+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+1\right)} \frac{1}{\alpha^2} \int_0^1 y^r \left[1 - y^{\alpha^{-2}}\right]^n \left[y\right]^{\frac{n+1}{\alpha^2} - 1} dy \\ &= \frac{\Gamma\left(2n+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+1\right)} \frac{1}{\alpha^2} \int_0^1 \left[1 - y^{\alpha^{-2}}\right]^n \left[y\right]^{\frac{n+1}{\alpha^2} - 1 + r} dy \\ &= \frac{\Gamma\left(2n+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+1\right)} \int_0^1 \left[1 - w\right]^n \left[w\right]^{n + r\alpha^2} dw \\ &= \frac{\Gamma\left(2n+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+1\right)} \frac{\Gamma\left(n+1\right)\Gamma\left(n+1+r\alpha^2\right)}{\Gamma\left(2n+2+r\alpha^2\right)} \\ &= \frac{\Gamma\left(2n+2\right)}{\Gamma\left(n+1\right)} \frac{\Gamma\left(n+1+r\alpha^2\right)}{\Gamma\left(2n+2+r\alpha^2\right)} \end{aligned} \quad (23)$$

Theorem 4: the rth raw moment of the second version of the distribution is given by

$$E\left(y^r\right) = \frac{\Gamma\left(n+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2} + r\alpha^2\right)}{\Gamma\left(n+1+r\alpha^2\right)}$$

Proof: the expectation of the rth moment in Equation 24 is obtained with the help of the transformation mentioned in

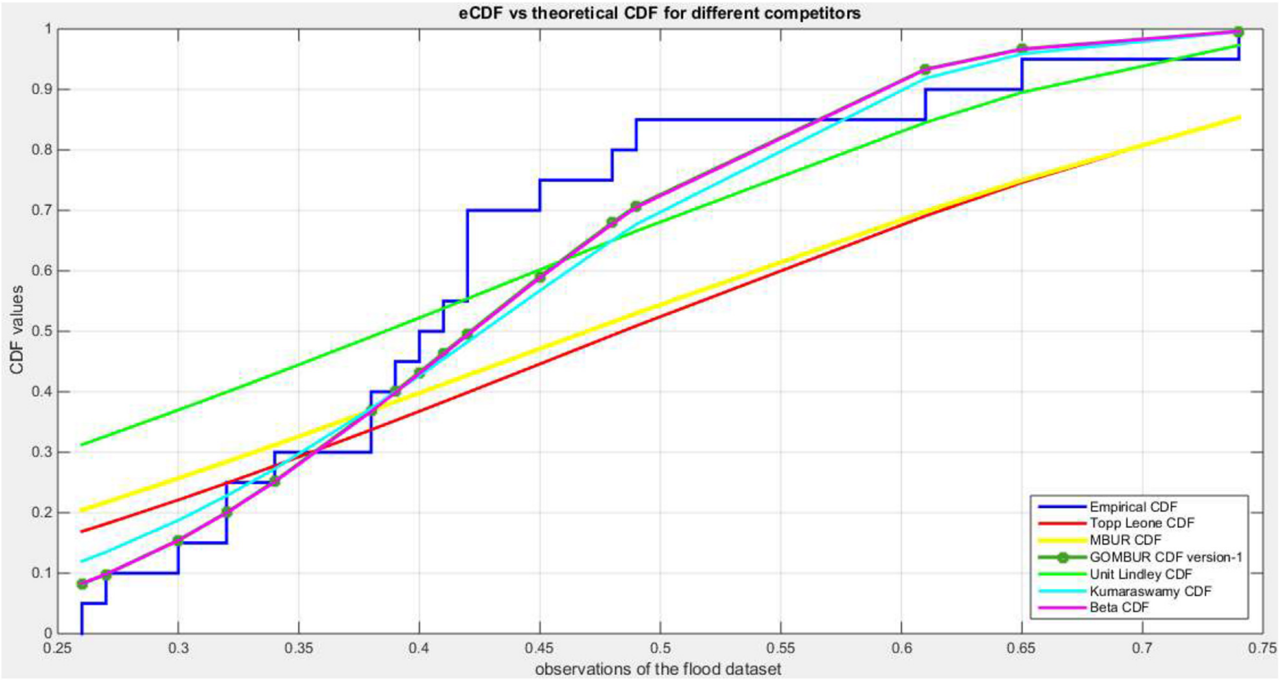

FIGURE 6
Shows the e-CDF and the theoretical CDF for the fitted distributions of flood data.

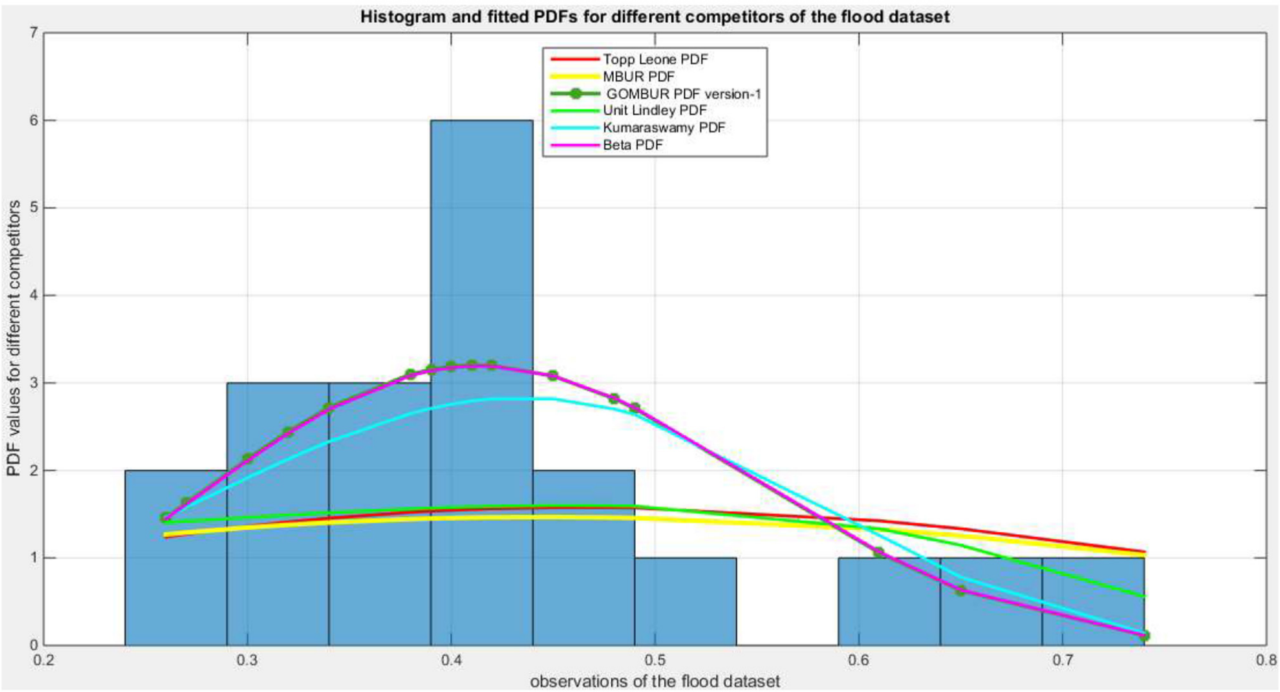

FIGURE 7
Shows the histogram of the flood data and the theoretical PDFs for the fitted distributions. The GOMBUR-1 perfectly aligns with the Beta distribution.

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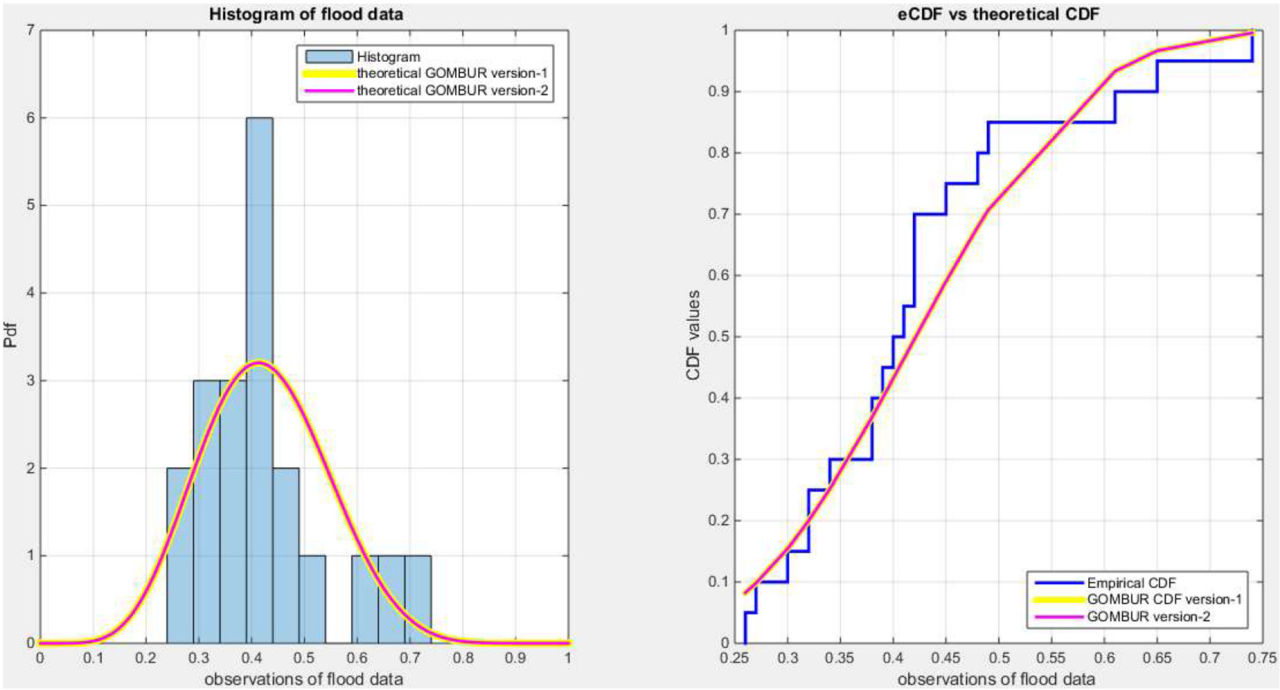

FIGURE 8
Shows on the left subplot the histogram of the flood data and the fitted PDFs of both GOMBUR-1 and GOMBUR-2 and on the right subplot the e-CDF and the theoretical CDF for both distributions. Both the fitted CDF and the fitted PDFs of both versions are identical.

TABLE 3A Shows the results of analysis of COVID-19 death rate analysis in Canada.

Results	Beta		Kumaraswamy		MBUR	Topp-Leone	Unit-Lindley
Theta	$\alpha = 14.5128$		$\alpha = 5.0309$		1.3479	1.0814	3.9381
	$\beta = 48.4899$		$\beta = 1049.6$				
Variance	7.839	27.531	0.2719	370.2768	0.0042	0.0209	0.203
	27.531	100.6504	370.2768	523950			
SE(a)	2.7998		0.5214		0.0648	0.1446	0.4506
	$P < 0.01$		$P < 0.01$		$P < 0.01$	$P < 0.01$	$P < 0.01$
SE(b)	10.0325		723.8439		–	-	-
	$P < 0.01$		$P > 0.025$				
AIC	−167.88		−169.2		−48.8337	−46.3748	−80.2707
CAIC	−167.6536		−168.9736		−48.7596	−46.3008	−80.1966
BIC	−163.8293		−165.1493		−46.8083	−44.3495	−78.2453
HQIC	−166.3096		−167.6296		−48.0485	−45.5896	−79.4855
LL	85.94		86.6		25.4168	24.1874	41.1353
K-S	0.0754		0.1029		0.429	0.4685	0.359
H_0	Fail to reject		Fail to reject		Reject	Reject	Reject
P-value	0.6802		0.5583		0	0	0
AD	0.4398		0.369		14.0394	15.8748	12.7087
CVM	0.0692		0.0686		2.8621	3.3539	2.5936
Determinant	31.0432		5348.9		–	-	-

Equation 12

$$E\left(Y^{r}\right)=\int_{0}^{1} y^{r} f_{Y}\left(y\right) dy=\frac{\Gamma\left(n+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\frac{1}{\alpha^{2}}$$
$$\int_{0}^{1} y^{r}\left[1-y^{\alpha^{-2}}\right]^{\frac{n-1}{2}}\left[y\right]^{\frac{n+1}{2\alpha^{2}}-1} dy \quad (24)$$
$$=\frac{\Gamma\left(n+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\frac{1}{\alpha^{2}}$$
$$\int_{0}^{1}\left[1-y^{\alpha^{-2}}\right]^{\frac{n-1}{2}}\left[y\right]^{\frac{n+1}{2\alpha^{2}}-1+r} dy$$
$$=\frac{\Gamma\left(n+1\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}$$
$$\int_{0}^{1}\left[1-w\right]^{\frac{n-1}{2}}\left[w\right]^{\frac{n-1}{2}+r\alpha^{2}} dw$$
$$=\frac{\Gamma\left(n+1\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}+r\alpha^{2}\right)}{\Gamma\left(n+1+r\alpha^{2}\right)}$$
$$=\frac{\Gamma\left(n+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}+r\alpha^{2}\right)}{\Gamma\left(n+1+r\alpha^{2}\right)}$$

Figures 1–4 show PDF, CDF, survival function, Hazard function for the distribution at different parameters values. See Supplementary material 1 for more figures.

4 Methods of estimation

4.1 Maximum likelihood estimation

Let Y be a random variable having the PDF of GOMBUR-1. To derive the MLE for version 1, for one observation, taking the log of Equation 7 results in Equation 25:

$$l\left(y;\ \alpha, n\right)=$$
$$\ln\Gamma\left(2n+2\right)-\ln\Gamma\left(n+1\right)-\ln\Gamma\left(n+1\right)+\ln\alpha^{-2}$$
$$+n\ln\left[1-y_{i}^{\alpha^{-2}}\right]+\left(\frac{n+1}{\alpha^{2}}-1\right)\ln y_{i} \quad (25)$$

Taking the first derivative of Equation 25 with respect to n and alpha parameter yields Equations 26 and 27, respectively:

$$\frac{\partial l}{\partial n}=2^{*}\psi\left(2n+2\right)-\psi\left(n+1\right)-\psi\left(n+1\right)$$
$$+\ln\left[1-y_{i}^{\alpha^{-2}}\right]+\alpha^{-2}\ln y_{i} \quad (26)$$
$$\frac{\partial l}{\partial\alpha}=\frac{-2}{\alpha}+\left(\frac{2n}{\alpha^{3}}\right)\frac{y_{i}^{\alpha^{-2}}\ln y_{i}}{1-y_{i}^{\alpha^{-2}}}-\left(\frac{2\left[n+1\right]}{\alpha^{3}}\right)\ln y_{i} \quad (27)$$

Let Y be a random variable having the PDF of GOMBUR-2. To derive the MLE for version 2, for one observation, taking the log of Equation 10 results into Equation 28:

$$l\left(y;\ \alpha, n\right)=$$
$$\ln\Gamma\left(n+1\right)-\ln\Gamma\left(\frac{n+1}{2}\right)-\ln\Gamma\left(\frac{n+1}{2}\right)+\ln\alpha^{-2}$$
$$+\left(\frac{n-1}{2}\right)\ln\left[1-y_{i}^{\alpha^{-2}}\right]+\left(\frac{n+1}{2\alpha^{2}}-1\right)\ln y_{i} \quad (28)$$

TABLE 3B To be continued: comparison between GOMBUR 1 and GOMBIR 2.

Results	GOMBUR-1		GOMBUR-2	
Theta	$n = 41.02961$		$n = 83.0593$	
	$\alpha = 1.4623$		$\alpha = 1.4623$	
Variance	62.6379	0.0083	250.5451	0.0166
	0.0083	0.0002376	0.0166	0.00023762
SE (n)	7.9144		15.8286	
SE (a)	0.0154		0.0154	
AIC	−167.5661		−167.5661	
CAIC	−167.3396		−167.3396	
BIC	−163.5154		−163.5154	
HQIC	−165.9956		−165.9956	
LL	85.783		85.783	
K-S Value	0.0781		0.0781	
H_0	Fail to reject		Fail to reject	
P-value	0.6461		0.6461	
AD	0.4653		0.4653	
CVM	0.0728		0.0728	
Determinant	0.0148		0.0593	
Significant (n)	$P < 0.025$		$P < 0.025$	
Significant (a)	$P < 0.01$		$P < 0.01$	

Taking the first derivative of Equation 28 with respect to n and alpha parameter yields Equations 29 and 30, respectively:

$$\frac{\partial l}{\partial n}=\psi\left(n+1\right)-\frac{1}{2}\psi\left(\frac{n+1}{2}\right)-\frac{1}{2}\psi\left(\frac{n+1}{2}\right)$$
$$+\frac{1}{2}\ln\left[1-y_{i}^{\alpha^{-2}}\right]+\frac{1}{2}\alpha^{-2}\ln y_{i} \quad (29)$$
$$\frac{\partial l}{\partial\alpha}=\frac{-2}{\alpha}+\left(\frac{n-1}{\alpha^{3}}\right)\frac{y_{i}^{\alpha^{-2}}\ln y_{i}}{1-y_{i}^{\alpha^{-2}}}-\left(\frac{n+1}{\alpha^{3}}\right)\ln y_{i} \quad (30)$$

For each version, set the above equations to zero, and since they are non-linear equations, numerical methods like quasi-Newton method can be used as a solution.

5 Real data analysis

5.1. Description of the real data

The real data used in this paper can be found in Appendix A in Supplementary material 2. These are 14 datasets. In the main manuscript, the author will discuss only two of them. The flood dataset was used by the author [32] in previous work. The second dataset is the COVID-19 death rate in Canada, previously analyzed by [33].

The analysis of the data sets aims to determine how these sets correspond to the following distributions: Beta, Topp-Leone, Unit Lindley, and Kumaraswamy. The author will compare the fitting of these data sets to the fitting of the new MBUR distribution

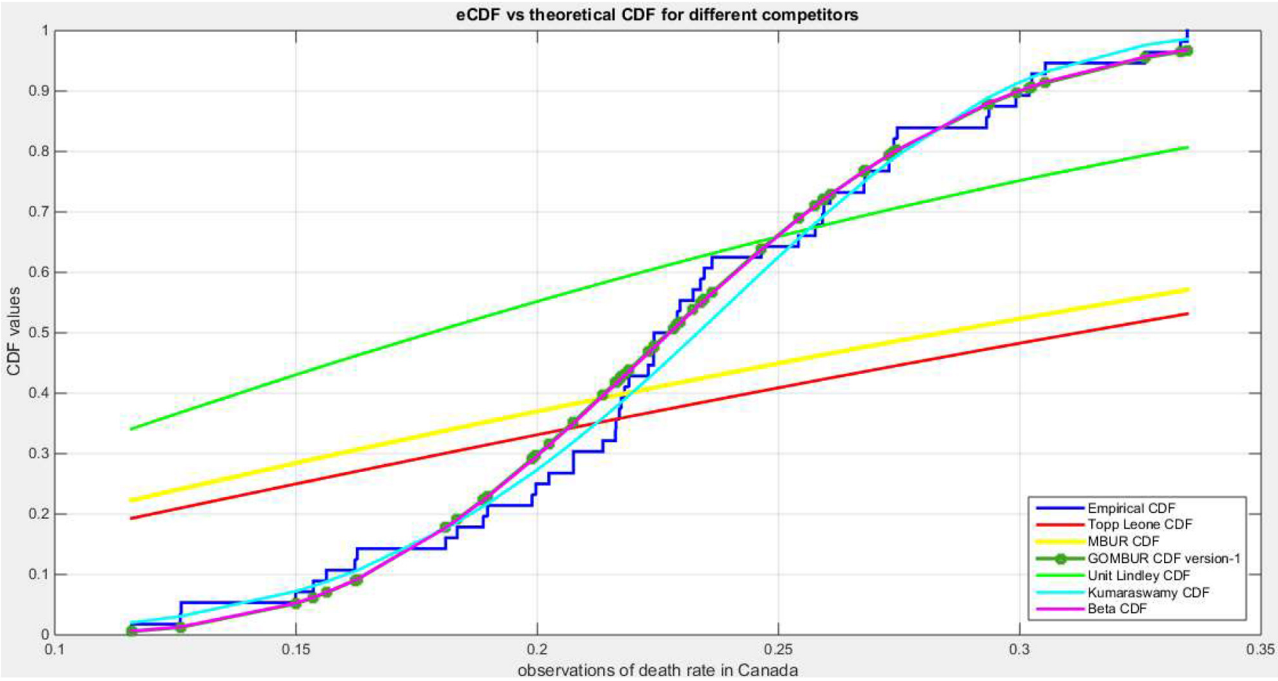

FIGURE 9
Shows the e-CDF and the theoretical CDF for the fitted distributions of death rate in Canada data.

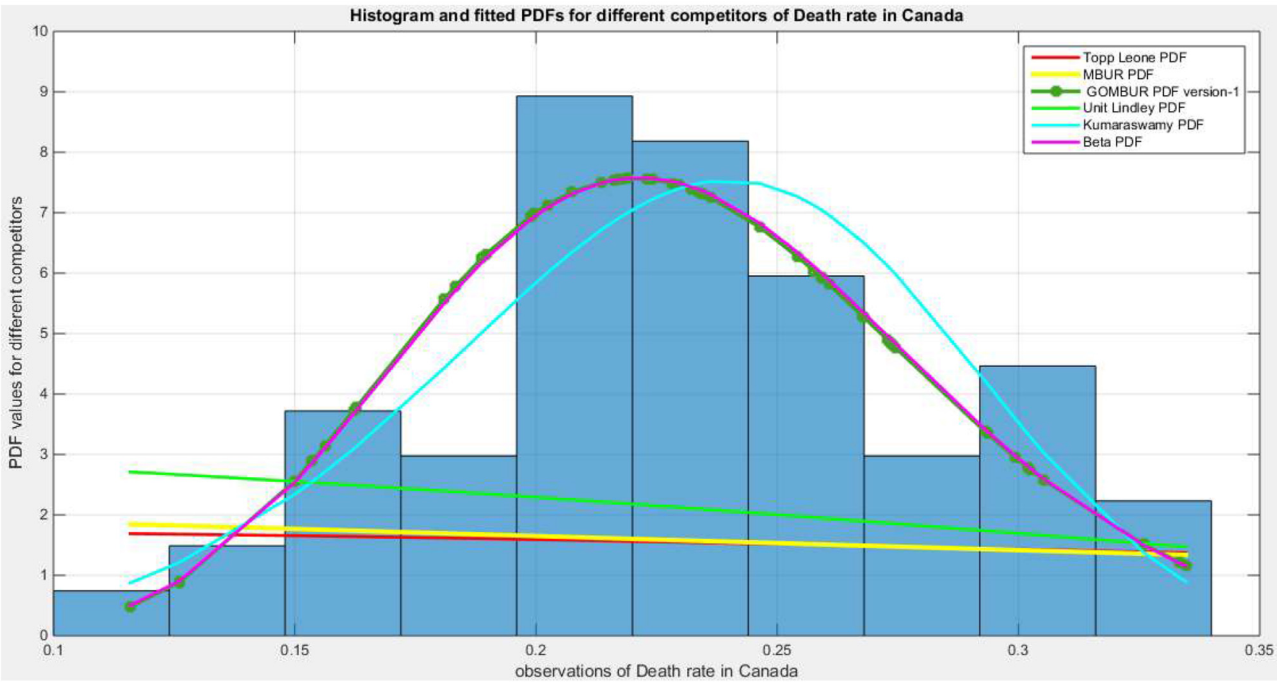

FIGURE 10
Shows the histogram of the COVID-19 death rate in Canada data and the theoretical PDFs for the fitted distributions. The GOMBUR-1 shows near-perfect alignments with Beta distribution. Kumaraswamy distribution fits the data. MBUR, Topp Leone, Unit Lindley distributions do not fit the data.

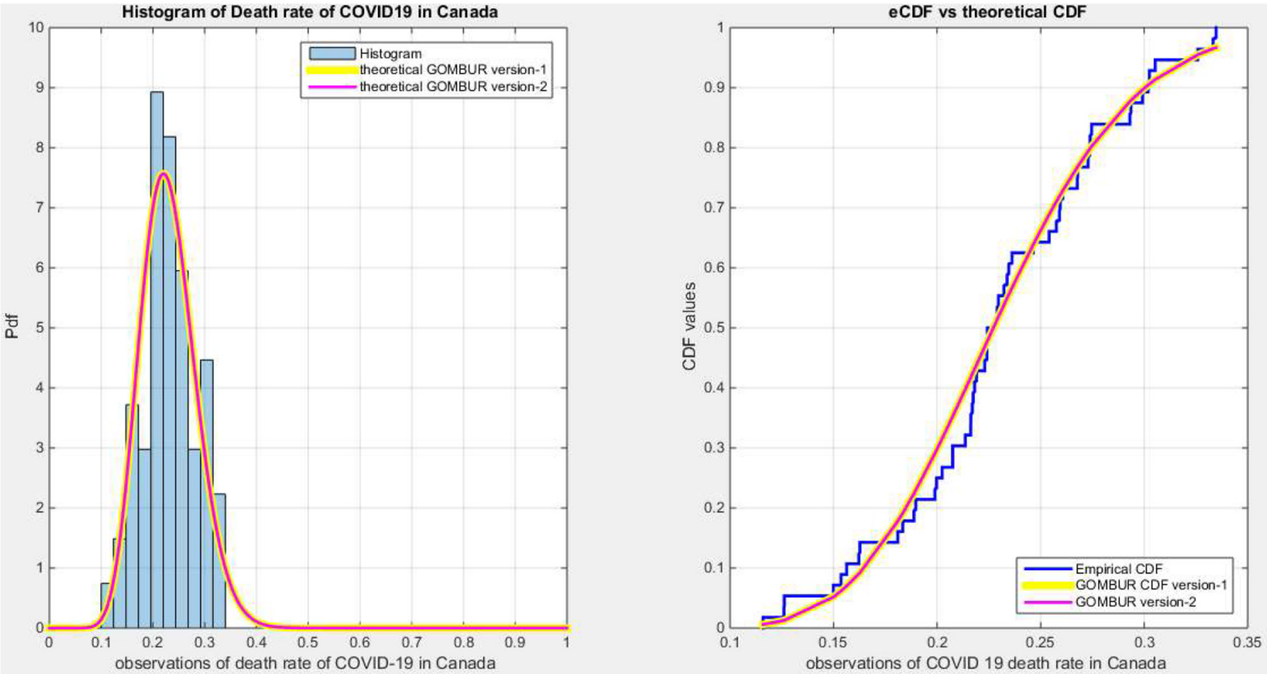

FIGURE 11
Shows on the left subplot the histogram of the death rate in Canada data and the fitted PDFs of both GOMBUR-1 and GOMBUR-2 and on the right subplot the e-CDF and the theoretical CDF for both distributions. Both the fitted CDF and the fitted PDFs of both versions are identical.

[32] and its generalized forms, GOMBUR-1 and GOMBUR-2. The evaluation will utilize several metrics, including the log-likelihood (LL), Akaike Information Criterion (AIC), corrected AIC (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). Additionally, the author will perform the Kolmogorov-Smirnov (K-S) test, documenting its value as well as the outcome of the null hypothesis (H0), which posits that the data set follows the investigated distribution. If the data do not support this assumption, the null hypothesis is rejected. The *P*-value for the test will also be recorded. Furthermore, the Cramér-von Mises test and the Anderson-Darling test will be conducted, with their respective values reported. Figures illustrating the empirical cumulative distribution function (eCDF) and the theoretical cumulative distribution functions (CDF) of the distributions will be included. The author will present the estimated parameter values, along with their variances and standard errors. MATLAB was used for analysis. The competing distributions are as follows:

1- Beta Distribution: $f(y;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$, $0<y<1$, $\alpha>0,\ \ \beta>0$

2- Kumaraswamy Distribution: $f(y;\alpha,\beta) = \alpha\beta y^{\alpha-1}(1-y^{\alpha})^{\beta-1}$, $0<y<1$, $\alpha>0,\ \ \beta>0$

3- Median Based Unit Rayleigh: $f(y;\ \alpha) = \frac{6}{\alpha^2}\left[1-y^{\frac{1}{\alpha^2}}\right] y^{\left(\frac{2}{\alpha^2}-1\right)}$, $0<y<1$, $\alpha>0$

4- Topp-Leone Distribution: $f(y;\theta) = \theta(2-2y)\ (2y-y^2)^{\theta-1}$, $0<y<1$, $\theta>0$

5- Unit-Lindley: $f(y;\theta) = \frac{\theta^2}{1+\theta}(1-y)^3 \exp\left(\frac{-\theta y}{1-y}\right)$, $0<y<1$, $\theta>0$

Comparison tools are: (k) is the number of parameter, (*n*) is the number of observations.

$$AIC = -2MLL + 2k, CAIC = -2MLL + \frac{2kn}{n-k-1},$$
$$BIC = -2MLL + k\log(n)$$
$$HQIC = -2\log L + 2k^{*}\ln\left[\ln(n)\right]$$
$$KS-test = Sup_n\ |F_n - F|, F_n = \frac{1}{n}\Sigma_{i=1}^{n} I_{x_i < x}$$
$$Cramer-Von-Mises-test(CVM) = \frac{1}{12n} + \sum_{i=1}^{n}\left\{F(x_i) - \frac{2i-1}{2n}\right\}^2$$
$$Anderson-Darling-test(AD) = -n - \Sigma_{i=1}^{n}\left(\frac{2i-1}{n}\right)\left\{\log\left[F(x_i)\right] + log\left[1-F(x_{n-i+1})\right]\ \right\}$$

5.2 Descriptive statistics of the datasets

Figure 5 and Table 1 shows that the flood data are right skewed and exhibits excess kurtosis (leptokurtic) while the COVID-19 death rate in Canada data rate are slightly left skewed and exhibit less than excess kurtosis (platykurtic).

5.3 Individual dataset analysis (results and discussion)

For each dataset, there will be a table displaying the results of the fitted distributions, along with figures that highlight the fitted (CDFs) and (PDFs) for the tested distributions. An extra figure will be provided to compare the fitted CDFs and PDFs of the two versions. Tables 2A, 2B presents the results of the analysis of the flood data with accompanied Figures 6–8.

The analysis shows that Beta distribution fits the data better than any other distribution. The Topp Leone did not fit the distribution. Generalization of MBUR using the GOMBUR-1 improves the fitting up to the level of the Beta distribution and slightly exceeding it. Marked increases in the negativity levels of AIC, CAIC, BIC & HQIC are obtained. The level of Log-likelihood shows marked improvement. Marked reduction in the levels of AD and CVM statistics are obvious. The variance of the estimated alpha shows marked reduction after fitting the GOMBUR-1. The determinant of GOMBUR-1&2 is far less than the determinant of the Beta distribution and Kumaraswamy distribution demonstrating more efficiency. GOMBUR-1 has lesser determinant than the GOMBUR-2. The estimates of alpha value, their variances and standard errors are identical for both versions. While the estimates of the n parameter, its variance and standard error obtained after fitting GOMBUR-2 is higher than the levels obtained after fitting GOMBUR-1 distribution.

Figure 6 shows the eCDF and the theoretical CDF for the fitted distributions. Figure 7 shows the fitted PDFs. Figure 8 illuminates the fitted CDFs and the fitted PDFs for both versions of the generalization expounding identity of the curves.

The next data is the COVID-19 death rate in Canada. The results are shown in Tables 3A, 3B with associated Figures 9–11.

The analysis indicates that both the Beta distribution and the Kumaraswamy distribution provide a good fit for the data. In contrast, the MBUR, Topp Leone, and Unit Lindley distributions do not fit the data well. The GOMBUR-1 and GOMBUR-2 distributions fit the data adequately, with their AIC, CAIC, BIC, and HQIC values significantly outperforming those of MBUR, although they are slightly less effective than the Kumaraswamy and Beta distributions. One key advantage of using the generalized form of MBUR is that it reduces variance, as demonstrated by the notably lower values of the determinants of the variance-covariance matrices obtained after fitting the GOMBUR-1 and GOMBUR-2 distributions. The estimated variance for alpha is significantly lower than that obtained from fitting the Beta and Kumaraswamy distributions. Additionally, the estimated alpha levels, their variances, and standard errors are consistent across both versions of the GOMBUR distributions. Although the Kumaraswamy fitting shows more negative values for AIC, BIC, and HQIC—indicating superior performance compared to both the Beta and GOMBUR-1 data fittings—it also has a higher value for the determinant of the variance-covariance matrix, leading to a less efficient fit of the data.

Figure 9 show the eCDF and theoretical CDF for the fitted distributions. Figure 10 depicts the fitted PDF. While Figure 11 displays the fitted (CDFs) and the fitted (PDFs) for both versions. The shapes of the PDFs are symmetrical and identical for both versions, which is expected given the large estimated values of "n." Generally, as the estimated "n" increases, the distribution becomes more symmetrical.

6 Conclusion

The addition of a new parameter to the previously studied MBUR distribution enhances the capability of MBUR to fit the data. Two versions for generalizing the MBUR were discussed by the author. Both have a non-explicit closed form of the CDF, but they can be expressed using special function. Subsequently, the quantile functions for both versions are not expressed in closed form. Also the rth moments of both versions are expressed with the aid of special function. This is considered as a limitation for being used in applications like median based quantile regression or mean based regression in generalized linear model. The advantage of adding this new parameter is that it helps MBUR to fit near symmetric data. Moreover, the two versions of this Generalized Odd MBUR exhibit a new shape for the hazard rate in the form of an oscillating pattern at the end of the distribution before approaching infinity and at different values of the random variable depending on the level of the alpha and the n parameter (See Supplementary materials 1).

The analysis of the datasets shows that adding a new parameter to the MBUR model increases its flexibility, allowing it to accommodate diverse shapes of data with different characteristics, such as skewness, kurtosis, and various tail behaviors. This newly introduced parameter enhances the estimation process by improving validity indices like AIC, CAIC, BIC, and HQIC. Additionally, it enhances the goodness of fit by reducing test statistics such as the (AD), (CVM), and (KS) tests. Furthermore, it increases the value of the Log-Likelihood. The two versions of the generalization yield different values for the parameter (n), but they have equal values for the parameter (alpha). The variances of the estimated (alpha) obtained from the two versions are identical, and the covariance between the two parameters is minimal, which is significantly lower than the covariance observed when fitting distributions like the Beta and the Kumaraswamy. The determinant of the estimated variance-covariance matrix obtained from fitting GOMBUR-1 is minimal, almost the lowest compared to that achieved after fitting the Beta and Kumaraswamy distributions. In Supplementary material 3, the author discusses various new generalized unit distributions utilizing the general formula for the order statistics. These distributions can be candidates for various applications in future work.

7 Future work

In this paper, the Nelder-Mead optimizer was employed for Maximum Likelihood Estimation (MLE) of the parameters. Future work may explore Bayesian inference procedures. Methods like Maximum product of spacing, least square and weighted least square methods can be attempted but the encountered limitation is that the CDF does not have a well-closed form. This limitation will impose some cumbersome calculations to estimate the parameters. The generalized method of moments using moment based approach by equating the sample moments with the population moments can offer an alternative for parameter estimation. But this will also face some difficult calculations as the moments require special functions. Applying derivative based algorithms like Quasi-Newton methods could be another solution for the optimization procedure.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

IA: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

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Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Supplementary material

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fams.2025.1648127/full#supplementary-material

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